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KODAI MATH. J.
36 (2013), 313–324

AREA INTEGRAL MEANS, HARDY AND WEIGHTED BERGMAN SPACES OF PLANAR HARMONIC MAPPINGS

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Abstract

In this paper, we investigate some properties of planar harmonic mappings. First, we generalize the main results in [2] and [10], and then discuss the relationship between area integral means and harmonic Hardy spaces or harmonic weighted Bergman spaces. At the end, coefficient estimates of mappings in weighted Bergman spaces are obtained.

1. Introduction and main results

For each $r \in (0, 1]$, we denote by $\mathbf{D}_r$ the open disk $\{z \in \mathbf{C} : |z| < r\}$ and by $\mathbf{D}$, the open unit disk $\mathbf{D}_1$. The *harmonic Hardy space* $\mathscr{H}_h^p(\mathbf{D})$ with $0 < p < \infty$ consists of all complex-valued functions f harmonic in $\mathbf{D}$ (i.e. $f_{z\bar{z}} = 0$ in $\mathbf{D}$) for which

$$\|f\|_p := \sup_{0<r<1} (I_p(r, f))^{1/p} < \infty, \quad I_p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

The classical analytic Hardy space over the unit disk $\mathbf{D}$, denoted usually by $\mathscr{H}^p(\mathbf{D})$, is obviously contained in $\mathscr{H}_h^p(\mathbf{D})$. We refer to [5, 7] for many basic analytic and geometric properties of univalent harmonic mappings, in particular.

In this paper, we call a complex-valued harmonic function as a harmonic mapping. For a harmonic mapping f in $\mathbf{D}$ and $0 \le r < 1$, the *generalized harmonic area function* $A_h(r)$ of f is defined by (cf. [2])

$$A_h(r) = A_h(r, f) = \int_{\mathbf{D}_r} |\widehat{\nabla} f(z)|^2 \, dA(z),$$

where dA denotes the normalized Lebesgue measure on $\mathbf{D}$,

$$\widehat{\nabla} f = (f_z, f_{\bar{z}}) \quad \text{and} \quad |\widehat{\nabla} f| = (|f_z|^2 + |f_{\bar{z}}|^2)^{1/2}.$$

2000 *Mathematics Subject Classification*. Primary: 30C65, 30C45; Secondary: 30C20.

Key words and phrases. Harmonic mapping, harmonic Hardy space, area integral means, and Bergmann spaces.

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Received April 3, 2012; revised December 11, 2012.

In particular, if f is analytic in $\mathbf{D}$, then we denote the analytic area function of f by $A(r) := A(r, f) = \int_{\mathbf{D}_r} |f'(z)|^2 \, dA(z)$.

In [10, 16], the authors discussed the relationship between (analytic) Hardy spaces and area functions. The main result in [10, Theorem 1] is as follows.

Theorem A. *Let f be analytic in $\mathbf{D}$. Then, if $1 < p \le 2$,*

$$f \in \mathscr{H}^p(\mathbf{D}) \Rightarrow \int_0^1 A^{p/2}(r)\, dr < \infty, \tag{1}$$

while if $p > 2$,

$$\int_0^1 A^{p/2}(r)\, dr < \infty \Rightarrow f \in \mathscr{H}^p(\mathbf{D}). \tag{2}$$

We refer to [6, 8, 9, 10, 12, 13, 14, 15, 16] for results related to the theory of analytic Hardy spaces, whereas for the harmonic Hardy spaces, the readers may refer to [2, 4, 11]. In the context of recent investigation and interest on harmonic mappings, it is natural to ask whether Theorem A continues to hold in the setting of planar harmonic mappings over the unit disk. In this note we show that the answer is yes.

Theorem 1. *Let f be harmonic in $\mathbf{D}$. Then, if $1 < p \le 2$,*

$$f \in \mathscr{H}_h^p(\mathbf{D}) \Rightarrow \int_0^1 A_h^{p/2}(r, f)\, dr < \infty, \tag{3}$$

while if $p > 2$,

$$\int_0^1 A_h^{p/2}(r, f)\, dr < \infty \Rightarrow f \in \mathscr{H}_h^p(\mathbf{D}). \tag{4}$$

As an application of Theorem 1, we obtain the following result.

Theorem 2. *Let $f \in \mathscr{H}_h^p(\mathbf{D})$. If $1 < p \le 2$, then $\lim_{r\to 1^-}(1 - r)^{2/p} A_h(r, f) = 0$.*

Remark 1. Theorems 1 and 2 show that the factor $(1 - r)^{\delta(2-p)/2}$ in [2, Theorem 3] and the one $(1 - r)^{\delta(2-p)/p}$ in [2, Theorem 4] are redundant. Later, it was brought to our attention that [2, Theorem 3] and [2, Theorem 4] were proved by Stević in 2004 [19] in a slightly different method of proof.

For a given real number α, we consider the weighted area measure $dA_\alpha^*(z) = (1 - |z|^2)^\alpha \, dA(z)$ on $\mathbf{D}$ (cf. [20]). For $0 < r < 1$ and $0 < p < \infty$, we define

$$M_{p,\alpha}(r, f) = \left[\frac{1}{A_\alpha^*(\mathbf{D}_r)} \int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z)\right]^{1/p},$$

where f is harmonic in $\mathbf{D}$ and

$$A_\alpha^*(\mathbf{D}_r) = \int_{\mathbf{D}_r} dA_\alpha^*(z).$$

We call $M_{p,\alpha}(r,f)$, the *area integral means* of f on $\mathbf{D}_r$.

It is well known that the measure A_α^* is finite on $\mathbf{D}$ if and only if $\alpha > -1$. In the following, for $\alpha > -1$, we normalize the measure dA_α^* by letting

$$dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha \, dA(z).$$

For a harmonic mapping f in $\mathbf{D}$, we denote

$$A_\alpha(\mathbf{D}_r) = \int_{\mathbf{D}_r} dA_\alpha(z),$$

where $\alpha > -1$.

For $\alpha > -1$ and $0 < p \le \infty$, the *weighted Bergman space* $A_{h,\alpha}^p(\mathbf{D})$ consists of all harmonic mappings f on $\mathbf{D}$ such that

$$\|f\|_{b^p,\alpha}^h = \begin{cases} \left(\int_{\mathbf{D}} |f(z)|^p \, dA_\alpha(z) \right)^{1/p} < \infty & \text{if } p \in (0,\infty), \\ \sup_{z \in \mathbf{D}} |f(z)| < \infty & \text{if } p = \infty. \end{cases}$$

Our next result provides the relationship between area integral means and harmonic Hardy spaces or harmonic weighted Bergman spaces.

THEOREM 3. *Suppose $1 < p < \infty$, α is real, and f is harmonic in $\mathbf{D}$. Then, we have the following*:

(a) *The function $M_{p,\alpha}(r,f)$ is strictly increasing in $[0,1)$ unless f is constant.*
(b) *For $\alpha > -1$, $M_{p,\alpha}(r,f)$ is bounded in $[0,1)$ if and only if $f \in A_{h,\alpha}^p(\mathbf{D})$.*
(c) *For $\alpha \le -1$, $M_{p,\alpha}(r,f)$ is bounded in $[0,1)$ if and only if $f \in \mathscr{H}_h^p(\mathbf{D})$.*

Our final result concerns the coefficient estimate on mappings in harmonic weighted Bergman spaces.

THEOREM 4. *For $1 \le p \le \infty$, let $f \in A_{h,\alpha}^p(\mathbf{D})$ with*

$$f(z) = \sum_{m=0}^\infty a_m z^m + \sum_{m=1}^\infty \overline{b}_m \overline{z}^m.$$

Then $|a_0| \le \|f\|_{b^p,\alpha}^h$, and for $m \ge 1$,

$$|a_m| + |b_m| \le \frac{4\|f\|_{b^p,\alpha}^h}{\pi} \inf_{0<r<1} \left\{ \frac{1}{r^m [1 - r^{\alpha+1}(2-r)^{\alpha+1}]^{1/p}} \right\}.$$

In particular, if $\alpha = 0$, then for $m \ge 1$,

$$|a_m| + |b_m| \le \frac{4\|f\|_{b^p,0}^h}{\pi} \left(\frac{2}{pm} + 1 \right)^m \left(1 + \frac{pm}{2} \right)^{2/p}.$$

Moreover, if $\alpha = 0$ *and* $p = \infty$, *then*

$$|a_m| + |b_m| \leq \frac{4\|f\|^h_{b^\infty,0}}{\pi}. \tag{5}$$

The estimate (5) *is sharp and the only extremal functions are*

$$f_m(z) = \frac{2\gamma\|f\|^h_{b^\infty,0}}{\pi} \arg\left(\frac{1+\beta z^m}{1-\beta z^m}\right),$$

where $|\gamma| = |\beta| = 1$, *and* $m \geq 1$.

2. Proofs of the main results

We begin this section with the following two basic lemmas which are useful in the proof of Theorem 1.

Lemma B. *Let* $a, b \in [0, \infty)$ *and* $p \in [1, \infty)$. *Then we have*

$$a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Lemma B is well-known (see for instance [18, Lemma 2.29]).

Lemma 1. *Let* f *be a complex-valued continuously differentiable function defined on* $\mathbf{D}$ *and* $f = u + iv$, *where* u *and* v *are real-valued functions. Then for* $z = x + iy \in \mathbf{D}$,

$$|f_z(z)| + |f_{\bar{z}}(z)| \leq |\nabla u(x, y)| + |\nabla v(x, y)|, \tag{6}$$

where $\nabla u = (u_x, u_y)$ *and* $\nabla v = (v_x, v_y)$.

Proof. From the triangle inequality, it follows that

$$|f_z| = \left|\frac{1}{2}\{u_x - iu_y + i(v_x - iv_y)\}\right| \leq \frac{1}{2}\{|u_x - iu_y| + |v_x - iv_y|\} = \frac{1}{2}\{|\nabla u| + |\nabla v|\}$$

and

$$|f_{\bar{z}}| = \left|\frac{1}{2}\{u_x + iu_y + i(v_x + iv_y)\}\right| \leq \frac{1}{2}\{|u_x - iu_y| + |v_x - iv_y|\} = \frac{1}{2}\{|\nabla u| + |\nabla v|\}$$

from which we easily obtain (6).

Finally, we remark that the equality sign in (6) does not always hold as the function $f(z) = z^2 + \bar{z}$ shows. $\square$

Proof of Theorem 1. We first prove the implication (3). Let $1 < p \leq 2$ and $f = u + iv \in \mathscr{H}_h^p(\mathbf{D})$. Then u and v are real harmonic functions in $\mathbf{D}$. By Lemma B, we deduce that $u, v \in \mathscr{H}_h^p(\mathbf{D})$. Let F_1 and F_2 be analytic functions defined on $\mathbf{D}$ such that $\operatorname{Re} F_1 = u$ and $\operatorname{Re} F_2 = v$. Riesz' theorem (cf. [6, Theorem 4.1]) shows that

$$\|F_k\|_p \leq \left(\frac{p}{p-1}\right)^{1/p} \|\operatorname{Re} F_k\|_p \quad \text{for } k = 1, 2,$$

which, in particular, implies that $F_k \in \mathscr{H}^p(\mathbf{D})$ for $k = 1, 2$. By the implication (1) in Theorem A, it follows that

$$\int_0^1 A_h^{p/2}(r, F_k)\, dr < \infty \quad \text{for } k = 1, 2. \tag{7}$$

By calculations, we see that for $r \in (0, 1)$,

$$A_h(r, F_1) = \int_{\mathbf{D}_r} |F_1'(z)|^2\, dA(z) = \int_{\mathbf{D}_r} |\nabla u(x, y)|^2\, dA(z) = A_h(r, u) \tag{8}$$

and similarly,

$$A_h(r, F_2) = \int_{\mathbf{D}_r} |F_2'(z)|^2\, dA(z) = \int_{\mathbf{D}_r} |\nabla v(x, y)|^2\, dA(z) = A_h(r, v). \tag{9}$$

The inequalities (7), (8), (9) and Lemmas B and 1 yield that

$$\begin{aligned}
\int_0^1 A_h^{p/2}(r, f)\, dr &= \int_0^1 \left[\int_{\mathbf{D}_r} (|f_z(z)|^2 + |f_{\bar z}(z)|^2)\, dA(z)\right]^{p/2} dr \\
&\leq \int_0^1 \left[\int_{\mathbf{D}_r} (|f_z(z)| + |f_{\bar z}(z)|)^2\, dA(z)\right]^{p/2} dr \\
&\leq \int_0^1 \left[\int_{\mathbf{D}_r} (|\nabla u(x, y)| + |\nabla v(x, y)|)^2\, dA(z)\right]^{p/2} dr \\
&\leq \int_0^1 \left[\int_{\mathbf{D}_r} 2(|\nabla u(x, y)|^2 + |\nabla v(x, y)|^2)\, dA(z)\right]^{p/2} dr \\
&\leq 2^{(2p-1)/2} \int_0^1 \left[\left(\int_{\mathbf{D}_r} |\nabla u(x, y)|^2\, dA(z)\right)^{p/2} \right. \\
&\qquad \left. + \left(\int_{\mathbf{D}_r} |\nabla v(x, y)|^2\, dA(z)\right)^{p/2}\right] dr \\
&= 2^{(2p-1)/2} \int_0^1 [A_h^{p/2}(r, F_1) + A_h^{p/2}(r, F_2)]\, dr \\
&< \infty
\end{aligned}$$

which proves the implication (3).

We next prove the implication (4). Let $p > 2$ and f be harmonic in $\mathbf{D}$. Then f admits the canonical decomposition $f = \phi + \overline{\psi}$, where ϕ and ψ are analytic in $\mathbf{D}$ with $\psi(0) = 0$. Then

$$\int_{\mathbf{D}_r} (|\phi'(z)|^2 + |\psi'(z)|^2)\, dA(z) = A_h(r, f),$$

which implies

$$\int_0^1 A^{p/2}(r, \phi)\, dr < \infty \quad \text{and} \quad \int_0^1 A^{p/2}(r, \psi)\, dr < \infty. \tag{10}$$

By the implication (2) in Theorem A and (10), we conclude that $\phi, \psi \in \mathscr{H}^p(\mathbf{D})$. But then by the Minkowski inequality, we deduce that

$$\begin{aligned}(I_p(r, f))^{1/p} &\le \left(\frac{1}{2\pi}\int_0^{2\pi} (|\phi(re^{i\theta})| + |\psi(re^{i\theta})|)^p\, d\theta\right)^{1/p} \\ &\le (I_p(r, \phi))^{1/p} + (I_p(r, \psi))^{1/p},\end{aligned}$$

which yields that $\|f\|_p < \infty$. □

Proof of Theorem 2. It is not difficult to see that

$$(1 - r)A_h^{p/2}(r, f) \le \int_r^1 A_h^{p/2}(\rho, f)\, d\rho, \text{ i.e. } (1 - r)^{2/p} A_h(r, f) \le \left(\int_r^1 A_h^{p/2}(\rho, f)\, d\rho\right)^{2/p}.$$

By the implication (3) in Theorem 1, we conclude

$$\int_0^1 A_h^{p/2}(r, f)\, dr < \infty$$

from which we obtain that $\lim_{r \to 1^-} (1 - r)^{2/p} A_h(r, f) = 0$. □

Lemma 2. *Suppose that f is harmonic on $\mathbf{D}$ and is constant in an open neighborhood of the origin. Then f is constant throughout the unit disk $\mathbf{D}$.*

Proof. As every harmonic function f in $\mathbf{D}$ admits the representation

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n,$$

we may assume that $f(z) = a_0$ in $\mathbf{D}_r$, for some $r \in (0, 1)$. But then the Parseval relation, for $0 < \rho < r$, gives

$$|a_0|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^2\, d\theta = |a_0|^2 + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)\rho^{2n},$$

which obviously implies $a_n = b_n = 0$ for all $n \ge 1$. Thus, $f(z) \equiv a_0$ for $z \in \mathbf{D}$. □

Green's theorem (cf. [2, 4, 17]) states that if $g \in C^2(\mathbf{D})$, i.e., twice continuously differentiable on $\mathbf{D}$, then

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,d\theta = g(0) + \frac{1}{2}\int_{\mathbf{D}_r} \Delta g(z)\log\frac{r}{|z|}\,dA(z), \quad r \in (0,1). \tag{11}$$

Lemma 3. *Let f be harmonic in $\mathbf{D}$. Then, for $p > 1$, $I_p(r,f)$ is a strictly increasing function of r on $(0,1)$ unless f is constant.*

Proof. By (11), we have

$$\begin{aligned}
r\frac{d}{dr}I_p(r,f) &= \frac{1}{2}\int_{\mathbf{D}_r} \Delta(|f(z)|^p)\,dA(z)\\
&= p\int_{\mathbf{D}_r}\left[\left(\frac{p}{2}-1\right)|f(z)|^{p-4}|f_z(z)\overline{f(z)} + f(z)\overline{f_{\bar z}(z)}|^2\right.\\
&\qquad\qquad \left. + |f(z)|^{p-2}|\widehat{\nabla} f(z)|^2\right] dA(z)\\
&\geq p\int_{\mathbf{D}_r}(|f_z(z)| + |f_{\bar z}(z)|)^2|f(z)|^{p-2}\,dA(z)\\
&\geq 0,
\end{aligned}$$

which implies $I_p(r,f)$ is increasing on r in $(0,1)$. Moreover, the last inequality implies that $\frac{d}{dr}I_p(r,f) = 0$ if and only if f is constant in $\mathbf{D}_r$. But then, in this case, Lemma 2 shows that f is constant on $\mathbf{D}$. □

Proof of Theorem 3. We first prove (a). Since

$$\int_{\mathbf{D}_r} |f(z)|^p\,dA_\alpha^*(z) = \int_0^r 2\rho(1-\rho^2)^\alpha I_p(\rho,f)\,d\rho, \tag{12}$$

we see that

$$\frac{d}{dr}\int_{\mathbf{D}_r} |f(z)|^p\,dA_\alpha^*(z) = 2r(1-r^2)^\alpha I_p(r,f). \tag{13}$$

Simple calculations gives

$$\frac{d}{dr}A_\alpha^*(\mathbf{D}_r) = 2r(1-r^2)^\alpha. \tag{14}$$

By (12), (14) and Lemma 3, we have

$$I_p(r,f) - M_{p,\alpha}^p(r,f) = \frac{1}{A_\alpha^*(\mathbf{D}_r)}\int_0^r \left[\frac{d}{dt}I_p(t,f)\right] A_\alpha^*(\mathbf{D}_t)\,dt \geq 0,$$

which implies

$$\int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z) \le A_\alpha^*(\mathbf{D}_r) I_p(r, f). \tag{15}$$

By Lemma 3, we know that the equality holds in (15) for some r only when f is constant. By (13), (15) and computations, we conclude that

$$\begin{aligned}\frac{d}{dr} M_{p,\alpha}^p(r, f) &= \frac{A_\alpha^*(\mathbf{D}_r) \frac{d}{dr} \int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z) - \int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z) \frac{d}{dr} A_\alpha^*(\mathbf{D}_r)}{A_\alpha^{*2}(\mathbf{D}_r)} \\ &= \frac{2r(1-r^2)^\alpha [A_\alpha^*(\mathbf{D}_r) I_p(r, f) - \int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z)]}{A_\alpha^{*2}(\mathbf{D}_r)} \\ &\ge 0.\end{aligned}$$

Hence, the function $M_{p,\alpha}(r, f)$ is strictly increasing on $r \in [0, 1)$ unless f is constant.

Next we prove (b). We assume that $\alpha > -1$ and $M_{p,\alpha}(r, f)$ is bounded. Then by (a), we have

$$\lim_{r \to 1-} \left[\frac{1}{A_\alpha(\mathbf{D}_r)} \int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha(z) \right] = \int_{\mathbf{D}} |f(z)|^p \, dA_\alpha(z), \tag{16}$$

which implies $f \in A_{h,\alpha}^p(\mathbf{D})$. On the other hand, if $f \in A_{h,\alpha}^p(\mathbf{D})$, then the boundedness of $M_{p,\alpha}(r, f)$ follows from (16).

In order to prove (c), we need some additional care.

Claim 1. *Suppose that $\alpha \le -1$, $1 \le p < \infty$, and that f is harmonic in $\mathbf{D}$. Then*

$$\int_{\mathbf{D}} |f(z)|^p \, dA_\alpha^*(z) = \sup_{r \in (0,1)} \left\{ \int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z) \right\} < \infty \Leftrightarrow f \equiv 0$$

Proof. Fix $\rho \in (0, 1)$ so that $\rho < r < 1$. Then (a) yields that

$$M_{p,\alpha}^p(R, f) \le \frac{\int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z)}{A_\alpha^*(\mathbf{D}_r)}.$$

It is not a difficult task to see that $A_\alpha^*(\mathbf{D}_r) \to \infty$ and

$$\int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z) \to \int_{\mathbf{D}} |f(z)|^p \, dA_\alpha^*(z) \quad \text{as } r \to 1-,$$

which gives $M_{p,\alpha}(\rho, f) \equiv 0$ for each $\rho \in (0, 1)$. Therefore, $f \equiv 0$ and the proof of the claim is finished. $\square$

Finally, we prove (c). We assume that $\alpha \le -1$, $1 < p < \infty$ and f is not identically zero. Then $A_\alpha^*(\mathbf{D}_r) \to \infty$ as $r \to 1-$ and so Claim 1 implies

$$\lim_{r\to 1-} \int_{\mathbf{D}_r} |f(z)|^p \, dA_\alpha^*(z) = \int_{\mathbf{D}} |f(z)|^p \, dA_\alpha^*(z) = \infty.$$

By (a) and calculations, we get

$$\begin{aligned}
\sup_{0<r<1} M_{p,\alpha}^p(r,f) &= \lim_{r\to 1-} M_{p,\alpha}^p(r,f) \\
&= \lim_{r\to 1-} \frac{\int_{\mathbf{D}_r} |f(z)|^p (1-|z|^2)^\alpha \, dA(z)}{\int_{\mathbf{D}_r} (1-|z|^2)^\alpha \, dA(z)} \\
&= \lim_{r\to 1-} \frac{2r(1-r^2)^\alpha \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta}{2r(1-r^2)^\alpha} \\
&= \lim_{r\to 1-} I_p(r,f) = \|f\|_p^p
\end{aligned}$$

and the proof of the theorem is complete. □

Proof of Theorem 4. It is not difficult to show that for $p \in [1,\infty)$, $|f|^p$ is subharmonic in $\mathbf{D}$. Then for $z \in \mathbf{D}$ and $r \in [0, 1-|z|)$, we have

$$|f(z)|^p \le \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta} + z)|^p \, d\theta.$$

Integration gives

$$\begin{aligned}
[1 - |z|^{\alpha+1}(2-|z|)^{\alpha+1}]|f(z)|^p &\le \frac{1+\alpha}{\pi} \int_0^{2\pi} \int_0^{1-|z|} r(1-r^2)^\alpha |f(z+re^{i\theta})|^p \, dr d\theta \\
&\le \int_{\mathbf{D}} |f(z)|^p \, dA_\alpha(z) = (\|f\|_{b^p,\alpha}^h)^p,
\end{aligned}$$

which implies

$$|f(z)| \le \frac{\|f\|_{b^p,\alpha}^h}{[1-|z|^{\alpha+1}(2-|z|)^{\alpha+1}]^{1/p}}. \tag{17}$$

For $\zeta \in \mathbf{D}$ and $r \in (0,1)$, let $F(\zeta) = f(r\zeta)/r$. Then

$$F(\zeta) = \frac{a_0}{r} + \sum_{m=1}^\infty A_m \zeta^m + \sum_{m=1}^\infty \bar{B}_m \bar{\zeta}^m,$$

where $A_m = a_m r^{m-1}$ and $B_m = b_m r^{m-1}$. Hence for $\zeta \in \mathbf{D}$,

$$|F(\zeta)| \le \frac{\|f\|^h_{b^p,\alpha}}{r[1 - r^{\alpha+1}(2-r)^{\alpha+1}]^{1/p}} = M(r).$$

By (17), we see that $|a_0| \le \|f\|^h_{b^p,\alpha}$. It follows from [1, Lemma 1] that for $m \ge 1$,

$$|A_m| + |B_m| \le \frac{4M(r)}{\pi}$$

which yields

$$|a_m| + |b_m| \le \frac{4\|f\|^h_{b^p,\alpha}}{\pi} \inf_{0<r<1} \left\{ \frac{1}{r^m[1 - r^{\alpha+1}(2-r)^{\alpha+1}]^{1/p}} \right\}.$$

If $\alpha = 0$, then

$$|a_m| + |b_m| \le \frac{4\|f\|^h_{b^p,0}}{\pi} \inf_{0<r<1} \left[\frac{1}{r^m(1-r)^{2/p}} \right] = \frac{4\|f\|^h_{b^p,0}}{\pi} \left(\frac{2}{pm} + 1 \right)^m \left(1 + \frac{pm}{2} \right)^{2/p}.$$

Since

$$\lim_{p\to\infty} \left(\frac{2}{pm} + 1 \right)^m \left(1 + \frac{pm}{2} \right)^{2/p} = 1,$$

we conclude that

$$|a_m| + |b_m| \le \frac{4\|f\|^h_{b^\infty,0}}{\pi}. \tag{18}$$

Thus, for $p = \infty$, the estimate (18) is sharp. By the subordination in the proof of [3, Theorem 1], we know that the only extreme functions are

$$f_m(z) = \frac{2\gamma\|f\|^h_{b^\infty,0}}{\pi} \operatorname{Im}\left(\log \frac{1 + \beta z^m}{1 - \beta z^m} \right) \quad (|\gamma| = |\beta| = 1),$$

whose values are confined to $\mathbf{D}_{\|f\|^h_{b^\infty,0}} = \{z : |z| < \|f\|^h_{b^\infty,0}\}$. □

Acknowledgements. The authors would like to thank the referee for his/her useful suggestions. This research was partly supported by NSF of China (No. 11071063). The work of the first author was also supported in part by the Construct Program of the Key Discipline in Hunan Province (No. [2011] 76) and the Start Project of Hengyang Normal University (No. 12B34).

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