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A NOTE ON THE GEOMETRICITY OF OPEN HOMOMORPHISMS BETWEEN THE ABSOLUTE GALOIS GROUPS OF p-ADIC LOCAL FIELDS

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Abstract

In the present paper, we prove that an open continuous homomorphism between the absolute Galois groups of p-adic local fields is *geometric* [i.e., roughly speaking, arises from an embedding of fields] if and only if the homomorphism is *HT-preserving* [i.e., roughly speaking, satisfies the condition that the pull-back by the homomorphism of every Hodge-Tate representation is Hodge-Tate].

Introduction

Let p be a prime number. Write $\mathbf{Q}_p$ for the p-adic completion of the field of rational numbers $\mathbf{Q}$. For $\square \in \{\circ, \bullet\}$, let $k_\square$ be a p-adic local field [i.e., a finite extension of $\mathbf{Q}_p$] and $\bar{k}_\square$ an algebraic closure of $k_\square$. Write $G_{k_\square} \stackrel{\text{def}}{=} \mathrm{Gal}(\bar{k}_\square/k_\square)$. Let

$$\alpha : G_{k_\circ} \to G_{k_\bullet}$$

be an open continuous homomorphism. In [1], [2], S. Mochizuki discussed the *geometricity* [cf. [2], Definition 3.1, (iv)] of such an α. In particular, Mochizuki proved that the following conditions are equivalent [cf. [2], Theorem 3.5, (i)]:

(i) α is *geometric*, i.e., arises from an isomorphism of fields $\bar{k}_\bullet \xrightarrow{\sim} \bar{k}_\circ$ that determines an embedding $k_\bullet \hookrightarrow k_\circ$.

(ii) α is *of CHT-type* [cf. [2], Definition 3.1, (iv)], i.e., α is *compatible* with the respective p-adic *cyclotomic characters* of $G_{k_\circ}$, $G_{k_\bullet}$, and, moreover, there exists an isomorphism of topological modules [but *not necessarily the topological fields*] $\bar{k}_\circ^\wedge \xrightarrow{\sim} \bar{k}_\bullet^\wedge$—where, for $\square \in \{\circ, \bullet\}$, we write $\bar{k}_\square^\wedge$ for the p-adic completion of $\bar{k}_\square$—that is *compatible* with the respective natural actions of $G_{k_\circ}$, $G_{k_\bullet}$ on $\bar{k}_\circ^\wedge$, $\bar{k}_\bullet^\wedge$ [relative to α].

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(iii) α is *of 01-qLT-type* [cf. [2], Definition 3.1, (iv)], i.e., for every pair of open subgroups $H_\circ \subseteq G_{k_\circ}$, $H_\bullet \subseteq G_{k_\bullet}$ of $G_{k_\circ}$, $G_{k_\bullet}$ such that $\alpha(H_\circ) \subseteq H_\bullet$, and every character $\phi : H_\bullet \to E^\times$ *of qLT-type* [cf. [2], Definition 3.1, (iii)]—where E is a p-adic local field all of whose $\mathbf{Q}_p$-conjugates are contained in the fixed fields $\bar{k}_\circ^{H_\circ}$, $\bar{k}_\bullet^{H_\bullet}$—the composite $H_\circ \xrightarrow{\alpha|_{H_\circ}} H_\bullet \xrightarrow{\phi} E^\times$ is *Hodge-Tate*, and the *set of Hodge-Tate weights of this composite is contained in* $\{0, 1\}$.

We shall say that α is *HT-preserving* [cf. Definition 1.3, (i)] if α preserves the Hodge-Tate-ness of p-adic representations, i.e., for every finite dimensional continuous representation $\phi : G_{k_\bullet} \to \mathrm{GL}_n(\mathbf{Q}_p)$ of $G_{k_\bullet}$, if ϕ is Hodge-Tate, then the composite $G_{k_\circ} \xrightarrow{\alpha} G_{k_\bullet} \xrightarrow{\phi} \mathrm{GL}_n(\mathbf{Q}_p)$ is Hodge-Tate. Then it is immediate that

if α is *of CHT-type*, then α is *HT-preserving*.

Moreover, since a character *of qLT-type* is *Hodge-Tate*, and its set of Hodge-Tate weights is *contained* in $\{0, 1\}$, one verifies easily that

if α is *not only HT-preserving but also preserves the sets of Hodge-Tate weights* of Hodge-Tate representations, then α is *of 01-qLT-type*.

On the other hand, it does not seem to be clear that the following assertion holds:

If α is *HT-preserving*, then α is either *of CHT-type* or *of 01-qLT-type*.

In particular, the following question may be regarded as a natural question concerning the *geometricity* of open continuous homomorphisms between the absolute Galois groups of p-adic local fields:

Is every *HT-preserving* open continuous homomorphism between the absolute Galois groups of p-adic local fields *geometric*?

In the present paper, we answer this question in the affirmative by refining the argument of Mochizuki applied in [1], [2]. The main consequence of the present paper is as follows [cf. Corollaries 3.4; 3.5].

Theorem. *Let p be a prime number. For $\square \in \{\circ, \bullet\}$, let $k_\square$ be a p-adic local field and $\bar{k}_\square$ an algebraic closure of $k_\square$. Write $G_{k_\square} \overset{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}_\square/k_\square)$. Let*

$$\alpha : G_{k_\circ} \to G_{k_\bullet}$$

be an open continuous homomorphism. Then α is **geometric** [*cf.* [2], *Definition* 3.1, (iv)] *if and only if α is* **HT-preserving** [*cf. Definition* 1.3, (i)]. *In particular, if we write*

$$\mathrm{Emb}(\bar{k}_\bullet/k_\bullet, \bar{k}_\circ/k_\circ)$$

for the set of isomorphisms of fields $\bar{k}_\bullet \xrightarrow{\sim} \bar{k}_\circ$ that determine embeddings $k_\bullet \hookrightarrow k_\circ$;

$$\mathrm{Emb}(k_\bullet, k_\circ)$$

for the set of embeddings of fields $k_\bullet \hookrightarrow k_\circ$;

$$\mathrm{Hom}^{\mathrm{open}}_{\mathrm{HT}}(G_{k_\circ}, G_{k_\bullet})$$

for the set of **HT-preserving** *open continuous homomorphisms* $G_{k_\circ} \to G_{k_\bullet}$, *then we have a commutative diagram of natural maps*

$$\begin{array}{ccc} \mathrm{Emb}(\overline{k}_\bullet/k_\bullet, \overline{k}_\circ/k_\circ) & \xrightarrow{\sim} & \mathrm{Hom}^{\mathrm{open}}_{\mathrm{HT}}(G_{k_\circ}, G_{k_\bullet}) \\ \downarrow & & \downarrow \\ \mathrm{Emb}(k_\bullet, k_\circ) & \xrightarrow{\sim} & \mathrm{Hom}^{\mathrm{open}}_{\mathrm{HT}}(G_{k_\circ}, G_{k_\bullet})/\mathrm{Inn}(G_{k_\bullet}) \end{array}$$

—where the vertical arrows are **surjective**, *and the horizontal arrows are* **bijective**.

Remark. The various discussions given in the present paper may be regarded as just slight modifications or improvements of the discussions of [1], [2]. From this point of view, one may consider that some arguments in §2 and the observation that a similar technique of [1], §4, can be available in the situation of the proof of Theorem 3.3 are essentially the only new contributions of the present paper.

1. HT-preserving homomorphisms

In the present §1, we define the notion of an *HT-preserving* [i.e., "Hodge-Tate-preserving"] homomorphism [cf. Definition 1.3, (i), below]. Let p be a prime number. Write $\mathbf{Q}_p$ for the p-adic completion of the field of rational numbers $\mathbf{Q}$. For $\square \in \{\circ, \bullet, \emptyset\}$, let $k_\square$ be a p-adic local field [i.e., a finite extension of $\mathbf{Q}_p$] and $\overline{k}_\square$ an algebraic closure of $k_\square$. Write $\mathfrak{o}_{k_\square}$ for the ring of integers of $k_\square$, $G_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square)$, $I_{k_\square} \subseteq G_{k_\square}$ for the inertia subgroup of $G_{k_\square}$, and $P_{k_\square} \subseteq I_{k_\square}$ for the wild inertia subgroup of $G_{k_\square}$. Now let us recall from *local class field theory* that we have a natural isomorphism

$$G_k^{\mathrm{ab}} \xrightarrow{\sim} (k^\times)^\wedge$$

—where we write $(k^\times)^\wedge$ for the profinite completion of the topological group $k^\times$—that determines an isomorphism

$$(G_k^{\mathrm{ab}} \supseteq) \quad \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}}) \xrightarrow{\sim} \mathfrak{o}_k^\times \quad (\subseteq (k^\times)^\wedge).$$

In the following, let us regard $\mathfrak{o}_k^\times$ as a closed subgroup of G_k^{ab} by means of this isomorphism, i.e., $\mathfrak{o}_k^\times \subseteq G_k^{\mathrm{ab}}$.

Proposition 1.1. *Let* $\alpha : G_{k_\circ} \to G_{k_\bullet}$ *be an open continuous homomorphism. Then* $\alpha(I_{k_\circ})$, $\alpha(P_{k_\circ}) \subseteq G_{k_\bullet}$ *are* **open** *subgroups of* $I_{k_\bullet}$, $P_{k_\bullet}$, *respectively. Moreover, it holds that* $\mathrm{Ker}(\alpha) \subseteq P_{k_\circ}$.

Proof. This follows immediately from [2], Proposition 3.4 [cf. also the proof of [2], Proposition 3.4]. □

DEFINITION 1.2.

(i) Let A be a topological group; $\phi_1, \phi_2 : G_k \to A$ continuous homomorphisms. Then we shall say that ϕ_1 is *inertially equivalent* to ϕ_2 if ϕ_1 and ϕ_2 coincide on an open subgroup of $I_k \subseteq G_k$ [cf. the discussion preceding [4], Chapter III, §A.5, Theorem 2].

(ii) Let E be a finite Galois extension of $\mathbf{Q}_p$ that admits an embedding $\sigma : E \hookrightarrow k$. Let $\pi \in \mathfrak{o}_k$ be a uniformizer of $\mathfrak{o}_k$. Then we shall write

$$\chi^{\mathrm{LT}}_{\sigma,\pi} : G_k \to E^\times$$

for the continuous character obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{\mathrm{ab}} \xrightarrow{\sim} (k^\times)^\wedge \xrightarrow{\sim} \mathfrak{o}_k^\times \times \hat{\mathbf{Z}} \twoheadrightarrow \mathfrak{o}_k^\times \to \mathfrak{o}_E^\times \xrightarrow{\sim} \mathfrak{o}_E^\times \hookrightarrow E^\times$$

—where the first arrow is the natural surjection, the second arrow is the natural isomorphism arising from *local class field theory*, the third arrow is the isomorphism determined by the uniformizer $\pi \in \mathfrak{o}_k$, the fourth arrow is the first projection, the fifth arrow is the homomorphism induced by the norm map $k^\times \to E^\times$ [with respect to the embedding σ], the sixth arrow is the isomorphism given by mapping a to a^{-1}, and the seventh arrow is the natural inclusion [cf. [4], Chapter III, §A.4]. Since $I_k \subseteq G_k$ surjects onto $\mathfrak{o}_k \times \{1\} \subseteq \mathfrak{o}_k \times \hat{\mathbf{Z}}$ [cf. the discussion at the beginning of §1], one verifies easily that the *inertial equivalence class* [cf. (i)] of $\chi^{\mathrm{LT}}_{\sigma,\pi}$ does *not depend* on the choice of $\pi \in \mathfrak{o}_k$. Thus, we shall often write $\chi^{\mathrm{LT}}_{\sigma}$ to denote $\chi^{\mathrm{LT}}_{\sigma,\pi}$ for some unspecified choice of $\pi \in \mathfrak{o}_k$.

DEFINITION 1.3. Let $\alpha : G_{k_\circ} \to G_{k_\bullet}$ be an open continuous homomorphism.

(i) We shall say that α is *HT-preserving* [i.e., "Hodge-Tate-preserving"] if, for every finite dimensional continuous representation $\phi : G_{k_\bullet} \to \mathrm{GL}_n(\mathbf{Q}_p)$ of $G_{k_\bullet}$ that is Hodge-Tate, the composite $G_{k_\circ} \xrightarrow{\alpha} G_{k_\bullet} \xrightarrow{\phi} \mathrm{GL}_n(\mathbf{Q}_p)$ is Hodge-Tate.

(ii) We shall say that α is *of HT-qLT-type* [i.e., "Hodge-Tate-quasi-Lubin-Tate" type] (respectively, *of weakly HT-qLT-type* [i.e., "weakly Hodge-Tate-quasi-Lubin-Tate" type]) if, for
- every pair of respective finite extensions $k'_\circ$ $(\subseteq \bar{k}_\circ)$, $k'_\bullet$ $(\subseteq \bar{k}_\bullet)$ of $k_\circ$, $k_\bullet$ such that $\alpha(G_{k'_\circ}) \subseteq G_{k'_\bullet}$,
- every finite Galois extension E of $\mathbf{Q}_p$ that admits a pair of embeddings $\sigma_\circ : E \hookrightarrow k'_\circ$, $\sigma_\bullet : E \hookrightarrow k'_\bullet$,

the composite

$$G_{k'_\circ} \xrightarrow{\alpha|_{G_{k'_\circ}}} G_{k'_\bullet} \xrightarrow{\chi^{\mathrm{LT}}_{\sigma_\bullet}} E^\times$$

[cf. Definition 1.2, (ii)] is Hodge-Tate (respectively, is inertially equivalent [cf. Definition 1.2, (i)] to a continuous character $G_{k'_\circ} \to E^\times$ that factors through the natural open injection $G_{k'_\circ} \hookrightarrow \mathrm{Gal}(\bar{k}_\circ/E)$ determined by the embeddings $E \overset{\sigma_\circ}{\hookrightarrow} k'_\circ \hookrightarrow \bar{k}_\circ$) [cf. Proposition 1.1]. [Here, we note that, as is well-known—cf., e.g., [4], Chapter III, §A.1, Corollary 2—the issue of whether or not a finite dimensional continuous representation is *Hodge-Tate depends only* on the *inertial equivalence class* of the given representation.]

LEMMA 1.4. *Let* $\alpha : G_{k_\circ} \to G_{k_\bullet}$ *be an open continuous homomorphism. Consider the following four conditions*:

(1) α *is* **HT-preserving** [*cf. Definition* 1.3, (i)].

(1′) *For every pair of respective finite extensions* $k'_\circ$ $(\subseteq \bar{k}_\circ)$, $k'_\bullet$ $(\subseteq \bar{k}_\bullet)$ *of* $k_\circ$, $k_\bullet$ *such that* $\alpha(G_{k'_\circ}) \subseteq G_{k'_\bullet}$, *the restriction* $\alpha|_{G_{k'_\circ}} : G_{k'_\circ} \to G_{k'_\bullet}$ *is* **HT-preserving**.

(2) α *is* **of HT-qLT-type** [*cf. Definition* 1.3, (ii)].

(3) α *is* **of weakly HT-qLT-type** [*cf. Definition* 1.3, (ii)].

Then we have an equivalence and implications

$$(1) \Leftrightarrow (1') \Rightarrow (2) \Rightarrow (3).$$

Proof. The implication $(1') \Rightarrow (1)$ is immediate. Next, let us verify that the implication $(1) \Rightarrow (1')$ follows from the following *well-known argument*: Let $k'_\circ$ $(\subseteq \bar{k}_\circ)$, $k'_\bullet$ $(\subseteq \bar{k}_\bullet)$ be respective finite extensions of $k_\circ$, $k_\bullet$ such that $\alpha(G_{k'_\circ}) \subseteq G_{k'_\bullet}$; $\phi : G_{k'_\bullet} \to \mathrm{GL}_n(\mathbf{Q}_p)$ a finite dimensional continuous representation of $G_{k_\bullet}$ that is *Hodge-Tate*. Now let us observe [cf., e.g., [4], Chapter III, §A.1, Corollary 2] that, to verify that the composite $\phi \circ \alpha|_{G_{k'_\circ}}$ is *Hodge-Tate*—by replacing $k'_\circ$, $k'_\bullet$ by suitable finite extensions of $k'_\circ$, $k'_\bullet$, respectively—we may assume without loss of generality that $k'_\circ$, $k'_\bullet$ are *Galois* over $k_\circ$, $k_\bullet$, respectively. Write $\phi_{k_\bullet}$ for the finite dimensional continuous representation of $G_{k_\bullet}$ obtained by inducing ϕ from $G_{k'_\bullet}$ to $G_{k_\bullet}$. Then since [one verifies easily that] $\phi_{k_\bullet}|_{G_{k'_\bullet}}$ is isomorphic to the direct product of $[k'_\bullet : k_\bullet]$ copies of ϕ, it holds that $\phi_{k_\bullet}$ is *Hodge-Tate*. Thus, since α is *HT-preserving*, it holds that $\phi_{k_\bullet} \circ \alpha$, hence also $(\phi_{k_\bullet} \circ \alpha)|_{G_{k'_\circ}}$, is *Hodge-Tate*. On the other hand, one verifies easily that $\phi \circ \alpha|_{G_{k'_\circ}}$ is isomorphic to a subrepresentation of $(\phi_{k_\bullet} \circ \alpha)|_{G_{k'_\circ}}$. In particular, we conclude that $\phi \circ \alpha|_{G_{k'_\circ}}$ is *Hodge-Tate*. This completes the proof of the implication $(1) \Rightarrow (1')$.

The implication $(1') \Rightarrow (2)$ follows from the fact that "$\chi^{\mathrm{LT}}_{\sigma,\pi}$" defined in Definition 1.2, (ii), is *Hodge-Tate* [cf. [4], Chapter III, §A.5, Corollary]. Finally, we verify the implication $(2) \Rightarrow (3)$. We shall apply the notational conventions established in Definition 1.3, (ii). Then since α is *of HT-qLT-type*, the character $\chi : G_{k'_\circ} \to E^\times$ obtained by forming the composite

$$G_{k'_\circ} \xrightarrow{\alpha|_{G_{k'_\circ}}} G_{k'_\bullet} \xrightarrow{\chi^{\mathrm{LT}}_{\sigma_\bullet}} E^\times$$

is *Hodge-Tate*. Thus, since E is *Galois* over $\mathbf{Q}_p$, it follows immediately from [4], Chapter III, §A.5, Corollary, that χ is *inertially equivalent* [cf. Definition 1.2, (i)] to the character

$$\prod_{\sigma \in \mathrm{Gal}(E/\mathbf{Q}_p)} (\chi^{\mathrm{LT}}_{\sigma_\circ \circ \sigma})^{n_\sigma} : G_{k'_\circ} \to E^\times$$

for some choices of integers n_σ. On the other hand, one verifies easily from *local class field theory* that this character is *inertially equivalent* to the restriction to $G_{k'_\circ} \subseteq \mathrm{Gal}(\bar{k}_\circ/E)$ of the character

$$\prod_{\sigma \in \mathrm{Gal}(E/\mathbf{Q}_p)} (\chi_\sigma^{\mathrm{LT}})^{n_\sigma} : \mathrm{Gal}(\bar{k}_\circ/E) \to E^\times.$$

This completes the proof of the implication (2) $\Rightarrow$ (3), hence also of Lemma 1.4. $\square$

Remark 1.4.1. In the notation of Lemma 1.4, consider the following four conditions:

(4) α is *of qLT-type* [cf. [2], Definition 3.1, (iv)].
(5) α is *of 01-qLT-type* [cf. [2], Definition 3.1, (iv)].
(6) α is *of CHT-type* [cf. [2], Definition 3.1, (iv)].
(7) α is *of HT-type* [cf. [2], Definition 3.1, (iv)].

Then we have equivalences and implications

$$(7) \Leftarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \ \ (\Rightarrow (1) \Leftrightarrow (1') \Rightarrow (2) \Rightarrow (3)).$$

Indeed, the equivalences (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follow from [2], Theorem 3.5, (i); the implications (6) $\Rightarrow$ (1) and (6) $\Rightarrow$ (7) are immediate. If, moreover, α is *injective*, then we have equivalences and implications

$$(4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \ \ (\Rightarrow (1) \Leftrightarrow (1') \Rightarrow (2) \Rightarrow (3)).$$

Indeed, the implication (7) $\Rightarrow$ (6) follows immediately from [1], Proposition 1.1.

2. Injectivity result

In the present §2, we prove that every open continuous homomorphism *of weakly HT-qLT-type* is *injective* [cf. Proposition 2.4 below]. We maintain the notation of the preceding §1.

Definition 2.1.

(i) Let G be a profinite group. Then we shall write

$$(G \twoheadrightarrow) \ \ G^{p\text{-ab-free}}$$

for the maximal pro-p abelian torsion-free quotient of G.

(ii) Let A be an *abelian* topological group and $\phi : G_k \to A$ a continuous homomorphism. Then we shall write

$$\text{iner-dim}(\phi) \overset{\text{def}}{=} \dim_{\mathbf{Q}_p}(\phi(I_k)^{p\text{-ab-free}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$$

[cf. (i)] and refer to iner-dim(ϕ) as the *inertial dimension* of ϕ.

Lemma 2.2. *Let A be an abelian topological group and $\phi : G_k \to A$ a continuous homomorphism. Then the following hold*:

(i) *It holds that*

$$0 \leq \text{iner-dim}(\phi) \leq [k : \mathbf{Q}_p]$$

[*cf. Definition* 2.1, (ii)].

(ii) *Let* $H \subseteq I_k$ *be a closed subgroup of* I_k. *Suppose that* H **contains** *an open subgroup of* P_k [*e.g.*, H *is an open subgroup of* I_k *or* P_k]. *Then*

$$\text{iner-dim}(\phi) = \dim_{\mathbf{Q}_p}(\phi(H)^{p\text{-ab-free}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$$

[*cf. Definition* 2.1, (i)].

(iii) *Let* $\phi' : G_k \to A$ *be a continuous homomorphism that is* **inertially equivalent** *to* ϕ [*cf. Definition* 1.2, (i)]. *Then*

$$\text{iner-dim}(\phi) = \text{iner-dim}(\phi').$$

(iv) *In the notation of Definition* 1.2, (ii), *it holds that*

$$\text{iner-dim}(\chi_\sigma^{\mathrm{LT}}) = [E : \mathbf{Q}_p]$$

[*cf.* (iii)].

(v) *Let* $\alpha : G_{k_\circ} \to G_k$ *be an* **open** *continuous homomorphism. Then it holds that*

$$\text{iner-dim}(\phi) = \text{iner-dim}(\phi \circ \alpha).$$

Proof. First, I claim that the following assertion holds:

Claim 2.2.A: The natural surjection $I_k \twoheadrightarrow \phi(I_k)^{p\text{-ab-free}}$ *factors* through the natural surjection $I_k \twoheadrightarrow \mathfrak{o}_k^\times \twoheadrightarrow (\mathfrak{o}_k^\times)^{p\text{-ab-free}}$ [cf. the discussion at the beginning of §1].

Indeed, this follows immediately from our assumption that A is *abelian*. This completes the proof of Claim 2.2.A.

Assertion (i) follows immediately from Claim 2.2.A, together with the fact that $(\mathfrak{o}_k^\times)^{p\text{-ab-free}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is of dimension $[k : \mathbf{Q}_p]$. Assertion (ii) follows immediately from Claim 2.2.A, together with the [easily verified] fact that the composite $P_k \hookrightarrow I_k \twoheadrightarrow \mathfrak{o}_k^\times$ is *open*. Assertion (iii) follows immediately from assertion (ii). Assertion (iv) follows immediately from the definition of the character $\chi_\sigma^{\mathrm{LT}}$, together with the fact that $(\mathfrak{o}_E^\times)^{p\text{-ab-free}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is of dimension $[E : \mathbf{Q}_p]$. Finally, we verify assertion (v). Let us first observe that it follows from Proposition 1.1 that α determines an *open* homomorphism $P_{k_\circ} \to P_k$. Thus, assertion (v) follows immediately from assertion (ii). This completes the proof of assertion (v). □

Lemma 2.3. *Let* $N \subseteq G_k$ *be a* **nontrivial** *normal closed subgroup of* G_k. *Then there exists an open subgroup* $H \subseteq G_k$ *of* G_k *such that the image of the composite* $N \cap H \hookrightarrow H \twoheadrightarrow H^{p\text{-ab-free}}$ [*cf. Definition* 2.1, (i)] *is* **nontrivial**.

Proof. Assume that, for every open subgroup $H \subseteq G_k$ of G_k, the image of the composite $N \cap H \hookrightarrow H \twoheadrightarrow H^{p\text{-ab-free}}$ is *trivial*, i.e., if we write $J_H \subseteq H$ for the kernel of the natural surjection $H \twoheadrightarrow H^{p\text{-ab-free}}$, then $N \cap H \subseteq J_H$. Now since N is *nontrivial*, it is immediate that there exists a normal open subgroup $H \subseteq G_k$ such that the composite $N \hookrightarrow G_k \twoheadrightarrow G_k/H$ is *nontrivial*. In particular, one verifies easily that, to verify Lemma 2.3, by replacing G_k by the inverse image of the image of N in G_k/H via $G_k \twoheadrightarrow G_k/H$, we may assume without loss of generality that the composite $N \hookrightarrow G_k \twoheadrightarrow G_k/H$ is [nontrivial and] *surjective*. Thus, since [we have assumed that] $N \cap H \subseteq J_H$, it follows immediately that the composite $N \hookrightarrow G_k \twoheadrightarrow G_k/J_H$ determines a *splitting* of the exact sequence of profinite groups

$$1 \to H^{p\text{-ab-free}} \to G_k/J_H \to G_k/H \to 1.$$

[Here, we note that since $H \subseteq G_k$ is *normal*, and $J_H \subseteq H$ is *characteristic*, one verifies easily that J_H is *normal* in G_k.] In particular, since $N \subseteq G_k$ is *normal*, the natural action [determined by the above exact sequence] of G_k/H on $H^{p\text{-ab-free}}$, hence also on $H^{p\text{-ab-free}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, is *trivial*. On the other hand, if we write k' $(\subseteq \bar{k})$ for the finite Galois extension of k corresponding to $H \subseteq G_k$, then it follows immediately from *local class field theory* that there exists a G_k/H $(= \mathrm{Gal}(k'/k))$-equivariant injection of $\mathbf{Q}_p$-vector spaces $k' \hookrightarrow H^{p\text{-ab-free}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, which *contradicts* the fact that the action of G_k/H on $H^{p\text{-ab-free}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is *trivial*. This completes the proof of Lemma 2.3. □

Next, we prove the main result of the present §2. Note that the *injectivity* result was shown in the proof of the implication (c) ⇒ (d) of [2], Theorem 3.5, (i), for homomorphisms *of qLT-type*, and that Proposition 2.4 is its improvement for homomorphisms *of weakly HT-qLT-type*.

Proposition 2.4. *Let* $\alpha : G_{k_\circ} \to G_{k_\bullet}$ *be an open continuous homomorphism. Suppose that* α *is* **of weakly HT-qLT-type** [*cf. Definition* 1.3, (ii)]. *Then* α *is* **injective**.

Proof. Assume that the homomorphism α is *not injective*. Then it follows immediately from Lemma 2.3 that there exists a finite Galois extension E of $\mathbf{Q}_p$ that admits a pair of embeddings $E \hookrightarrow \bar{k}_\circ$, $E \hookrightarrow \bar{k}_\bullet$ such that if we write $E_\circ \subseteq \bar{k}_\circ$, $E_\bullet \subseteq \bar{k}_\bullet$ for the respective images of these embeddings [so $E_\circ \xleftarrow{\sim} E \xrightarrow{\sim} E_\bullet$], then $k_\circ \subseteq E_\circ$, $k_\bullet \subseteq E_\bullet$, and, moreover, the image of the composite $\mathrm{Ker}(\alpha) \cap G_{E_\circ} \hookrightarrow G_{E_\circ} \twoheadrightarrow G_{E_\circ}^{p\text{-ab-free}}$ [cf. Definition 2.1, (i)] is *nontrivial*.

Let $k'_\circ$ $(\subseteq \bar{k}_\circ)$ be a finite extension of $k_\circ$ such that $E_\circ \subseteq k'_\circ$, and, moreover, $\alpha(G_{k'_\circ}) \subseteq G_{E_\bullet}$. Write χ for the composite

$$G_{k'_\circ} \xrightarrow{\alpha|_{G_{k'_\circ}}} G_{E_\bullet} \xrightarrow{\chi_{\mathrm{id}}^{\mathrm{LT}}} E_\bullet^\times \quad (\xleftarrow{\sim} E^\times \xrightarrow{\sim} E_\circ^\times)$$

[cf. Definition 1.2, (ii)]. Then since $\alpha|_{G_{k'_\circ}}$ is *open*, it follows from Lemma 2.2, (iv), (v), that

$$\text{iner-dim}(\chi) = \text{iner-dim}(\chi_{\mathrm{id}}^{\mathrm{LT}}) = [E_\bullet : \mathbf{Q}_p]$$

[cf. Definition 2.1, (ii)]. On the other hand, since α is *of weakly HT-qLT-type*, the character χ is *inertially equivalent* to the continuous character factors as the composite

$$G_{k'_\circ} \longrightarrow G_{E_\circ} \xrightarrow{\chi_{E_\circ}} E_\circ^\times \quad (\xleftarrow{\sim} E^\times \xrightarrow{\sim} E_\bullet^\times)$$

of the natural open injection $G_{k'_\circ} \hookrightarrow G_{E_\circ}$ and a continuous character $\chi_{E_\circ} : G_{E_\circ} \to E_\circ^\times$. Thus, it follows from Lemma 2.2, (iii), (v), that

$$([E_\bullet : \mathbf{Q}_p] =) \quad \text{iner-dim}(\chi) = \text{iner-dim}(\chi_{E_\circ}).$$

Now let us recall from Proposition 1.1 that $\mathrm{Ker}(\alpha) \subseteq P_{k_\circ}$. In particular, it holds that $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\alpha) \cap I_{k_\circ}$, which thus implies that $\mathrm{Ker}(\alpha) \cap I_{k'_\circ}$ is *open* in $\mathrm{Ker}(\alpha)$. On the other hand, it follows from the definition of χ that $\mathrm{Ker}(\alpha) \cap I_{k'_\circ}$ $(= \mathrm{Ker}(\alpha) \cap G_{k'_\circ}) \subseteq \mathrm{Ker}(\chi)$. Thus, since χ is *inertially equivalent* to $\chi_{E_\circ}|_{G_{k'_\circ}}$, we conclude that there exists an *open* subgroup $J \subseteq \mathrm{Ker}(\alpha)$ of $\mathrm{Ker}(\alpha)$ such that $J \subseteq \mathrm{Ker}(\chi_{E_\circ}) \subseteq G_{E_\circ}$. Now since $J \subseteq \mathrm{Ker}(\alpha)$ is *open* in $\mathrm{Ker}(\alpha)$, and [we have assumed that] the image of the composite $\mathrm{Ker}(\alpha) \cap G_{E_\circ} \hookrightarrow G_{E_\circ} \twoheadrightarrow G_{E_\circ}^{p\text{-ab-free}}$ is *nontrivial*, it follows that the image of the composite $J \hookrightarrow G_{E_\circ} \twoheadrightarrow G_{E_\circ}^{p\text{-ab-free}}$ is *nontrivial*. Thus, one verifies easily that the image of the homomorphism $J \to \mathfrak{o}_{E_\circ}^\times$ $(\subseteq G_{E_\circ}^{\mathrm{ab}})$ [cf. the discussion at the beginning of §1] determined by the composite $J \hookrightarrow G_{E_\circ} \twoheadrightarrow G_{E_\circ}^{\mathrm{ab}}$ [where we recall that $J \subseteq I_{E_\circ}$] is *infinite*. In particular, since $J \subseteq \mathrm{Ker}(\chi_{E_\circ})$, we conclude that the kernel of the character $(I_{E_\circ} \twoheadrightarrow)$ $\mathfrak{o}_{E_\circ}^\times \to E_\circ^\times$ determined by the restriction of $\chi_{E_\circ}$ to $I_{E_\circ} \subseteq G_{E_\circ}$ is *infinite*. Thus, we obtain an inequality

$$([E_\bullet : \mathbf{Q}_p] =) \quad \text{iner-dim}(\chi_{E_\circ}) < \dim_{\mathbf{Q}_p}((\mathfrak{o}_{E_\circ}^\times)^{p\text{-ab-free}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p) = [E_\circ : \mathbf{Q}_p],$$

which *contradicts* the fact that $E_\circ \xleftarrow{\sim} E \xrightarrow{\sim} E_\bullet$. This completes the proof of Proposition 2.4. □

3. The main results

In the present §3, we prove the main theorem of the present paper [cf. Theorem 3.3 below]. We maintain the notation of §1.

Definition 3.1. Let $\alpha : G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ be a continuous *isomorphism* and $\beta : k_\bullet \xrightarrow{\sim} k_\circ$ an isomorphism of fields. Then we shall say that β is *inertially compatible* with α if the composite

$$\mathfrak{o}_{k_\bullet}^\times \hookrightarrow k_\bullet^\times \xrightarrow{\sim} k_\circ^\times \hookrightarrow (k_\circ^\times)^\wedge$$

—where the second arrow is the isomorphism determined by β—and the composite

$$\mathfrak{o}_{k_\bullet}^\times \hookrightarrow G_{k_\bullet}^{\mathrm{ab}} \xrightarrow{\sim} G_{k_\circ}^{\mathrm{ab}} \xrightarrow{\sim} (k_\circ^\times)^\wedge$$

—where the first arrow is the natural inclusion arising from local class field theory [cf. the discussion at the beginning of §1], the second arrow is the isomorphism determined by α^{-1}, and the third arrow is the isomorphism arising from local class field theory—coincide on an open subgroup of $\mathfrak{o}_{k_\bullet}^\times$.

Lemma 3.2. *Let $\alpha : G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ be a continuous isomorphism; $\beta_1, \beta_2 : k_\bullet \xrightarrow{\sim} k_\circ$ isomorphisms of fields. Suppose that β_1, β_2 are* **inertially compatible** *with α [cf. Definition 3.1]. Then $\beta_1 = \beta_2$.*

Proof. Since β_1, β_2 are *inertially compatible* with α, one verifies easily from the various definitions involved that there exists an open subgroup $S_\bullet \subseteq \mathfrak{o}_{k_\bullet}^\times$ of $\mathfrak{o}_{k_\bullet}^\times$ such that $\beta_1|_{S_\bullet} = \beta_2|_{S_\bullet}$. On the other hand, let us recall from [1], Lemma 4.1, that the sub-$\mathbf{Q}_p$-vector space of $k_\bullet$ generated by $S_\bullet$ *coincides* with $k_\bullet$. Thus, the equality $\beta_1|_{S_\bullet} = \beta_2|_{S_\bullet}$ implies the equality $\beta_1 = \beta_2$. This completes the proof of Lemma 3.2. □

Next, we prove the main theorem of the present paper. Note that the argument given in the proof of Theorem 3.3 is essentially the same as the argument applied in [1] to prove the main theorem of [1].

Theorem 3.3. *Let p be a prime number. For $\square \in \{\circ, \bullet\}$, let $k_\square$ be a p-adic local field and $\bar{k}_\square$ an algebraic closure of $k_\square$. Write $G_{k_\square} \overset{\text{def}}{=} \mathrm{Gal}(\bar{k}_\square/k_\square)$. Let*

$$\alpha : G_{k_\circ} \to G_{k_\bullet}$$

be an open continuous homomorphism. Suppose that α is **of HT-qLT-type** *[cf. Definition 1.3, (ii)]. Then α is* **geometric** *[cf. [2], Definition 3.1, (iv)], i.e., arises from an isomorphism of fields $\bar{k}_\bullet \xrightarrow{\sim} \bar{k}_\circ$ that determines an embedding $k_\bullet \hookrightarrow k_\circ$.*

Proof. First, let us observe that it follows from Proposition 2.4, together with the implication (2) $\Rightarrow$ (3) of Lemma 1.4, that α is *injective*. Next, let us observe that, to verify Theorem 3.3, by replacing $G_{k_\bullet}$ by the image of α, we may assume without loss of generality that α is an *isomorphism*.

Now I claim that the following assertion holds:

Claim 3.3.A: Suppose that $k_\circ$ is *Galois* over $\mathbf{Q}_p$. Then there exists a(n) [necessarily *unique*—cf. Lemma 3.2] isomorphism of fields $\beta_{k_\bullet,k_\circ} : k_\bullet \xrightarrow{\sim} k_\circ$ that is *inertially compatible* with α [cf. Definition 3.1].

Indeed, let E be a finite Galois extension of $\mathbf{Q}_p$ that admits embeddings $E \hookrightarrow \bar{k}_\circ$, $E \hookrightarrow \bar{k}_\bullet$ such that if we write $E_\circ \subseteq \bar{k}_\circ$, $E_\bullet \subseteq \bar{k}_\bullet$ for the respective images of these embeddings [so $E_\circ \xleftarrow{\sim} E \xrightarrow{\sim} E_\bullet$], then $k_\circ \subseteq E_\circ$, $k_\bullet \subseteq E_\bullet$. Let $k'_\circ$ $(\subseteq \bar{k}_\circ)$ be a finite Galois extension of $k_\circ$ such that $k'_\circ$ *contains* $E_\circ$, and, moreover, the finite [necessarily Galois] extension $k'_\bullet$ $(\subseteq \bar{k}_\bullet)$ of $k_\bullet$ corresponding to the open subgroup $\alpha(G_{k'_\circ}) \subseteq G_{k_\bullet}$ *contains* $E_\bullet$. For $\square \in \{\circ, \bullet\}$, write $\sigma_\square : E_\square \hookrightarrow k'_\square$ for the natural inclusion. Write χ for the composite

$$G_{k'_\circ} \xrightarrow[\sim]{\alpha|_{G_{k'_\circ}}} G_{k'_\bullet} \xrightarrow{\chi^{\mathrm{LT}}_{\sigma_\bullet}} E_\bullet^\times \quad (\xleftarrow{\sim} E^\times \xrightarrow{\sim} E_\circ^\times).$$

Then since α is *of HT-qLT-type*, it holds that χ is *Hodge-Tate*. Thus, since $E_\circ$ is *Galois* over $\mathbf{Q}_p$, it follows from [4], Chapter III, §A.5, Corollary, that χ is *inertially equivalent* to the character

$$\prod_{\sigma \in \mathrm{Gal}(E_\circ/\mathbf{Q}_p)} (\chi^{\mathrm{LT}}_{\sigma_\circ \circ \sigma})^{n_\sigma} : G_{k'_\circ} \to E_\circ^\times \quad (\xleftarrow{\sim} E^\times \xrightarrow{\sim} E_\bullet^\times)$$

for some choices of integers n_σ.

For $\square \in \{\circ, \bullet\}$, write $\mathrm{Ver}_{k'_\square/k_\square} : G^{\mathrm{ab}}_{k_\square} \to G^{\mathrm{ab}}_{k'_\square}$ for the *Verlagerung map* with respect to the finite Galois extension $k'_\square/k_\square$. Then since χ is *inertially equivalent* to $\prod_{\sigma \in \mathrm{Gal}(E_\circ/\mathbf{Q}_p)} (\chi^{\mathrm{LT}}_{\sigma_\circ \circ \sigma})^{n_\sigma}$, and [one verifies easily from *local class field theory* that] $\mathrm{Ver}_{k'_\square/k_\square}$ maps $\mathfrak{o}^\times_{k_\square} \subseteq G^{\mathrm{ab}}_{k_\square}$ [cf. the discussion at the beginning of §1] to $\mathfrak{o}^\times_{k'_\square} \subseteq G^{\mathrm{ab}}_{k'_\square}$, we conclude that there exists an open subgroup $S_\circ \subseteq \mathfrak{o}^\times_{k_\circ}$ $(\subseteq G^{\mathrm{ab}}_{k_\circ})$ of $\mathfrak{o}^\times_{k_\circ}$ such that if we write $S_\bullet \subseteq \mathfrak{o}^\times_{k_\bullet}$ for the image of $S_\circ \subseteq \mathfrak{o}^\times_{k_\circ}$ by the isomorphism

$$(G^{\mathrm{ab}}_{k_\circ} \supseteq) \quad \mathfrak{o}^\times_{k_\circ} \xrightarrow{\sim} \mathfrak{o}^\times_{k_\bullet} \quad (\subseteq G^{\mathrm{ab}}_{k_\bullet})$$

induced by α [where let us recall from Proposition 1.1 that α induces an isomorphism $I_{k_\circ} \xrightarrow{\sim} I_{k_\bullet}$], then the diagram of topological modules

$$\begin{array}{ccccccc}
S_\circ & \longrightarrow & G^{\mathrm{ab}}_{k_\circ} & \xrightarrow{\mathrm{Ver}_{k'_\circ/k_\circ}} & G^{\mathrm{ab}}_{k'_\circ} & \xrightarrow{\prod_{\sigma \in \mathrm{Gal}(E_\circ/\mathbf{Q}_p)} (\chi^{\mathrm{LT}}_{\sigma_\circ \circ \sigma})^{n_\sigma}} & E_\circ^\times \xleftarrow{\sim} E^\times \\
\wr\downarrow & & & & & & \| \\
S_\bullet & \longrightarrow & G^{\mathrm{ab}}_{k_\bullet} & \xrightarrow{\mathrm{Ver}_{k'_\bullet/k_\bullet}} & G^{\mathrm{ab}}_{k'_\bullet} & \xrightarrow{\chi^{\mathrm{LT}}_{\sigma_\bullet}} & E_\bullet^\times \xleftarrow{\sim} E^\times
\end{array}$$

—where the left-hand vertical arrow is the isomorphism induced by α, and the left-hand horizontal arrows are the natural inclusions—*commutes*. On the other hand, it follows immediately from *local class field theory*, together with Definition 1.2, (ii), that, for $\square \in \{\circ, \bullet\}$, if we write $\mathrm{Im}(I_{k_\square}) \subseteq G^{\mathrm{ab}}_{k_\square}$ for the image of the composite $I_{k_\square} \hookrightarrow G_{k_\square} \twoheadrightarrow G^{\mathrm{ab}}_{k_\square}$ [i.e., "$\mathfrak{o}^\times_{k_\square}$" $\subseteq G^{\mathrm{ab}}_{k_\square}$—cf. the discussion at the beginning of §1], then we have commutative diagrams of topological modules

$$\begin{array}{ccccc}
\mathrm{Im}(I_{k_\circ}) & \xrightarrow{\mathrm{Ver}_{k'_\circ/k_\circ}} & \mathrm{Im}(I_{k'_\circ}) & \xrightarrow{\prod_{\sigma \in \mathrm{Gal}(E_\circ/\mathbf{Q}_p)} (\chi^{\mathrm{LT}}_{\sigma_\circ \circ \sigma})^{n_\sigma}} & E_\circ^\times \xleftarrow{\sim} E^\times \\
\wr\downarrow & & \wr\downarrow & & \| \\
\mathfrak{o}^\times_{k_\circ} & \longrightarrow & \mathfrak{o}^\times_{k'_\circ} & \xrightarrow{\prod_{\sigma \in \mathrm{Gal}(E_\circ/\mathbf{Q}_p)} (\sigma^{-1} \circ \mathrm{Nm}_{k'_\circ/E_\circ})^{n_\sigma}} & E_\circ^\times \xleftarrow{\sim} E^\times,
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{Im}(I_{k_\bullet}) & \xrightarrow{\mathrm{Ver}_{k'_\bullet/k_\bullet}} & \mathrm{Im}(I_{k'_\bullet}) & \xrightarrow{\chi^{\mathrm{LT}}_{\sigma_\bullet}} & E_\bullet^\times \xleftarrow{\sim} E^\times \\
\wr\downarrow & & \wr\downarrow & & \| \\
\mathfrak{o}^\times_{k_\bullet} & \longrightarrow & \mathfrak{o}^\times_{k'_\bullet} & \xrightarrow{\mathrm{Nm}_{k'_\bullet/E_\bullet}} & E_\bullet^\times \xleftarrow{\sim} E^\times
\end{array}$$

—where the left-hand and middle vertical arrows are isomorphisms that arise from *local class field theory*; the lower left-hand horizontal arrows are the homomorphisms induced by the natural inclusions $k_\circ \hookrightarrow k'_\circ$, $k_\bullet \hookrightarrow k'_\bullet$, respectively; we write "Nm" for the *norm map*. In particular, if, for $\square \in \{\circ, \bullet\}$, we write $\mathrm{Im}(S_\square) \subseteq E_\square^\times$ for the image of $S_\square$ in $E_\square^\times$, then the following hold:

(a) Since $k_\circ \subseteq E_\circ \subseteq k'_\circ$, and $k_\circ$ is *Galois* over $\mathbf{Q}_p$ [which thus implies that every $\sigma \in \mathrm{Gal}(E_\circ/\mathbf{Q}_p)$ *preserves* $k_\circ \subseteq E_\circ$], it holds that

$$\begin{aligned}\mathrm{Im}(S_\circ) &= \prod_{\sigma \in \mathrm{Gal}(E_\circ/\mathbf{Q}_p)} (\sigma^{-1} \circ \mathrm{Nm}_{k'_\circ/E_\circ})(S_\circ)^{n_\sigma} \\ &= \prod_{\sigma \in \mathrm{Gal}(E_\circ/\mathbf{Q}_p)} \sigma^{-1}(S_\circ^{n_\sigma \cdot [k'_\circ : E_\circ]}) \subseteq k_\circ^\times,\end{aligned}$$

i.e., that the subgroup $\mathrm{Im}(S_\circ) \subseteq E_\circ^\times$ is *contained* in $k_\circ^\times \subseteq E_\circ^\times$.

(b) Since $k_\bullet \subseteq E_\bullet \subseteq k'_\bullet$, it holds that the subgroup $\mathrm{Im}(S_\bullet) \subseteq E_\bullet^\times$ *coincides* with the subgroup $(\mathfrak{o}_{k_\bullet}^\times)^{[k'_\bullet : E_\bullet]} \subseteq E_\bullet^\times$, which thus implies that the subgroup $\mathrm{Im}(S_\bullet) \subseteq E_\bullet^\times$ is an *open* subgroup of $\mathfrak{o}_{k_\bullet}^\times \subseteq E_\bullet^\times$.

For each $\square \in \{\circ, \bullet\}$, write $V_\square \subseteq E_\square$ for the sub-$\mathbf{Q}_p$-vector space of $E_\square$ generated by $\mathrm{Im}(S_\square) \subseteq E_\square$. Now we have a commutative diagram of topological modules

$$\begin{array}{ccccc} \mathrm{Im}(S_\circ) & \longrightarrow & E_\circ^\times & \stackrel{\sim}{\leftarrow} & E^\times \\ \wr\downarrow & & & & \| \\ \mathrm{Im}(S_\bullet) & \longrightarrow & E_\bullet^\times & \stackrel{\sim}{\leftarrow} & E^\times \end{array}$$

—where the left-hand vertical arrow is the isomorphism induced by α, and the left-hand horizontal arrows are the natural inclusions. Thus, it is immediate that the isomorphisms of fields $E_\bullet \stackrel{\sim}{\leftarrow} E \stackrel{\sim}{\rightarrow} E_\circ$ determine an isomorphism $V_\bullet \stackrel{\sim}{\rightarrow} V_\circ$, which thus implies that $\dim_{\mathbf{Q}_p}(V_\circ) = \dim_{\mathbf{Q}_p}(V_\bullet)$. Moreover, it follows from (a) (respectively, (b), together with [1], Lemma 4.1) that $V_\circ \subseteq k_\circ \subseteq E_\circ$ (respectively, $V_\bullet = k_\bullet \subseteq E_\bullet$). Thus, since $[k_\circ : \mathbf{Q}_p] = [k_\bullet : \mathbf{Q}_p]$ [cf. [1], Proposition 1.2], we conclude that $V_\circ = k_\circ$, $V_\bullet = k_\bullet$, and, moreover, the isomorphism of $\mathbf{Q}_p$-vector spaces $V_\bullet \stackrel{\sim}{\rightarrow} V_\circ$ [determined by the *isomorphisms of fields* $E_\bullet \stackrel{\sim}{\leftarrow} E \stackrel{\sim}{\rightarrow} E_\circ$] is *compatible* with the structures of fields of $k_\circ$, $k_\bullet$. In particular, we obtain an *isomorphism of fields* $\beta_{k_\bullet, k_\circ} : k_\bullet = V_\bullet \stackrel{\sim}{\rightarrow} V_\circ = k_\circ$. On the other hand, it follows from the definition of $\beta_{k_\bullet, k_\circ}$, together with the above discussion concerning $\mathrm{Im}(S_\square)$, that $\beta_{k_\bullet, k_\circ}$ is *inertially compatible* with α. This completes the proof of Claim 3.3.A.

Next, I claim that the following assertion holds:

Claim 3.3.B: For every pair of respective finite extensions $k'_\circ$ $(\subseteq \bar{k}_\circ)$, $k'_\bullet$ $(\subseteq \bar{k}_\bullet)$ of $k_\circ$, $k_\bullet$ such that $\alpha(G_{k'_\circ}) = G_{k'_\bullet}$, there exists a(n) [necessarily *unique*—cf. Lemma 3.2] isomorphism of fields $\beta_{k'_\bullet, k'_\circ} : k'_\bullet \stackrel{\sim}{\rightarrow} k'_\circ$ that is *inertially compatible* with the restriction $\alpha|_{G_{k'_\circ}} : G_{k'_\circ} \stackrel{\sim}{\rightarrow} G_{k'_\bullet}$.

Indeed, let $k''_\circ$ $(\subseteq \bar{k}_\circ)$ be a finite extension of $k'_\circ$ that is *Galois* over $\mathbf{Q}_p$. Write $k''_\bullet$ $(\subseteq \bar{k}_\bullet)$ for the finite [necessarily Galois] extension of $k'_\bullet$ corresponding to the open subgroup $\alpha(G_{k''_\circ}) \subseteq G_{k_\bullet}$. Then it follows from Claim 3.3.A that there exists an isomorphism of fields $\beta_{k''_\bullet, k''_\circ} : k''_\bullet \xrightarrow{\sim} k''_\circ$ that is *inertially compatible* with the restriction $\alpha|_{G_{k''_\circ}} : G_{k''_\circ} \xrightarrow{\sim} G_{k''_\bullet}$. Then one verifies easily from Lemma 3.2, together with the fact that $\beta_{k''_\bullet, k''_\circ}$ is *inertially compatible* with the restriction $\alpha|_{G_{k''_\circ}}$, that $\beta_{k''_\bullet, k''_\circ}$ is *compatible* with the respective natural actions of $\mathrm{Gal}(k''_\circ/k'_\circ)$, $\mathrm{Gal}(k''_\bullet/k'_\bullet)$ on $k''_\circ$, $k''_\bullet$ [relative to the isomorphism $\mathrm{Gal}(k''_\circ/k'_\circ) = G_{k'_\circ}/G_{k''_\circ} \xrightarrow{\sim} G_{k'_\bullet}/G_{k''_\bullet} = \mathrm{Gal}(k''_\bullet/k'_\bullet)$ induced by $\alpha|_{G_{k'_\circ}}$]. Thus, we conclude that the isomorphism $\beta_{k''_\bullet, k''_\circ}$ determines an isomorphism $\beta_{k'_\bullet, k'_\circ} : k'_\bullet \xrightarrow{\sim} k'_\circ$. On the other hand, again by Lemma 3.2, together with the fact that $\beta_{k''_\bullet, k''_\circ}$ is *inertially compatible* with the restriction $\alpha|_{G_{k''_\circ}}$, it follows immediately that this isomorphism $\beta_{k'_\bullet, k'_\circ}$ is *inertially compatible* with the restriction $\alpha|_{G_{k'_\circ}}$. This completes the proof of Claim 3.3.B.

Now, by applying Claim 3.3.B to the various finite extensions of $k_\circ$, we obtain an isomorphism of fields $\beta_{\bar{k}_\bullet, \bar{k}_\circ} : \bar{k}_\bullet \xrightarrow{\sim} \bar{k}_\circ$ that determines an isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$. Moreover, again by applying Claim 3.3.B, one verifies easily that α arises from this isomorphism $\beta_{\bar{k}_\bullet, \bar{k}_\circ}$. This completes the proof of Theorem 3.3. □

Remark 3.3.1. Theorem 3.3 leads naturally to the following observation:

> Let p be an *odd* prime number and $\overline{\mathbf{Q}}_p$ an algebraic closure of the p-adic completion $\mathbf{Q}_p$ of the field of rational numbers $\mathbf{Q}$. Write $G_{\mathbf{Q}_p} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Then there exist an automorphism α of $G_{\mathbf{Q}_p}$ and a finite dimensional continuous representation $\phi : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(\mathbf{Q}_p)$ of $G_{\mathbf{Q}_p}$ such that ϕ is *potentially locally algebraic*, i.e., the restriction of ϕ to an open subgroup of $G_{\mathbf{Q}_p}$ is *locally algebraic* [cf. [4], Chapter III, §1, Definition] [hence *Hodge-Tate*], the set of Hodge-Tate weights of ϕ is *contained* in $\{0, 1\}$, but $\phi \circ \alpha$ is *not Hodge-Tate*.

Indeed, let us first observe that it follows immediately from the discussion given at the final part of [3], Chapter VII, §5, that we have an automorphism α of $G_{\mathbf{Q}_p}$ that is *not geometric* [cf. [2], Definition 3.1, (iv)]. Thus, it follows from Theorem 3.3 that α is *not of HT-qLT-type* [cf. Definition 1.3, (ii)]. In particular, since the character "$\chi_\sigma^{\mathrm{LT}}$" defined in Definition 1.2, (ii), is *locally algebraic* [cf. [4], Chapter III, §1, Example (2)], and the set of Hodge-Tate weights is *contained* in $\{0, 1\}$ [cf., e.g., [4], Chapter III, §A.5, Theorem 2], it follows from the definition of a homomorphism *of HT-qLT-type* that there exist normal open subgroups $H_1, H_2 \subseteq G_{\mathbf{Q}_p}$ and a finite dimensional continuous representation $\phi_{H_2} : H_2 \to \mathrm{GL}_n(\mathbf{Q}_p)$ of H_2 such that $\alpha(H_1) \subseteq H_2$, ϕ_{H_2} is *locally algebraic*, the set of Hodge-Tate weights of ϕ_{H_2} is *contained* in $\{0, 1\}$, and, moreover, $\phi_{H_2} \circ \alpha : H_1 \to \mathrm{GL}_n(\mathbf{Q}_p)$ is *not Hodge-Tate*. Thus, it follows immediately from a similar argument to the argument applied in the proof of the implication $(1) \Rightarrow (1')$ of Lemma 1.4 that if we write ϕ for the finite dimensional continuous representation of $G_{\mathbf{Q}_p}$ obtained by inducing ϕ_{H_2} from H_2 to $G_{\mathbf{Q}_p}$, then ϕ is *potentially locally algebraic* [cf. also [4], Chapter III, §A.7, Theorem 3], the set of Hodge-Tate weights of ϕ is *contained* in $\{0, 1\}$, but $\phi \circ \alpha$ is *not Hodge-Tate*.

COROLLARY 3.4. *In the notation of Theorem* 3.3, *consider the following nine conditions*:

(1) α *is* **HT-preserving** [*cf. Definition* 1.3, (i)].
(2) α *is* **of HT-qLT-type** [*cf. Definition* 1.3, (ii)].
(3) α *is* **geometric** [*cf.* [2], *Definition* 3.1, (iv)].
(4) α *is* **of qLT-type** [*cf.* [2], *Definition* 3.1, (iv)].
(5) α *is* **of 01-qLT-type** [*cf.* [2], *Definition* 3.1, (iv)].
(6) α *is* **of CHT-type** [*cf.* [2], *Definition* 3.1, (iv)].
(7) α *is* **of HT-type** [*cf.* [2], *Definition* 3.1, (iv)].
(8) α *is* [*an isomorphism and*] **RF-preserving** [*cf.* [2], *Definition* 3.6, (iii)].
(9) α *is* [*an isomorphism and*] **uniformly toral** [*cf.* [2], *Definition* 3.6, (iii)].

Then we have equivalences and implications

$$(8) \Leftrightarrow (9) \Rightarrow (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Rightarrow (7).$$

If, moreover, α *is an* **isomorphism**, *then the above nine conditions are* **equivalent**.

Proof. Let us recall from Remark 1.4.1 that we have implications

$$(4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1) \Rightarrow (2) \quad \text{and} \quad (6) \Rightarrow (7).$$

The implication (2) $\Rightarrow$ (3) follows from Theorem 3.3. The implication (3) $\Rightarrow$ (4) follows from [2], Theorem 3.5, (i). The equivalence (8) $\Leftrightarrow$ (9) and the implication (8) $\Rightarrow$ (3) follow from [2], Corollary 3.7. Finally, the implication (7) $\Rightarrow$ (6) (respectively, (3) $\Rightarrow$ (8)) in the case where α is an *isomorphism* follows immediately from [1], Proposition 1.1 (respectively, [2], Corollary 3.7). This completes the proof of Corollary 3.4. $\square$

COROLLARY 3.5. *Let p be a prime number. For $\square \in \{\circ, \bullet\}$, let $k_\square$ be a p-adic local field and $\bar{k}_\square$ an algebraic closure of $k_\square$. Write* $G_{k_\square} \stackrel{\text{def}}{=} \mathrm{Gal}(\bar{k}_\square / k_\square)$;

$$\mathrm{Emb}(\bar{k}_\bullet / k_\bullet, \bar{k}_\circ / k_\circ)$$

for the set of isomorphisms of fields $\bar{k}_\bullet \xrightarrow{\sim} \bar{k}_\circ$ that determine embeddings $k_\bullet \hookrightarrow k_\circ$;

$$\mathrm{Emb}(k_\bullet, k_\circ)$$

for the set of embeddings of fields $k_\bullet \hookrightarrow k_\circ$;

$$\mathrm{Hom}^{\mathrm{open}}_{\mathrm{HT}}(G_{k_\circ}, G_{k_\bullet})$$

for the set of open continuous homomorphisms $\alpha : G_{k_\circ} \to G_{k_\bullet}$ that are **HT-preserving** [*cf. Definition* 1.3, (i)], *i.e., for every finite dimensional continuous representation $\phi : G_{k_\bullet} \to \mathrm{GL}_n(\mathbf{Q}_p)$ of $G_{k_\bullet}$, if ϕ is* **Hodge-Tate**, *then $\phi \circ \alpha$ is* **Hodge-Tate**. *Then we have a commutative diagram of natural maps*

$$\begin{array}{ccc}
\mathrm{Emb}(\bar{k}_\bullet / k_\bullet, \bar{k}_\circ / k_\circ) & \xrightarrow{\sim} & \mathrm{Hom}^{\mathrm{open}}_{\mathrm{HT}}(G_{k_\circ}, G_{k_\bullet}) \\
\downarrow & & \downarrow \\
\mathrm{Emb}(k_\bullet, k_\circ) & \xrightarrow{\sim} & \mathrm{Hom}^{\mathrm{open}}_{\mathrm{HT}}(G_{k_\circ}, G_{k_\bullet}) / \mathrm{Inn}(G_{k_\bullet})
\end{array}$$

—*where the vertical arrows are* **surjective**, *and the horizontal arrows are* **bijective**.

Proof. The *injectivity* of the horizontal arrows follow immediately from the *injectivity* portion of [1], Theorem 4.2 [cf. also the proof of [1], Theorem 4.2]. The *surjectivity* of the horizontal arrows follow immediately from Theorem 3.3, together with the implication (1) $\Rightarrow$ (2) of Lemma 1.4. This completes the proof of Corollary 3.5. $\square$

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