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ON THE SCALAR CURVATURE ESTIMATES FOR GRADIENT YAMABE SOLITONS*

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Abstract

Let (M^n, g) be a gradient Yamabe soliton $Rg + \text{Hess}\, f = \lambda g$ with $\text{Ric}_{f_1} \geq K$ (see (1.3) for f_1) and $\lambda, K \in \mathbf{R}$ are constants. In this paper, it is showed that for gradient shrinking Yamabe solitons, the scalar curvature $R > 0$ unless $R \equiv 0$ and (M^n, g) is the Gaussian soliton, and for gradient steady and expanding Yamabe solitons, $R > \lambda$ unless $R \equiv \lambda$ and (M^n, g) is either trivial or a Riemannian product manifold. Replacing the assumptions $\text{Ric}_{f_1} \geq K$ by $R \geq \lambda$, we also derive the corresponding scalar curvature estimates. In particular, we show that any shrinking gradient Yamabe soliton with $R \geq \lambda$ must have constant scalar curvature $R \equiv \lambda$. Moreover, the lower bounds of scalar curvature for quasi gradient Yamabe solitons are obtained.

1. Introduction

In this paper, we mainly consider the scalar curvature estimates for gradient Yamabe solitons. Recall that an n-dimensional complete Riemannian manifold (M^n, g) is called a Yamabe soliton if there exists a smooth vector field X and a constant $\lambda \in \mathbf{R}$ such that

$$Rg + \mathscr{L}_X g = \lambda g,$$

where R is the scalar curvature of g, and $\mathscr{L}_X$ the Lie derivative with respect to X. For $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, (M^n, g, X) is called shrinking, steady, or expanding, respectively.

If $X = \nabla f$, the equation above can also be written as

$$Rg + \text{Hess}\, f = \lambda g, \tag{1.1}$$

where $\text{Hess}\, f$ is the Hessian of f. When the potential function f is a constant, we say M^n is trivial. Obviously, Einstein manifolds are trivial gradient Yamabe solitons. Another interesting example is the Gaussian soliton (see [6]). Namely,

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Euclidean space $\mathbf{R}^n$ with the standard flat metric $g_{ij} = \delta_{ij}$ and

$$f = \frac{\lambda}{2}|x|^2.$$

Yamabe solitons play an important role in the study of the Yamabe flow

$$\frac{\partial}{\partial t} g = -Rg,$$

which was introduced by Hamilton in [10] to construct Yamabe metrics on compact Riemannian manifolds, and has been studied by many authors (see [1], [2], [3], [5], [13], [15], [18], [22] and the references therein).

For gradient Yamabe solitons, it is proved that any compact gradient Yamabe soliton has constant scalar curvature (see [7], [8], or [14]), Therefore, in this paper, we only discuss the noncompact case. In 2011, Ma and Miquel [16] showed that a complete noncompact non-expanding gradient Yamabe soliton has nonnegative scalar curvature, provided that $\lim_{x\to\infty} R(x) \geq 0$. Recently, Wu [21] studied the lower bounds of scalar curvature for a class of complete noncompact gradient Yamabe solitons and derived

Theorem A. *Assume that (M^n, g) is a complete noncompact gradient Yamabe soliton satisfying*

$$\lim_{r(x)\to\infty} \frac{1}{r(x)} \int_1^{r(x)} \mathrm{Ric}(\gamma'(s), \gamma'(s))\, ds \geq 0,$$

where $r(x)$ is the distance function from a fixed point $p \in M^n$ and $\gamma : [0, r(x)] \to M^n$ is a unit speed minimal geodesic joining p to x.

(i) *If the gradient Yamabe soliton is steady or shrinking, then $R \geq 0$.*

(ii) *If the gradient Yamabe soliton is expanding, then $R \geq \lambda$.*

Note that the curvature assumption of Theorem A exclude Einstein solitons with negative constant curvature, as a question proposed in [21], it is hoped that the curvature condition of Theorem A will be improved.

In this paper, from the smooth metric measure space point of view, we study the lower bound estimates on the scalar curvature of gradient Yamabe solitons, and obtain unform lower bounds of R under a natural condition. In order to state our results, we recall the following terminologies and notations.

Let $(M^n, g, e^{-f}\, dvol_g)$ be an n-dimensional smooth metric measure space, where $dvol_g$ is the Riemannian volume form on M^n. The so-called f-Laplacian Δ_f on $C^2(M^n)$ (see [20]) is

$$\Delta_f = e^f\, div(e^{-f}\nabla) := \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint in $L^2(M^n, g, e^{-f}\, dvol_g)$. The corresponding Ricci tensor to $(M^n, g, e^{-f}\, dvol_g)$ is the Bakry-Émery Ricci tensor, which is defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f.$$

In [12], Huang and Li introduce the concept of quasi gradient Yamabe soliton. Recall that an n-dimensional complete Riemannian manifold (M^n, g, f) is called a quasi Yamabe soliton if it satisfies

$$Rg + \operatorname{Hess} f - \frac{df \otimes df}{m} = \lambda g \tag{1.2}$$

for some constant $\lambda, m \in \mathbf{R}$ and $m > 0$. (M^n, g) is said to be shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. By taking $M^2 = \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\} \subset \mathbf{R}^2$, $g_{ij} = \delta_{ij}$ and $f = -\log(x + y)$, we see easily that $R = 0 = \lambda$ and $\operatorname{Hess} f = df \otimes df$. So (M^2, g, f) is an example of quasi gradient steady Yamabe soliton with $m = 1$.

Clearly, when $m \to \infty$, equations (1.2) become gradient Yamabe soliton equations (1.1). Denoting

$$f_1 = \frac{f}{2(n-1)} \quad \text{and} \quad f_2 = \frac{m + 4(n-1)}{2m(n-1)} f \tag{1.3}$$

from now on, we first give our main result for the scalar curvature estimates for gradient Yamabe solitons.

THEOREM 1.1. *Let (M^n, g) be an n-dimensional complete noncompact gradient Yamabe soliton with $Ric_{f_1} \geq K$ for some constant $K \in \mathbf{R}$ and $R_* = \inf_{M^n} R$.*

(1) *If M^n is shrinking, then $0 \leq R_* \leq \lambda$ and $R > 0$ unless $R \equiv 0$ and M^n is isometric to the Gaussian soliton.*

(2) *If M^n is steady, then $R_* = 0$ and $R > 0$ unless $R \equiv 0$ and M^n is either trivial or isometric to a Riemannian product mainfold.*

(3) *If M^n is expanding, then $\lambda \leq R_* \leq 0$ and $R > \lambda$ unless $R \equiv \lambda$ and M^n is either trivial or isometric to a warped product mainfold.*

As for quasi gradient Yamabe solitons, it is proved by Huang and Li [12] that any compact quasi gradient Yamabe soliton has constant scalar curvature. In the noncompact case, we can get

PROPOSITION 1.2. *Let (M^n, g) be an n-dimensional complete noncompact quasi gradient Yamabe soliton with $Ric_{f_2} \geq K$ for some constant $K \in \mathbf{R}$ and $R_* = \inf_{M^n} R$.*

(1) *If M^n is shrinking, then $0 \leq R_* \leq \lambda$ and $R > 0$, i.e., R can not attain its minimum 0.*

(2) *If M^n is steady, then $R_* = 0$ and $R > 0$ unless $R \equiv 0$.*

(3) *If M^n is expanding, then $\lambda \leq R_* \leq 0$ and $R > \lambda$ unless $R \equiv \lambda$.*

Remark 1.3. Theorem 1.1 and Theorem 1.2 include the Einstein solitons with negative constant curvature.

By Theorem 1.1 and equation (1.1), we obtain

THEOREM 1.4. *Let* (M^n, g) *be an n-dimensional complete noncompact gradient Yamabe soliton with* $Ric_{f_1} \geq K$ *for some constant* $K \in \mathbf{R}$. *Then there exist two constant* c_1 *and* c_2, *such that*

(1) *If* M^n *is shrinking or steady, then* $f(x) \leq \frac{\lambda}{2} r^2 + c_1 r(x) + c_2$.

(2) *If* M^n *is expanding, then* $f(x) \leq c_1 r(x) + c_2$.

Without the assumptions that $Ric_{f_i} \geq K$, $i = 1, 2$, we can derive the following results by the minimum principle.

THEOREM 1.5. *Let* (M^n, g) *be an n-dimensional complete noncompact gradient Yamabe soliton.*

(1) *If* $R \geq \lambda > 0$, *then* $R \equiv \lambda$ *and* M^n *is either trivial or isometric to a Riemannian product manifold.*

(2) *If* $R \geq \lambda = 0$, *then* $R > 0$ *and* M^n *is a warped product manifold, unless* $R \equiv 0$ *and* M^n *is either trivial or isometric to a Riemannian product soliton.*

(3) *If* $R \geq 0 > \lambda$, *then* $R > 0$ *and* M^n *is a warped product manifold, unless* $R \equiv 0$ *and* M^n *is isometric to the Gaussian soliton.*

Remark 1.6. It is known for gradient shrinking Ricci solitons $\mathrm{Ric} + \mathrm{Hess}\, f = \rho g$ that the scalar curvature $R \equiv n\rho$ provided that $R \geq n\rho > 0$. Theorem 1.5 states that gradient Yamabe solitons also have the similar property. We would like to thank the referee for bringing us the question that whether $R \equiv \lambda$ is true for complete gradient Yamabe solitons with $R \geq \lambda > 0$ and for very helpful suggestions which helped us with the proof of Theorem 1.5 and with the improvements of our work.

PROPOSITION 1.7. *Let* (M^n, g) *be an n-dimensional complete noncompact quasi gradient Yamabe soliton.*

(1) *If* $R \geq \lambda > 0$, *then* $R > \lambda$ *unless* $R \equiv \lambda$.

(2) *If* $R \geq \lambda = 0$, *then* $R > 0$ *unless* $R \equiv 0$.

(3) *If* $R \geq 0 > \lambda$, *then* $R > 0$, *namely*, R *can not attain its minimum* 0.

The paper is organized as follows: In section 2, we first compute the f_2-Laplacian of R of quasi gradient Yamabe solitons, then employing the weak maximum principle at infinity in smooth metric measure space, we give the proof of Theorem 1.1 and proposition 1.2. In section 3, we complete the proof of Theorem 1.4. The last section is devoted to proving Theorem 1.5 and Proposition 1.7.

2. Proofs of Theorem 1.1 and Proposition 1.2

To begin with, we compute the f_2-Laplacian of R of quasi gradient Yamabe solitons and get

LEMMA 2.1. *Let (M^n, g) be an n-dimensional quasi gradient Yamabe soliton satisfying* (1.2). *Then*

$$\nabla R = \frac{\mathrm{Ric}(\nabla f)}{n-1} + \frac{(R-\lambda)\nabla f}{m} \tag{2.1}$$

and

$$\Delta_{f_2} R = -\frac{n(R-\lambda)^2}{m} + \frac{R\lambda - R^2}{n-1}, \tag{2.2}$$

where $\Delta f_2 = \Delta - \nabla f_2 \cdot \nabla$ and f_2 is defined by (1.3).

Proof. In local coordinates, we denote by $g = (g_{ij})$ the Riemannian metric on M^n with coefficients g_{ij}, and denote the inverse matrix by $(g^{ij}) = (g_{ij})^{-1}$. Throughout this paper we adopt the Einstein summation convention. Taking the trace of (1.2), we get

$$n(R-\lambda) + \Delta f - \frac{1}{m}|\nabla f|^2 = 0. \tag{2.3}$$

By taking covariant derivatives of (1.2), we arrive at

$$\begin{aligned} 0 &= \nabla^i R g_{ij} + \nabla^i \nabla_i \nabla_j f - \frac{\nabla^i \nabla_i f \nabla_j f}{m} - \frac{\nabla^i \nabla_j f \nabla_i f}{m} \\ &= \nabla_j R + \nabla_j \Delta f + R_{ij} \nabla^i f - \frac{\Delta f \nabla_j f}{m} - \frac{\nabla^i \nabla_j f \nabla_i f}{m}, \end{aligned}$$

which together with (1.2) and (2.3) yields

$$\begin{aligned} 0 &= (1-n)\nabla_j R + \frac{\nabla_j \nabla^i f \nabla_i f}{m} + R_{ij}\nabla^i f + \frac{\nabla_j f}{m}\left(n(R-\lambda) - \frac{1}{m}|\nabla f|^2\right) \\ &= (1-n)\nabla_j R + \frac{\nabla_i f}{m}\left(\frac{\nabla_j f \nabla^i f}{m} - (R-\lambda)\delta_{ij}\right) + R_{ij}\nabla^i f + \frac{\nabla_j f}{m}\left(n(R-\lambda) - \frac{|\nabla f|^2}{m}\right) \\ &= (1-n)\nabla_j R + \frac{(n-1)(R-\lambda)\nabla_j f}{m} + R_{ij}\nabla^i f. \end{aligned}$$

Thus

$$\nabla_j R = \frac{\mathrm{Ric}_{ij}\, \nabla^i f}{n-1} + \frac{(R-\lambda)\nabla_j f}{m}. \tag{2.4}$$

Noting that $\nabla_i R = 2\nabla^j R_{ij}$, we conclude from (1.2), (2.3) and (2.4) that

$$\begin{aligned}
(2.5)\qquad \Delta R &= \frac{1}{n-1}\left(\frac{1}{2}\nabla_i R\cdot\nabla^i f+\mathrm{Ric}_{ij}\,\nabla^j\nabla^i f\right)+\frac{(R-\lambda)\Delta f}{m}+\frac{\nabla^j R\cdot\nabla_j f}{m}\\
&=\frac{\nabla_i R\cdot\nabla^i f}{2(n-1)}+\frac{\mathrm{Ric}_{ij}\,\nabla^j f\nabla^i f}{m(n-1)}-\frac{R(R-\lambda)}{n-1}+\frac{(R-\lambda)|\nabla f|^2}{m^2}\\
&\quad-\frac{n(R-\lambda)^2}{m}+\frac{\nabla^j R\cdot\nabla_j f}{m}\\
&=\frac{\nabla_i R\cdot\nabla^i f}{2(n-1)}+\frac{2\nabla^j R\cdot\nabla_j f}{m}-\frac{R(R-\lambda)}{n-1}-\frac{n(R-\lambda)^2}{m}\\
&=\frac{m+4(n-1)}{2m(n-1)}\nabla_i R\cdot\nabla^i f-\frac{R(R-\lambda)}{n-1}-\frac{n(R-\lambda)^2}{m},
\end{aligned}$$

where (2.4) is used in the third line of (2.5). So (2.2) follows from (2.5) by setting $f_2=[m+4(n-1)]f/2m(n-1)$. □

For quasi gradient Yamabe solitons, equation (2.2) implies

$$(2.6)\qquad \Delta_{f_2}R\le\frac{R\lambda-R^2}{n-1}.$$

Since $f_1=\lim_{m\to\infty}f_2$, for gradient Yamabe solitons, we can get from (2.5)

$$(2.7)\qquad \Delta_{f_1}R=\frac{R\lambda-R^2}{n-1}.$$

Recall that if $(M^n,g,e^{-f}\,dvol_g)$ is a smooth metric measure space, we say that the weak maximum principle at infinity for f-Laplacian holds if given a function $h\in C^2(M^n)$ satisfying $\sup_{M^n}h=h^*<+\infty$, there exists a sequence $\{x_n\}\subset M^n$ along which

$$\text{(i)}\ \ h(x_n)\ge h^*-\frac{1}{n}\quad\text{and}\quad\text{(ii)}\ \ \Delta_f h(x_n)\le\frac{1}{n}.$$

In [20], Wei and Wylie established the following volume comparison theorem for any smooth metric measure space $(M^n,g,e^{-f}\,dvol_g)$.

Lemma 2.2 ([20]). *Let $(M^n,g,e^{-f}\,dvol_g)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f\ge K$ for some constant $K\in\mathbf{R}$. Then for any $r_0>0$, there are constants $A,B,C>0$ such that for every $r\ge r_0$,*

$$vol_f(B_r)\le A+B\int_{r_0}^{r}\mathrm{e}^{-Kt^2+Ct}\,dt.$$

It follows from Lemma 2.2 and Theorem 9 in [19] that

LEMMA 2.3. *Let $(M^n, g, e^{-f}\, dvol_g)$ be an n-dimensional complete smooth metric measure space with* $\mathrm{Ric}_f \geq K$ *for some constant* $K \in \mathbf{R}$. *Then the weak maximum principle at infinity for the f-Laplacian always holds on* M^n.

The following Lemma is also needed in the proof of Theorem 1.1.

LEMMA 2.4 ([19], Theorem 1). *Let (M^n, g) be a complete manifold. Suppose that there exists a smooth function $f : M^n \to R$ satisfying* $\mathrm{Hess}\, f = \lambda g$ *for some constant $\lambda \neq 0$. Then M^n is isometric to R^n.*

Proof of Theorem 1.1. Let $f_1 = f/2(n-1)$, we see from the assumption $\mathrm{Ric}_{f_1} \geq K$ in Theorem 1.1 and Lemma 2.3 that the weak maximum principle at infinity for the f_1-Laplacian holds. Set $R_-(x) = \max\{-R(x), 0\}$, equality (2.7) implies

$$\Delta_{f_1} R_- = \frac{\lambda R_- + R_-^2}{n-1}.$$

Therefore, by Corollary 13 in [19], we obtain that R_- is bounded from above, or equivalently, $R_* = \inf_{M^n} R > -\infty$, which together with Lemma 2.3 and equation (2.7) produces a sequence $\{x_p\} \subset M^n$, $0 < p \in \mathbf{Z}$ such that

$$R(x_p) \to R_* \quad (p \to \infty)$$

and

$$-\frac{1}{p} \leq \Delta_{f_1} R(x_p) = \frac{\lambda R(x_p) - R(x_p)^2}{n-1}. \tag{2.8}$$

Taking the limit in (2.8) as $p \to \infty$ gives

$$R_*(\lambda - R_*) \geq 0. \tag{2.9}$$

Now we divide our proof into the following three cases:

CASE (i) $\lambda > 0$. Obviously, inequality (2.9) implies $0 \leq R_* \leq \lambda$ and consequently $R \geq 0$. Assume that R attains its minimum, namely, there exists a point $x_0 \in M^n$ such that $R(x_0) = 0$, then the non-negative function R satisfies

$$\begin{aligned} \Delta_{f_1} R - \frac{\lambda R}{n-1} &= \frac{-R^2}{n-1} \\ &\leq 0. \end{aligned}$$

and attains its minimum at $R(x_0) = 0$. Since $\dfrac{\lambda}{n-1} > 0$, it may be concluded from the minimum principle for elliptic equations (see [9], p. 35) that R is a constant on M^n, so M^n is scalar flat. Combing $R \equiv 0$ with equation (1.1) gives

$$\mathrm{Hess}\, f = \lambda g,$$

which together with Lemma 2.4 yields (1) in Theorem 1.2.

Case (ii) $\lambda = 0$. In this case, inequality (2.9) leads to $R_* = 0$, thus $R \geq 0$. Assume that there exists a minimum point $x_0 \in M^n$ such that $R(x_0) = 0$, using the same argument as in case (i), we obtain that R is a constant and M^n is scalar flat, which produce the equation:

$$\operatorname{Hess} f = 0.$$

Thus either f is a constant or $\nabla f \neq 0$ and the manifold splits along the gradient of f, the results of Cao-Sun-Zhang [4] and He [11] yield that (M^n, g) is isometric to the Riemannian product.

Case (iii) $\lambda < 0$. Similarly, we arrive at $\lambda \leq R_* \leq 0$ from (2.9) and consequently $R \geq \lambda$. Suppose that R can attain its minimum at point $x_0 \in M^n$ such that $R(x_0) = \lambda$, then the non-negative function $u(x) = R(x) - \lambda$ satisfies

$$\begin{aligned} \Delta_{f_1} u + \frac{\lambda u}{n-1} &= -\frac{u^2}{n-1} \\ &\leq 0 \end{aligned}$$

and attains its minimum $u(x_0) = 0$. Since $\dfrac{\lambda}{n-1} < 0$, we conclude from the minimum principle that u vanishes identically. Hence $R = \lambda$ is a constant, this together with equations (1.1) yields

$$\operatorname{Hess} f = 0.$$

Thus either f is a constant or the manifold splits along the gradient of f. □

Proof of Proposition 1.2. Under the assumption that $\mathrm{Ric}_{f_2} \geq K$ for some constant $K \in \mathbf{R}$, using a similar argument as in the proof of Theorem 1.1, we can deduce equation (2.9) by (2.5), Lemma 2.3 and the weak maximum principle at infinity for Δ_{f_2}. So the conclusions that $0 \leq R_* \leq \lambda$ for $\lambda \geq 0$ and $\lambda \leq R_* \leq 0$ for $\lambda \leq 0$ are still true for quasi gradient Yamabe solitons. In the following, the proof will be divided into two cases:

Case (i) $\lambda > 0$. If there exists a point $x_0 \in M^n$, such that $R(x_0) = 0$, then the minimum principle and equation (2.6) tell us that the non-negative function R must be a constant, namely, $R \equiv 0$ on M^n, contrary to (2.2).

Case (ii) $\lambda \leq 0$. Since the non-negative function $u(x) = R(x) - \lambda$ satisfies

$$\begin{aligned} \Delta_{f_2} u + \frac{\lambda u}{n-1} &= -\frac{(n(n-1)+m)u^2}{m(n-1)} \\ &\leq 0. \end{aligned}$$

The same proof as in Case (iii) of Theorem 1.1 still goes when R attained its minimum, which gives (2) and (3) in Proposition 1.2. □

3. Proof of Theorem 1.4

In this section, we are going to prove Theorem 1.4.

Proof of Theorem 1.4. Let $r(x) = d(x, x_0)$ be a distance function from a fixed point $x_0 \in M^n$ and $\gamma(s) : [0, r(x)] \to M^n$ be a minimizing normal geodesic starting from $x_0 = \gamma(0)$ to x. Then

$$\begin{aligned}\left.\frac{d}{ds}\right|_r f(\gamma(s)) &= \langle \nabla f, \nabla\gamma(r)\rangle = \int_0^r \frac{d}{ds}\langle \nabla f, \nabla\gamma(s)\rangle\, ds + \langle \nabla f, \nabla\gamma(0)\rangle \\ &= \int_0^r \mathrm{Hess}\, f(\nabla\gamma, \nabla\gamma)\, ds + \langle \nabla f, \nabla\gamma(0)\rangle.\end{aligned}$$

While Theorem 1.1 says that $R \ge 0$ for $\lambda \ge 0$ and $R \ge \lambda$ for $\lambda < 0$, which together with the soliton equation (1.1) gives

$$f_{ij} = (\lambda - R)g_{ij} \le \begin{cases} \lambda g_{ij}, & \text{if } \lambda \ge 0, \\ 0, & \text{if } \lambda < 0. \end{cases}$$

Therefore,

$$\left.\frac{d}{ds}\right|_r f(\gamma(s)) \le \begin{cases} \lambda r + \langle \nabla f, \nabla\gamma(0)\rangle, & \text{if } \lambda \ge 0, \\ \langle \nabla f, \nabla\gamma(0)\rangle, & \text{if } \lambda < 0. \end{cases}$$

Integrating the above inequalities along $\gamma(s)$ yields Theorem 1.4. □

4. Proofs of Theorem 1.5 and Proposition 1.7

In order to prove Theorem 1.5 and Proposition 1.7, we need the following classical result of Riemannian geometry (see [17]).

LEMMA 4.1 ([17], Theorem 3.2). *Let (M^n, g) be a complete manifold with smooth functions f, v satisfying*

$$\mathrm{Hess}\, f = vg,$$

then the Riemannian structure is a warped product around any point where $\nabla f \ne 0$.

Proof of Theorem 1.5. Let (M^n, g) be an n-dimensional complete noncompact gradient Yamabe soliton. Now we divide our proof into the following three cases:

CASE (1) $R \ge \lambda > 0$. Suppose that f is not a constant. By Cao-Sun-Zhang's result (see Theorem 1.2 in [4]), we know that (M^n, g) is isometric to a warped product $(I, dr^2) \times_{|\nabla f|} (\overline{N}, \bar{g})$ for some interval I and $(\overline{N}, \bar{g})$ with constant scalar curvature $\overline{R} > 0$. Fix $(r_0, \bar{x}) \in I \times \overline{N}$ with $\nabla f(r_0, \bar{x}) \ne 0$, and consider a curve $[r_0, \infty) \ni r \mapsto (r, \bar{x}) \in M^n = I \times \overline{N}$. It follows from $R \ge \lambda > 0$ and equation (1.1) that the function f is concave, i.e., $\mathrm{Hess}\, f \le 0$, and

$$f'(r) \le f'(r_0) < 0 \quad \text{for any } r \in [r_0, \infty)$$

along this curve. Since

$$\bar{R} = (f')^2 R + (n-1)[(n-2)(f'')^2 + 2f'f'''] \geq \lambda(f')^2 + 2(n-1)f'f''',$$

e.g. (2.20) in [4], we have

$$f'''(r) \geq \frac{1}{2(n-1)}[\bar{R}/f'(r) - \lambda f'(r)] \geq \frac{-\lambda}{2(n-1)}[f'(r) - \bar{R}/\lambda f'(r_0)].$$

Let $\phi(r)$, $r \geq r_0$ be the solution to

$$\phi''(r) = \frac{-\lambda}{2(n-1)}\phi(r); \quad \phi(r_0) = f'(r_0) - \bar{R}/\lambda f'(r_0); \quad \phi'(r_0) = f''(r_0).$$

Then, $f'(r) - \bar{R}/\lambda f'(r_0) \geq \phi(r)$ for any $r \geq r_0$. Since $\phi(r)$ is a periodic function, there exists $r_1 > r_0$ such that $\phi(r_1) = \phi(r_0)$. Then, $f'(r_1) - \bar{R}/\lambda f'(r_0) \geq \phi(r_1) = \phi(r_0)$. This means $f'(r_0) = f'(r_1)$. Since r_0 was chosen arbitrarily, $f'(r)$ is a constant and hence (M^n, g) is isometric to $\mathbf{R} \times \bar{N}$ and $R \equiv \lambda$.

Case (2) $R \geq \lambda = 0$. By (2.7), we get

$$\Delta_{f_1} R = \frac{-R^2}{n-1} \leq 0. \tag{4.1}$$

Assume that R attains its minimum, namely, there exists a point $x_0 \in M^n$ such that $R(x_0) = \lambda$, then inequality (4.1) and the minimum principle imply that $R \equiv \lambda$. Therefore, equation (1.1) becomes

$$\text{Hess } f = 0.$$

Thus either f is a constant or (M^n, g) is isometric to the Riemannian product manifold.

If $R > \lambda$ on (M^n, g), (M^n, g) is a warped product has been already proved by Cao-Sun-Zhang [4]. In fact, by setting $v := \lambda - R < 0$, we conclude from Lemma 4.1 and equation (1.1) that M^n is a warped product manifold.

Case (3) $R \geq 0 > \lambda$. It is easy to see from (2.7) that

$$\Delta_{f_1} R = \frac{R(\lambda - R)}{n-1} \leq 0.$$

If there exists a point $x_0 \in M^n$ such that $R(x_0) = 0$, then $R \equiv 0$ and $\text{Hess } f = \lambda g \neq 0$, which together with Lemma 2.4 gives that M^n is isometric to the Gaussian soliton. If $R > 0$, then the result Cao-Sun-Zhang [4] implies that M^n is a warped product manifold, and (3) is proved. □

Proof of Proposition 1.7. Let (M^n, g) be an n-dimensional complete non-compact quasi gradient Yamabe soliton. For (1) and (2) in Theorem 1.7, we see from (2.6) and $R \geq \lambda \geq 0$ that

$$\Delta_{f_2} R \leq \frac{R(\lambda - R)}{n-1} \leq 0. \tag{4.2}$$

If R attains its minimum, namely, there exists a point $x_0 \in M^n$ such that $R(x_0) = \lambda$, then inequality (4.2) and the minimum principle imply that $R \equiv \lambda$. For $R \geq 0 > \lambda$, using the same approach, we can see that $R > 0$ or $R \equiv 0$, but the later case contradicts to equation (2.2), which gives (3), and the proof is complete. $\square$

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