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WEIGHTED NORM INEQUALITIES FOR DERIVATIVES AND THEIR APPLICATIONS

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Abstract

In this paper we establish several weighted norm inequalities for derivatives of products of composition and Laplace convolution of functions. Some generalizations including Yamada's and Opial-type inequalities are discussed. As applications we give L_p-weighted estimates for solutions of some integral equations.

1. Introduction

For an absolutely continuous complex valued function f on $[0, b]$ with $f(0) = 0$ the following Opial's inequality holds (see [1] and their references):

$$\int_0^b |f(x)f'(x)|\,dx \le \frac{b}{2}\int_0^b |f'(x)|^2\,dx. \tag{1.1}$$

Opial-type inequalities involving integrals of functions and their derivatives are essential and in fact indispensable in the theory of differential equations. In [12], Yamada considered an elementary integral inequality which extended a norm inequality of Saitoh [11] and yielded some well-known Opial-type inequalities (see [1]).

Independently, some results of Saitoh in [11] were also generalized by the authors [8], in which we presented some weighted norm inequalities for a non-linear transform in different normed spaces.

Recently, the first two authors [4] considered some norm inequalities for derivatives of products of functions which generalize Saitoh's result [10] and provide an effective tool to estimate Fourier sine and Fourier cosine transforms.

In this paper, we will obtain weighted norm inequalities for derivatives of products of composition of functions which generalize some results from ([4], [8]),

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Yamada [12] and [1]. Moreover, we will also establish a norm inequality for derivatives of Laplace convolution and its applications to obtain L_p-weighted estimates for solutions of some integral equations.

Let ρ be a positive and continuous function on $[a,b]$. Let $1 < p, q < \infty$ be conjugate exponents $1/p + 1/q = 1$. By $L_p(\rho)$ we denote the Lebesgue space of complex valued measurable functions f on $[a,b]$ such that $\|f\|_{L_p(\rho)} < \infty$, where

$$\|f\|_{L_p(\rho)} := \left(\int_a^b |f(x)|^p \rho(x)\, dx\right)^{1/p}. \tag{1.2}$$

Let $\mathscr{AC}[a,b]$ be the space of functions f which are complex valued and absolutely continuous on $[a,b]$. For a positive integer n we denote by $\mathscr{AC}^n[a,b]$ the space of complex valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $D^{n-1}f \in \mathscr{AC}[a,b]$, where $D = \dfrac{d}{dx}$.

For a weight ρ and $n \in \mathbf{N}$, we denote by $W_p^n(\rho)$ the weighted space consisting of all functions $f \in \mathscr{AC}^n[a,b]$ such that $D^k f(a) = 0$ for all derivatives of order $0 \le k \le n-1$ and $D^n f \in L_p(\rho)$. This space is characterized by the following [5, Lemma 1.1].

Lemma 1.1. *The space $W_p^n(\rho)$ consists of those and only those functions $f(x)$ which can be represented in the form*

$$f(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\varphi(t)\, dt, \tag{1.3}$$

where $\varphi \in L_p(\rho)$.

It follows from (1.3) that

$$\varphi(x) = D^n f(x), \quad x \in [a,b]. \tag{1.4}$$

Hence, if we define on $W_p^n(\rho)$ a norm

$$\|f\|_{W_p^n(\rho)} := \|D^n f\|_{L_p(\rho)} = \left(\int_a^b |D^n f(x)|^p \rho(x)\, dx\right)^{1/p}, \tag{1.5}$$

then, $(W_p^n(\rho), \|.\|_{W_p^n(\rho)})$ is a Banach space.

2. Norm inequalities for products of composition of functions

In this section, we combine the methods of [4] with the ideas of Yamada [12] to yield a norm inequality for derivatives of products of compositions of functions which extends and generalizes some recent results of Nhan, Duc and

Tuan [8], Nhan and Duc [4], Yamada [12], Saitoh [11, 10], and some earlier results on Opial-type inequalities [1].

THEOREM 2.1. *Let m and n be two positive integers. Let G_j, $1 \le j \le m$, be of class C^n on an interval $(-R, R)$, $0 < R \le \infty$, satisfying the following conditions*
(i) $D^k G_j(0) = 0$ *for all* $0 \le k \le n-1$,
(ii) $|D^n G_j(x)| \le D^n G_j(|x|)$ *for all* $x \in (-R, R)$, *and*
(iii) *if* $x \le y^{1/p} z^{1/q}$, $0 < x, y, z < R$, *then* $0 < D^n G_j(x) \le [D^n G_j(y)]^{1/p} \cdot [D^n G_j(z)]^{1/q}$.

Then, for $f_j \in W_p^n(\tau_n \rho_j)$ *such that* $f_j([a,b]) \subset (-R, R)$ *and* $\|f\|^p_{W_p^n(\tau_n\rho_j)} < R$, *where* ρ_j, $1 \le j \le m$, *are some weights and*

$$\tau_n(x) = \frac{(b-x)^{n-1}}{(n-1)!}, \quad x \in [a,b],$$

we have $\prod_{j=1}^m G_j \circ f_j \in W_p^n(\tau_n \gamma)$, *and moreover,*

$$\left\| \prod_{j=1}^m G_j \circ f_j \right\|^p_{W_p^n(\tau_n\gamma)} \le \prod_{j=1}^m G_j(\|f_j\|^p_{W_p^n(\tau_n\rho_j)}), \tag{2.1}$$

where

$$\gamma(x) = \left[D^n \left\{ \prod_{j=1}^m G_j \left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt \right) \right\} \right]^{1-p}, \quad x \in [a,b].$$

If, in addition, we assume that $D^n G_j$, $1 \le j \le m$, *are strictly increasing on* $(0, R)$, *then the equality in* (2.1) *holds only if*

$$f_j(x) = E_j G_j \left(C_j \int_a^{\min(x,y)} \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt \right), \quad x \in [a,b], \tag{2.2}$$

where C_j *and* E_j *are complex constants, and* y *is an arbitrary, but fixed real number on* $[a,b]$.

The proof of Theorem 2.1 is derived from Theorems 2.4 and 2.7.

COROLLARY 2.2. *Let* $r, s > 0$ *be such that* $1/p + 1/r = 1/s$. *We assume that the hypotheses in Theorem* 2.1 *are valid. Then, for a weight* φ *such that*

$$K = \left[\int_a^b \tau_n(x) [\gamma(x)]^{-r/p} [\varphi(x)]^{r/s}\, dx \right]^{1/r} < \infty,$$

we have

$$\left[\int_a^b \tau_n(x) \left| D^n \left(\prod_{j=1}^m G_j \circ f_j \right)(x) \right|^s \varphi(x)\, dx \right]^{1/s} \le K \left[\prod_{j=1}^m G_j(\|f_j\|^p_{W_p^n(\tau_n\rho_j)}) \right]^{1/p}. \tag{2.3}$$

Proof. Applying Hölder's inequality with conjugate exponents p/s and r/s we have

$$\int_a^b \tau_n(x)\left|D^n\left(\prod_{j=1}^m G_j\circ f_j\right)(x)\right|^s \varphi(x)\,dx$$

$$= \int_a^b [\tau_n(x)]^{s/p}\left|D^n\left(\prod_{j=1}^m G_j\circ f_j\right)(x)\right|^s [\gamma(x)]^{s/p}[\tau_n(x)]^{s/r}[\gamma(x)]^{-s/p}\varphi(x)\,dx$$

$$\le K^s\left[\int_a^b \tau_n(x)\left|D^n\left(\prod_{j=1}^m G_j\circ f_j\right)(x)\right|^p \gamma(x)\,dx\right]^{s/p},$$

which is, by using (2.1),

$$\le K^s\left[\prod_{j=1}^m G_j\left(\int_a^b \tau_n(x)|D^n f_j(x)|^p\rho_j(x)\,dx\right)\right]^{s/p}.$$

This concluses the proof of the corollary. □

Remark 2.3. From Corollary 2.2 we can derive many known results. These include the results of Opial (1.1) (for $n = m = 1$, $p = r = 2$, $s = 1$, $\varphi = \rho_1 \equiv 1$ and $G_1(x) \equiv x^2$ on $[0, b]$), Hua [1, Theorem 2.3.1] (for $n = m = 1$, $s = 1$, $\varphi = \rho_1 \equiv 1$ and $G_1(x) \equiv |x|^p$ on $[a, b]$), Yang [1, Theorem 2.5.6] (for $n = m = 1$, $\varphi = \rho_1$ be increasing on $[a, b]$ and $G_1(x) \equiv |x|^{p/s}$ on $[a, b]$), Beesack [1, Theorem 2.10.1] (for $n = m = 1$ and $G_1(x) \equiv |x|^{p/s}$ on $[a, b]$), Yamada [12, Theorem 4.1] (for $n = m = 1$), Lin and Yang [1, Theorem 2.15.1] (by letting $n = 1$, $m = 2$, $\varphi = \rho_1 = \rho_2$ be decreasing on $[a, b]$, $G_1(x) = G_2(x) = |x|^{p/s}$ on $[a, b]$ and by using the Cauchy-Schwarz inequality).

THEOREM 2.4. *Let m and n be two positive integers, and ρ_j, $1 \le j \le m$, be some weights on $[a, b]$. Put*

$$\rho(x) = \left[D^n\left(\prod_{j=1}^m \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\rho_j^{1/(1-p)}(t)\,dt\right)\right]^{1-p}, \quad x \in [a, b].$$

Then, for $f_j \in W_p^n(\tau_n\rho_j)$, $1 \le j \le m$, we have $\prod_{j=1}^m f_j \in W_p^n(\tau_n\rho)$, and moreover,

$$\left\|\prod_{j=1}^m f_j\right\|_{W_p^n(\tau_n\rho)} \le \prod_{j=1}^m \|f_j\|_{W_p^n(\tau_n\rho_j)}. \tag{2.4}$$

Unless $m = 1$, *the equality in* (2.4) *holds if, and only if*

$$f_j(x) = C_j \int_a^{\min(x,y)} \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt, \quad x \in [a,b], \tag{2.5}$$

where C_j *are complex constants and* y *is an arbitrary, but fixed number on* $[a,b]$.

Proof. First, we need to prove that ρ is a positive and continuous weight on $[a,b]$. Since ρ_j are positive and continuous on $[a,b]$, and

$$D^{n_j}\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt\right) = \begin{cases} \rho_j^{1/(1-p)}(t) & \text{if } n_j = n \\ \displaystyle\int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!} \rho_j^{1/(1-p)}(t)\, dt & \text{if } 0 \le n_j \le n-1 \end{cases},$$

we derive that $D^{n_j}\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt\right)$, for $0 \le n_j \le n$, $1 \le j \le m$, are positive and continuous on $[a,b]$. By the fact that

$$D^n\left(\prod_{j=1}^m \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt\right) = \sum_{n_1+\cdots+n_m=n} \binom{n}{n_1,\ldots,n_m} \prod_{j=1}^m D^{n_j}\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt\right),$$

where $\binom{n}{n_1,\ldots,n_m} = \frac{n!}{n_1!\cdots n_m!}$ is the multinomial coefficient, we imply ρ is positive and continuous on $[a,b]$.

Next, for $f_j \in W_p^n(\tau_n \rho_j)$, $0 \le n_j \le n-1$, $1 \le j \le m$, it follows from Lemma 1.1 that

$$D^{n_j} f_j(x) = \int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!} D^n f_j(t)\, dt, \quad x \in [a,b],$$

and therefore,

$$|D^{n_j} f_j(x)| \le \int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!} |D^n f_j(t)|\, dt, \quad x \in [a,b]. \tag{2.6}$$

By applying Hölder's inequality to (2.6), we get

$$(2.7)\qquad |D^{n_j}f_j(x)| \le \left(\int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!}\rho_j^{1/(1-p)}(t)\,dt\right)^{1/q} \times \left(\int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!}|D^nf_j(t)|^p\rho_j(t)\,dt\right)^{1/p},$$

for all $x \in [a,b]$.

Now, from the generalized Leibniz rule for m functions $f_1,\dots,f_m$, we have

$$(2.8)\qquad D^n\left(\prod_{j=1}^m f_j\right) = \sum_{n_1+\cdots+n_m=n}\binom{n}{n_1,\dots,n_m}\prod_{j=1}^m D^{n_j}f_j = \sum_{j=1}^m D^nf_j\prod_{k\ne j}f_k + \sum_{\substack{n_1+\cdots+n_m=n\\0\le n_j\le n-1}}\binom{n}{n_1,\dots,n_m}\prod_{j=1}^m D^{n_j}f_j.$$

Using (2.7) in (2.8) to give

$$\left|D^n\left(\prod_{j=1}^m f_j(x)\right)\right| \le \sum_{j=1}^m |D^nf_j(x)|\prod_{k\ne j}\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\rho_k^{1/(1-p)}(t)\,dt\right)^{1/q} \times \left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}|D^nf_k(t)|^p\rho_k(t)\,dt\right)^{1/p}$$
$$+ \sum_{\substack{n_1+\cdots+n_m=n\\0\le n_j\le n-1}}\binom{n}{n_1,\dots,n_m}\prod_{j=1}^m\left(\int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!}\rho_j^{1/(1-p)}(t)\,dt\right)^{1/q} \times \left(\int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!}|D^nf_j(t)|^p\rho_j(t)\,dt\right)^{1/p},$$

that, by Hölder's inequality, yields

$$\le \left[\sum_{j=1}^m \rho_j^{1/(1-p)}(x)\prod_{k\ne j}\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\rho_k^{1/(1-p)}(t)\,dt + \sum_{\substack{n_1+\cdots+n_m=n\\0\le n_j\le n-1}}\binom{n}{n_1,\dots,n_m}\prod_{j=1}^m\int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!}\rho_j^{1/(1-p)}(t)\,dt\right]^{1/q}$$

$$\times \left[\sum_{j=1}^{m} |D^n f_j(x)|^p \rho_j(x) \prod_{k\neq j} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f_k(t)|^p \rho_k(t)\, dt \right.$$

$$\left. + \sum_{\substack{n_1+\cdots+n_m=n \\ 0\le n_j\le n-1}} \binom{n}{n_1,\ldots,n_m} \prod_{j=1}^{m} \int_a^x \frac{(x-t)^{n-1-n_j}}{(n-1-n_j)!} |D^n f_j(t)|^p \rho_j(t)\, dt \right]^{1/p}$$

$$= \left[\sum_{j=1}^{m} D^n \left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt \right) \prod_{k\neq j} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_k^{1/(1-p)}(t)\, dt \right.$$

$$\left. + \sum_{\substack{n_1+\cdots+n_m=n \\ 0\le n_j\le n-1}} \binom{n}{n_1,\ldots,n_m} \prod_{j=1}^{m} D^{n_j} \left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt \right) \right]^{1/q}$$

$$\times \left[\sum_{j=1}^{m} D^n \left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f_j(t)|^p \rho_j(t)\, dt \right) \prod_{k\neq j} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f_k(t)|^p \rho_k(t)\, dt \right.$$

$$\left. + \sum_{\substack{n_1+\cdots+n_m=n \\ 0\le n_j\le n-1}} \binom{n}{n_1,\ldots,n_m} \prod_{j=1}^{m} D^{n_j} \left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f_j(t)|^p \rho_j(t)\, dt \right) \right]^{1/p},$$

which is, in view of the generalized Leibniz rule,

$$= \left[D^n \left(\prod_{j=1}^{m} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho_j^{1/(1-p)}(t)\, dt \right) \right]^{1/q}$$

$$\times \left[D^n \left(\prod_{j=1}^{m} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f_j(t)|^p \rho_j(t)\, dt \right) \right]^{1/p}.$$

Therefore,

$$\left| D^n \left(\prod_{j=1}^{m} f_j(x) \right) \right|^p \rho(x) \le D^n \left(\prod_{j=1}^{m} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f_j(t)|^p \rho_j(t)\, dt \right),$$

and so

$$\left|D^n\left(\prod_{j=1}^m f_j(x)\right)\right|^p \tau_n(x)\rho(x) \le \tau_n(x)D^n\left(\prod_{j=1}^m \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}|D^n f_j(t)|^p \rho_j(t)\,dt\right), \tag{2.9}$$

for all $x \in [a,b]$. Taking integration with respect to x over $[a,b]$ to both sides of (2.9) and using the formula for integration by parts we obtain

$$\begin{aligned}\int_a^b &\left|D^n\left(\prod_{j=1}^m f_j(x)\right)\right|^p \tau_n(x)\rho(x)\,dx \\ &\le \int_a^b \tau_n(x)D^n\left(\prod_{j=1}^m \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}|D^n f_j(t)|^p \rho_j(t)\,dt\right)dx \\ &= \prod_{j=1}^m \int_a^b |D^n f_j(t)|^p \tau_n(t)\rho_j(t)\,dt,\end{aligned}$$

which yields (2.4).

Finally, we determine under what conditions equality can hold in (2.4). The case of $m = 1$ is trivial. Let $m \ge 2$. A straightforward calculation shows that equality holds in (2.4) if f_j is given by (2.5). Conversely, equality in (2.4) implies that equalities hold in (2.6) and (2.7) for each $x \in [a,b]$, $0 \le n_j \le n-1$, and $1 \le j \le m$. Suppose $D^k f_j \ne 0$ for all $0 \le k \le n-1$ and $1 \le j \le m$ (otherwise, it is suffices to take $C_j = 0$). Putting

$$y = \operatorname{ess\,sup}\{x \in [a,b) : D^n f_j(x) \ne 0 \text{ for all } 1 \le j \le m\},$$

we have $y \in (a,b]$. If equality holds in (2.4), then y must be a cluster point of the set $\{x \in [a,b) : D^n f_j(x) \ne 0 \text{ for all } 1 \le j \le m\}$. Hence, by continuity

$$|D^{n_j} f_j(y)| = \int_a^y \frac{(y-t)^{n-1-n_j}}{(n-1-n_j)!}|D^n f_j(t)|\,dt$$

and

$$\begin{aligned}|D^{n_j} f_j(y)| &= \left(\int_a^y \frac{(y-t)^{n-1-n_j}}{(n-1-n_j)!}\rho_j^{1/(1-p)}(t)\,dt\right)^{1/q} \\ &\quad\times \left(\int_a^y \frac{(y-t)^{n-1-n_j}}{(n-1-n_j)!}|D^n f_j(t)|^p \rho_j(t)\,dt\right)^{1/p}\end{aligned}$$

for all $0 \le n_j \le n-1$ and $1 \le j \le m$. Then, from the equality condition of Hölder's inequality, these above equalities happen only if for $0 \le n_j \le n-1$, $1 \le j \le m$,

$$\left|\int_a^y \frac{(y-t)^{n-1-n_j}}{(n-1-n_j)!}D^n f_j(t)\,dt\right| = \int_a^y \frac{(y-t)^{n-1-n_j}}{(n-1-n_j)!}|D^n f_j(t)|\,dt \tag{2.10}$$

and

$$A_j|D^n f_j(t)|^p \rho_j(t) = B_j \rho_j^{1/(1-p)}(t) \quad \text{a.e. } t \in [a, y], \tag{2.11}$$

where A_j and B_j are complex constants. By continuity of ρ_j on $[a, b]$ and the fact that $D^n f_j(x) = 0$ a.e. $x \in [y, b]$ (by definition of y), the conditions (2.10) and (2.11) imply that there exist some complex constants $C_j \neq 0$, $1 \leq j \leq m$, such that

$$D^n f_j(x) = C_j \rho_j^{1/(1-p)}(x) \chi(x; [a, y]) \quad \text{a.e. } x \in [a, b],$$

which, by Lemma 1.1, establishes the formula (2.5). □

Remark 2.5. Theorem 2.4 generalizes a recent result of the first two authors [4] and an earlier result of Saitoh [10].

Now, we consider the following transform

$$f \mapsto G \circ f$$

for a suitable function f, and a function G of class C^n on the interval $(-R, R) \supset f([a, b])$ satisfying the conditions (i), (ii), and (iii) as in Theorem 2.1.

Remark 2.6. Since $D^n G$ is continuous, from (iii) we notice that the function $D^n G$ is positive and increasing on the interval $(0, R)$. Moreover, for $0 \leq k \leq n$,

$$|D^k G(x)| \leq D^k G(|x|), \quad x \in (-R, R), \tag{2.12}$$

and if $x \leq y^{1/p} z^{1/q}$, $0 < x, y, z < R$, then

$$0 < D^k G(x) \leq [D^k G(y)]^{1/p} [D^k G(z)]^{1/q}. \tag{2.13}$$

Indeed, we only need to prove (2.12) and (2.13) in the case of $k = n - 1$.

Since $D^n G$ is positive and increasing on the interval $(0, R)$ and

$$D^{n-1} f(x) = \int_0^x D^n G(t)\, dt,$$

we have

$$|D^{n-1} G(x)| = \left| \int_0^x D^n G(t)\, dt \right| \leq \int_0^{|x|} D^n G(t)\, dt = D^{n-1} G(|x|), \quad x \in (-R, R).$$

Assume that $x \leq y^{1/p} z^{1/q}$, $0 < x, y, z < R$. Then,

$$D^{n-1} G(x) = \int_0^x D^n G(t)\, dt \leq \int_0^{y^{1/p} z^{1/q}} D^n G(t)\, dt.$$

By letting $t = \dfrac{y^{1/p}z^{1/q}}{x}s$, we have

$$\int_0^{y^{1/p}z^{1/q}} D^n G(t)\,dt = \frac{y^{1/p}z^{1/q}}{x}\int_0^x D^n G\left(\frac{y^{1/p}z^{1/q}}{x}s\right) ds,$$

which is, from (iii),

$$\leq \frac{y^{1/p}z^{1/q}}{x}\int_0^x \left[D^n G\left(\frac{y}{x}s\right)\right]^{1/p}\left[D^n G\left(\frac{z}{x}s\right)\right]^{1/q} ds,$$

that is, by Hölder's inequality,

$$\leq \frac{y^{1/p}z^{1/q}}{x}\left[\int_0^x D^n G\left(\frac{y}{x}s\right) ds\right]^{1/p}\left[\int_0^x D^n G\left(\frac{z}{x}s\right) ds\right]^{1/q}$$

concluding that

$$\int_0^{y^{1/p}z^{1/q}} D^n G(t)\,dt \leq \left[\int_0^y D^n G(t)\,dt\right]^{1/p}\left[\int_0^z D^n G(t)\,ds\right]^{1/q},$$

which gives

$$D^{n-1}G(x) \leq [D^{n-1}G(y)]^{1/p}[D^{n-1}G(z)]^{1/q}.$$

The following theorem generalizes a recent result of Yamada [12, Theorem 2.1].

THEOREM 2.7. *Let G be of class C^n on an interval $(-R, R) \supset f([a,b])$, $0 < R \leq \infty$, satisfying the conditions* (i), (ii), *and* (iii). *For a weight ρ we define a new weight*

$$\omega(x) = \left[D^n\left\{G\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\rho^{1/(1-p)}(t)\,dt\right)\right\}\right]^{1-p}, \quad x \in [a,b].$$

Then, for $f \in W_p^n(\tau_n\rho)$ satisfying $\|f\|^p_{W_p^n(\tau_n\rho)} < R$, we have

$$\|G \circ f\|^p_{W_p^n(\tau_n\omega)} \leq G(\|f\|^p_{W_p^n(\tau_n\rho)}). \tag{2.14}$$

If, in addition, we assume that $D^n G$ is strictly increasing on $(0, R)$, then the equality holds in (2.14) *only if*

$$f(x) = C\int_a^{\min(x,y)} \frac{(x-t)^{n-1}}{(n-1)!}\rho^{1/(1-p)}(t)\,dt, \quad x \in [a,b], \tag{2.15}$$

where C is a complex constant, and $y \in [a,b]$.

Proof. It is easy to see that ω is really a weight, i.e. positive and continuous, on $[a,b]$. Now, for $0 \le k \le n-1$, we have

$$|D^k f(x)| = \left| \int_a^x \frac{(x-t)^{n-1-k}}{(n-1-k)!} D^n f(t)\, dt \right| \le \int_a^x \frac{(x-t)^{n-1-k}}{(n-1-k)!} |D^n f(t)|\, dt, \tag{2.16}$$

that, by Hölder's inequality, yields

$$|D^k f(x)| \le \left[\int_a^x \frac{(x-t)^{n-1-k}}{(n-1-k)!} |D^n f(t)|^p \rho(t)\, dt \right]^{1/p} \times \left[\int_a^x \frac{(x-t)^{n-1-k}}{(n-1-k)!} \rho^{1/(1-p)}\, dt \right]^{1/q}. \tag{2.17}$$

From (2.12), we see that

$$|(D^k G)(f(x))| \le (D^k G)(|f(x)|),$$

which is, in view of (2.17),

$$\le (D^k G)\left(\left[\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f(t)|^p \rho(t)\, dt \right]^{1/p} \left[\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho^{1/(1-p)}\, dt \right]^{1/q} \right),$$

and so, by using (2.13), we obtain

$$|(D^k G)(f(x))| \le \left\{ (D^k G)\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho^{1/(1-p)}\, dt \right) \right\}^{1/q} \times \left\{ (D^k G)\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f(t)|^p \rho(t)\, dt \right) \right\}^{1/p} \tag{2.18}$$

for all $0 \le k \le n$. Notice that

$$D^n(G \circ f) = [(D^n G) \circ f][(Df)^n] + \cdots + [(DG) \circ f][D^n f] \tag{2.19}$$

is the sum of positive coefficients. So, using (2.17) and (2.18) in (2.19) and then applying Hölder's inequality, we get

$$|D^n(G \circ f)(x)| \le \left[D^n \left\{ G\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \rho^{1/(1-p)}\, dt \right) \right\} \right]^{1/q} \times \left[D^n \left\{ G\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |D^n f(t)|^p \rho(t)\, dt \right) \right\} \right]^{1/p},$$

which gives

$$\tau_n(x)|D^n(G\circ f)(x)|^p\omega(x) \le \tau_n(x)D^n\left\{G\left(\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}|D^nf(t)|^p\rho(t)\,dt\right)\right\} \tag{2.20}$$

for all $x \in [a,b]$. Taking integration with respect to x over $[a,b]$ to both sides of (2.20) and using the formula for integration by parts and the monotonicity of D^nG we obtain (2.14).

If, D^nG is strictly increasing on $(0,R)$ and so are D^kG for $0 \le k \le n-1$, then the equality holds in (2.14) only if equality hold in (2.16) and (2.17). Hence, by putting

$$y = \operatorname{ess\,sup}\{x \in [a,b) : D^nf(x) \ne 0\},$$

and by an argument analogous to that used for the proof of Theorem 2.4 we see that f has the form (2.15) as required. □

Remark 2.8. In Theorem 2.7, by taking

$$G(x) = \int_0^x \left[\frac{(x-t)^{n-1}}{(n-1)!}\sum_{k=0}^{\infty}|a_k|t^k\right]dt, \quad x \in (-R,R),$$

where $\sum_{k=0}^{\infty} a_kx^k$ is an absolutely convergent power series with radius of convergence R, we derive previous result of [8] and an earlier result of Saitoh [11].

3. A convolution norm inequality

In this section, let us consider $\Omega = (0,b)$ and the Laplace convolution product $\prod_{j=1}^m * f_j$ defined by

$$\left[\prod_{j=1}^{1} * f_j\right](x) = f_1(x), \quad \left[\prod_{j=1}^{2} * f_j\right](x) = [f_1 * f_2](x) = \int_0^x f_1(t)f_2(x-t)\,dt, \quad x \in \Omega,$$

and for each $m > 2$,

$$\prod_{j=1}^{m} * f_j = \left[\prod_{j=1}^{m-1} * f_j\right] * f_m.$$

The main result in this section is the following.

THEOREM 3.1. *Let n_j, $1 \le j \le m$, be positive integers, $n = n_1 + \cdots + n_m$, and let ρ_j, $1 \le j \le m$, be some weights on Ω. Define $\omega_1 = \rho_1$, and set*

$$\omega_j(x) = [\omega_{j-1}^{1/(1-p)} * \rho_j^{1/(1-p)}]^{1-p}(x), \quad x \in \Omega,$$

for $j \geq 2$. *Then, for* $f_j \in W_p^{n_j}(\rho_j)$, $1 \leq j \leq m$, *we have* $\prod_{j=1}^m * f_j \in W_p^n(\omega_m)$, *and moreover,*

$$\left\| \prod_{j=1}^{m} * f_j \right\|_{W_p^n(\omega_m)} \leq \prod_{j=1}^{m} \|f_j\|_{W_p^{n_j}(\rho_j)}. \tag{3.1}$$

Unless $m = 1$, *the equality holds in* (3.1) *if, and only if,*

$$f_j(x) = C_j \int_0^{\min(x,y)} \frac{(x-t)^{n_j-1}}{(n_j-1)!} e^{\alpha t} \rho_j^{1/(1-p)}(t)\, dt, \quad x \in \Omega, \tag{3.2}$$

where $\alpha \in \mathbf{R}$ *is a constant and* y *is an arbitrary point on* Ω, *which is independent of* j.

Proof. Since $D^i f_j(0) = 0$ for $i = 0, 1, \ldots, n_j - 1$ and $1 \leq j \leq m$, we have

$$D^i \left[\prod_{j=1}^{m} * f_j \right](0) = 0, \quad i = 0, 1, \ldots, n-1,$$

and

$$D^n \left[\prod_{j=1}^{m} * f_j \right](x) = \left[\prod_{j=1}^{m} * D^{n_j} f_j \right](x). \tag{3.3}$$

We prove (3.1) by induction. With $\prod_{j=1}^{1} * D^{n_j} f_j = D^{n_1} f_1$, inequality (3.1) holds for $m = 1$, so we suppose that $k \geq 2$, $\int_\Omega |D^{n_j} f_j(x)|^p \rho_j(x)\, dx < \infty$, $j = 1, 2, \ldots, k$, and that inequality (3.1) has been verified for $m = k-1$. Set $H_{k-1}(x) = [\prod_{j=1}^{k-1} * D^{n_j} f_j](x)$. Then $\int_\Omega |H_{k-1}(x)|^p \omega_{k-1}(x)\, dx < \infty$, and

$$\begin{aligned} &\left| \int_0^x H_{k-1}(t) D^{n_k} f_k(x-t)\, dt \right|^p \\ &\quad = \left| \int_0^x H_{k-1}(t) \omega_{k-1}^{1/p}(t) D^{n_k} f_k(x-t) \rho_k^{1/p}(x-t) \frac{1}{[\omega_{k-1}(t)\rho_k(x-t)]^{1/p}}\, dt \right|^p \\ &\quad \leq \left[\int_0^x |H_{k-1}(t)|^p \omega_{k-1}(t) |D^{n_k} f_k(x-t)|^p \rho_k(x-t)\, dt \right] \frac{1}{\omega_k(x)} \end{aligned} \tag{3.4}$$

by Hölder's inequality. Fubini's theorem shows that the last integral is finite

$$\begin{aligned} &\int_\Omega \int_0^x |H_{k-1}(t)|^p \omega_{k-1}(t) |D^{n_k} f_k(x-t)|^p \rho_k(x-t)\, dt dx \\ &\quad = \int_\Omega \int_t^b |H_{k-1}(t)|^p \omega_{k-1}(t) |D^{n_k} f_k(x-t)|^p \rho_k(x-t)\, dx dt \\ &\quad \leq \int_\Omega |H_{k-1}(t)|^p \omega_{k-1}(t)\, dt \int_\Omega |D^{n_k} f_k(x)|^p \rho_k(x)\, dx < \infty. \end{aligned} \tag{3.5}$$

Thus, (3.4) shows that $H_{k-1} * D^{n_k} f$ exists. Combining (3.3), (3.4), and (3.5) yields

$$\int_\Omega \left| D^n \left[\prod_{j=1}^{k} * f_j \right](x) \right|^p \omega_k(x)\, dx \le \left[\prod_{j=1}^{k-1} \int_\Omega |D^{n_j} f_k(x)|^p \rho_j(x)\, dx \right] \int_\Omega |D^{n_k} f_k(x)|^p \rho_k(x)\, dx,$$

which gives inequality (3.1) for $m = k$. This completes the proof by induction of (3.1).

Next, we determine under what conditions equality holds in (3.1). Equality in (3.1) implies that equality holds in (3.4) for each positive integer k. This happens only if equality holds in Hölder's inequality, i.e. only if for a.e. $x > 0$ there is a complex valued function g_k such that for almost $x \in \Omega$

$$\frac{H_{k-1}(t)}{\omega_{k-1}^{1/(1-p)}(t)} \frac{D^{n_k} f_k(x-t)}{\rho_k^{1/(1-p)}(x-t)} = g_k(x) \quad \text{a.e. } t \in (0, x]. \tag{3.6}$$

The condition (3.6) implies that (see [3] or [2, Lemma]) there exists a constant $\alpha \in \mathbf{C}$ such that

$$\frac{H_{k-1}(t)}{\omega_{k-1}^{1/(1-p)}(t)} = E_{k-1} e^{\alpha t} \quad \text{a.e. } t \in (0, x], \tag{3.7}$$

and

$$\frac{D^{n_k} f_k(t)}{\rho_k^{1/(1-p)}(t)} = C_k e^{\alpha t} \quad \text{a.e. } t \in (0, x], \tag{3.8}$$

where E_{k-1} and C_k are complex constants. Then, in the same way as in the proof of the equality statement in Theorem 2.4, we obtain

$$f_k(x) = C_k \int_0^{\min(x,y)} \frac{(x-t)^{n_k - 1}}{(n_k - 1)!} e^{\alpha t} \rho_k^{1/(1-p)}(t)\, dt, \quad x \in \Omega, \tag{3.9}$$

where

$$y = \operatorname{ess\,sup}\{x \in (0, b) : D^{n_k} f_k(x) \ne 0\}.$$

The necessity of (3.2) may now be proved by induction. Taking $k = 2$ in (3.6) shows, in view of (3.7), (3.8) and (3.10), that (3.2) holds for $j = 1, 2$. To complete the induction, suppose (3.2) has been verified for $j = 1, \ldots, k-1$. Then,

$$H_{k-1}(t) = \left[\prod_{j=1}^{k-1} * D^{n_j} f_j \right](t) = \left(\prod_{j=1}^{k-1} C_j \right) e^{\alpha t} \omega_{k-1}^{1/(1-p)}(t) \quad \text{a.e. } t \in \Omega,$$

and hence (3.6), (3.8) and (3.10) show that

$$f_k(x) = C_k \int_0^{\min(x,y)} \frac{(x-t)^{n_k - 1}}{(n_k - 1)!} e^{\alpha t} \rho_k^{1/(1-p)}(t)\, dt, \quad x \in \Omega. \tag{3.10}$$

Thus (3.2) holds for $j = k$. □

Remark 3.2. Theorem 3.1 also holds for $n_j = 0$ for one or more integers j in the sense that

$$W_p^{n_j=0}(\rho_j) \equiv L_p(\rho_j).$$

See [2] and [7] for more details.

4. Applications to integral equations

We now apply Theorem 3.1 to obtain L_p-weighted estimates for solutions of some integral equations.

Example 4.1 (The Riemann-Liouville fractional derivative [5]). Let $\alpha > 0$ be such that $\beta = 1 - p(\alpha - [\alpha]) > 0$, and $f \in W_p^n(\rho)$, $n = [\alpha] + 1$, where $[\alpha]$ denotes the integral part of α. Then, the Riemann-Liouville fractional derivative

$$(D_+^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} D^n \left(\int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt \right) \tag{4.1}$$

satisfies the following estimate

$$\|D_+^\alpha f\|_{L_p(\omega)} \le \frac{b^{\beta/p} \beta^{-1/p}}{\Gamma(n-\alpha)} \|f\|_{W_p^n(\rho)}, \tag{4.2}$$

where

$$\omega(x) = \left[\int_0^x \rho^{1/(1-p)}(t)\, dt \right]^{1-p}, \quad x \in \Omega.$$

Example 4.2 (Volterra integral equations of the first kind [9]). Let us consider the integral transform

$$f(x) = \left(\frac{2}{\lambda} \right)^n D^{n+2} \left(\int_0^x I_0(\lambda \sqrt{x-t}) g(t)\, dt \right) \tag{4.3}$$

which yields the solution $f(x)$ of the integral equation

$$\int_0^x (x-t)^{n/2} J_n(\lambda \sqrt{x-t}) f(t)\, dt = g(x), \quad (n = 0, 1, 2, \ldots). \tag{4.4}$$

Here, $J_\nu(z)$ is the Bessel function of the first kind and $I_\nu(z)$ is the modified Bessel function of the first kind.

By the inequality of Hardy, Littlewood and Pólya (see [6, p. 106]), we have

$$\left(\int_0^b |I_0(\lambda \sqrt{x})|^p\, dx \right)^{1/p} \le \sum_{k=0}^\infty \frac{|\lambda/2|^{2k}}{k!\Gamma(k+1)} \left(\frac{b^{kp+1}}{kp+1} \right)^{1/p}$$
$$\le b^{1/p} I_0(|\lambda| \sqrt{b}).$$

Therefore, the formal solution f satisfies the following inequality

$$\|f\|_{L_p(\omega)} \le \left|\frac{2}{\lambda}\right|^n b^{1/p} I_0(|\lambda|\sqrt{b})\|g\|_{W_p^{n+2}(\rho)}, \tag{4.5}$$

provided $g \in W_p^{n+2}(\rho)$ and

$$\omega(x) = \left[\int_0^x \rho^{1/(1-p)}(t)\,dt\right]^{1-p}, \quad x \in \Omega.$$

Similarly, the solution $h(x)$ of the integral equation

$$\int_0^x (x-t)^{n/2} I_n(\lambda\sqrt{x-t})h(t)\,dt = u(x) \tag{4.6}$$

is given by

$$h(x) = \left(\frac{2}{\lambda}\right)^n D^{n+2}\left(\int_0^x J_0(\lambda\sqrt{x-t})u(t)\,dt\right), \tag{4.7}$$

which also satisfies the following estimate

$$\|h\|_{L_p(\omega)} \le \left|\frac{2}{\lambda}\right|^n b^{1/p} I_0(|\lambda|\sqrt{b})\|u\|_{W_p^{n+2}(\rho)}, \tag{4.8}$$

provided $u \in W_p^{n+2}(\rho)$ and

$$\omega(x) = \left[\int_0^x \rho^{1/(1-p)}(t)\,dt\right]^{1-p}, \quad x \in \Omega.$$

Example 4.3 (A Volterra integral equation of the second kind [9]). Let us consider the Volterra convolution integral equation of the second kind

$$f(x) + \int_0^x K(x-t)f(t)\,dt = g(x), \tag{4.9}$$

where K and g are given functions.

Let $h = h(x)$ be the solution of the simpler auxiliary equation with $g \equiv 1$:

$$h(x) + \int_0^x K(x-t)h(t)\,dt = 1. \tag{4.10}$$

Then, the solution of the original integral equation with arbitrary $g = g(x)$ is expressed via the solution of the auxiliary equation (4.10), which can be derived by the use of Laplace transform, as

$$f(x) = D\left(\int_0^x h(x-t)g(t)\,dt\right). \tag{4.11}$$

For $h \in L_p(\rho)$ and $g \in W_p^1(\gamma)$, we have $f \in L_p(\omega)$ and

$$\|f\|_{L_p(\omega)} \leq \|h\|_{L_p(\rho)} \|g\|_{W_p^1(\gamma)}, \tag{4.12}$$

where

$$\omega(x) = \left[\int_0^x \rho^{1/(1-p)}(t) \gamma^{1/(1-p)}(x-t)\, dt \right]^{1-p}, \quad x \in \Omega.$$

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