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CONFORMALLY NATURAL EXTENSIONS IN VIEW OF DYNAMICS

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Abstract

We give an easy description of the barycentric extension of a map of the unit circle to the closed unit disk using some ideas from dynamical systems. We then prove that every circle endomorphism of the unit circle of degree $d \geq 2$ (with a topological expansion condition) has a conformally natural extension to the closed unit disk which is real analytic on the open unit disk. If the endomorphism is uniformly quasisymmetric, then the extension is quasiconformal.

1. Conformal barycenter

Let $\mathbf{C}$ denote the complex plane, $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ the unit circle, $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ the open unit disk, and $\overline{\Delta} = \Delta \cup \mathbf{T}$ the closed unit disk.

Let μ be a probability measure on $\mathbf{T}$. The *barycenter* of μ, by definition, is

$$BC(\mu) = \int_{\mathbf{T}} \xi \, d\mu(\xi).$$

Suppose $f : \mathbf{T} \to \mathbf{T}$ is a μ-measurable map. The pushforward measure $f_*\mu$ of μ by f, by definition, is

$$f_*\mu(E) = \mu(f^{-1}(E))$$

for any μ-measurable set E on $\mathbf{T}$. For two μ-measurable maps f and g from $\mathbf{T}$ to $\mathbf{T}$, the composition $f \circ g$ is also a μ-measurable map from $\mathbf{T}$ to $\mathbf{T}$. Then $(f \circ g)_*\mu = f_*g_*\mu$ since

$$f_*g_*\mu(E) = g_*\mu(f^{-1}(E)) = \mu(g^{-1}(f^{-1}(E))) = \mu((f \circ g)^{-1}(E)) = (f \circ g)_*\mu(E)$$

for any μ-measurable set $E \subseteq \mathbf{T}$.

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The conformal barycentric extension of f to a point $z \in \Delta$ can be viewed by using some ideas from dynamical systems as follows.

For a Möbius transformation preserving the unit circle

$$\tau = \eta_{z,t}(\xi) = e^{2\pi it}\frac{\xi - z}{1 - \bar{z}\xi} : \Delta \to \Delta; \quad z \mapsto 0,$$

where $0 \le t < 1$. We use η_z to denote $\eta_{z,0}$. The inverse is

$$\xi(\tau) = \eta_z^{-1}(\tau) : \Delta \to \Delta; \quad 0 \mapsto z.$$

Let m_0 be the probability Lebesgue measure on $\mathbf{T}$, that is, $dm_0 = d\theta$ for $z = e^{2\pi i\theta}$, $0 \le \theta \le 1$. Let $m_z = (\eta_z^{-1})_* m_0$ be the harmonic measure at $z \in \Delta$. Consider $\mu_{f,z} = f_* m_z$. For any $w \in \Delta$, define $\mu_{f,w,z} = (\eta_w)_* \mu_{f,z}$. Then

$$\mu_{f,w,z} = (\eta_w)_* f_* (\eta_z^{-1})_* m_0 = (\eta_w \circ f \circ \eta_z^{-1})_* m_0.$$

Definition 1. If there is a unique $w \in \Delta$ such that $BC(\mu_{f,w,z}) = 0$, then we define $Ext_{bc}(f)(z) = w$ as the conformal barycentric extension of f at z.

Remark 1. From the work of Douady and Earle [2], we know that when f is a homeomorphism of the unit circle $\mathbf{T}$, there is always a unique solution w of the equation $BC(\mu_{f,w,z}) = 0$ for every $z \in \Delta$. Moreover, $Ext_{bc}(f)(z) := w$ extends f as a homeomorphism of the closed unit disk $\bar{\Delta}$. See Theorem 1.

Let $Ext_{cn}(f)$ denote a process of extending a map f of the unit circle to the closed unit disk. We say that it is conformally natural if it satisfies the following three properties:

(1) $Ext_{cn}(id) = Id$, where id means the identity of the unit circle and Id means the identity of the closed unit disk.
(2) $Ext_{cn}(A \circ f \circ B) = A \circ Ext_{cn}(f) \circ B$ for any two Möbius transformations preserving the unit disk.
(3) In addition, if f is a homeomorphism and $\int_0^1 f(e^{2\pi i\theta})\, d\theta = 0$, then $Ext_{cn}(f)(0) = 0$.

We will now prove that the process $Ext_{bc}(f)$ defined in Definition 1 is a conformally natural extension $Ext_{cn}(f)$.

(3) can be easily verified as follows. Suppose $\int_0^1 f(e^{2\pi i\theta})\, d\theta = 0$. Remember $\mu_{f,0,0} = f_* m_0$. Let $\xi = f(\eta) = f(e^{2\pi\theta})$. Then we have $w = 0$ is the unique point in Δ such that

$$BC(\mu_{f,0,0}) = \int_{\mathbf{T}} \xi\, d\mu_{f,0,0} = \int_{\mathbf{T}} \xi\, df_* m_0(\xi) = \int_{\mathbf{T}} f(e^{2\pi i\theta})\, d\theta = 0.$$

Thus $Ext_{bc}(f)(0) = 0$.

(1) can also be easily verified as follows: $\mu_{id,z,z} = m_0$ for any $z \in \Delta$. Since $BC(\mu_{id,z,z}) = BC(m_0) = 0$, we get $Ext_{bc}(id) = Id$.

Finally, (2) can be verified by the following proposition.

PROPOSITION 1. *Suppose f has the conformal barycentric extension $Ext_{bc}(f)(z)$ at $z \in \Delta$. Suppose A and B are two Möbius transformations preserving the unit circle. Then $A \circ f \circ B$ has the conformal barycentric extension*

$$Ext_{bc}(A \circ f \circ B)(B^{-1}(z)) = A(Ext_{bc}(f)(z)) = A \circ Ext_{bc}(f) \circ B(B^{-1}(z))$$

at $B^{-1}(z)$.

Proof. We need to prove that $A(Ext_{bc}(f)(z))$ is the unique point in Δ such that

$$BC(\mu_{A\circ f\circ B,\, A(Ext_{bc}(f)(z)),\, B^{-1}(z)}) = 0$$

for any two Möbius transformations A and B preserving the unit circle. We can write

$$A(z) = e^{2\pi it}\frac{z-a}{1-\bar{a}z} \quad \text{and} \quad B(z) = e^{2\pi is}\frac{z-b}{1-\bar{b}z}, \quad |a|, |b| < 1,\ 0 \le t, s < 1.$$

In the case $t = s = 0$, since

$$\begin{aligned}\mu_{A\circ f\circ B,\, A(Ext_{bc}(f)(z)),\, B^{-1}(z)} &= (\eta_{A(Ext_{bc}(f)(z))})_*(A\circ f\circ B)_*(\eta^{-1}_{B^{-1}(z)})_* m_0\\ &= (\eta_{A(Ext_{bc}(f)(z))})_* A_* f_* B_*(\eta^{-1}_{B^{-1}(z)})_* m_0\\ &= (\eta_{A(Ext_{bc}(f)(z))}\circ A)_* f_*(B\circ \eta^{-1}_{B^{-1}(z)})_* m_0\end{aligned}$$

and since

$$(\eta_{A(Ext_{bc}(f)(z))}\circ A)_* = (\eta_{Ext_{bc}(f)(z)})_* \quad \text{and} \quad (B\circ \eta^{-1}_{B^{-1}(z)})_* = (\eta_z^{-1})_*,$$

we have that

$$\mu_{A\circ f\circ B,\, A(Ext_{bc}(f)(z)),\, B^{-1}(z)} = (\eta_{Ext_{bc}(f)(z)})_* f_*(\eta_z^{-1})_* m_0 = \mu_{f,\, Ext_{bc}(f)(z),\, z}.$$

Thus the barycenter of $\mu_{A\circ f\circ B,\, A(Ext_{bc}(f)(z)),\, B^{-1}(z)}$ is zero (and $A(Ext_{bc}(f)(z))$ is the unique point in Δ having this property since $Ext_{bc}(f)(z)$ is unique). This implies that

$$Ext_{bc}(A\circ f\circ B)(B^{-1}(z)) = A(Ext_{bc}(f)(z)) = A\circ Ext_{bc}(f)\circ B(B^{-1}(z)).$$

Now consider the rotation $e^{2\pi it}z$. A calculation shows that

$$\eta_{e^{2\pi it}w}\circ(e^{2\pi it}f)\circ\eta_z^{-1} = e^{2\pi it}\eta_w\circ f\circ \eta_z^{-1}.$$

This implies $Ext_{bc}(e^{2\pi it}f)(z) = e^{2\pi it}\, Ext_{bc}(f)(z)$.

Finally consider the rotation $e^{2\pi is}\xi$. A calculation shows that

$$\eta^{-1}_{e^{2\pi is}z}(e^{2\pi is}\xi) = e^{2\pi is}\eta_z^{-1}(\xi).$$

This implies that $Ext_{bc}(f\circ e^{2\pi is})(z) = Ext_{bc}(f)(e^{2\pi is}z)$. This completes the proof. □

Remark 2. We showed above that $Ext_{bc}(f)$ is conformally natural. Furthermore, if f is a homeomorphism of the unit circle $\mathbf{T}$, then any Ext_{cn} is also Ext_{bc} (see Theorem 1).

Now we will study the barycentric extension for Blaschke products. A finite Blaschke product BP is

$$(1) \qquad BP(\xi) = e^{2\pi i t} \prod_{i=1}^{n} \frac{\xi - a_i}{1 - \overline{a_i}\xi}, \quad |a_i| < 1,\ 1 \le i \le n,\ 0 \le t < 1.$$

Then BP is a map from $\overline{\Delta}$ into itself and the restriction of BP to $\mathbf{T}$ is a circle endomorphism which we denote as $f_{BP} = BP \,|\, \mathbf{T}$.

PROPOSITION 2. *$Ext_{bc}(f_{BP})(z) = BP(z)$ for any $z \in \overline{\Delta}$.*

Proof. For any $z \in \Delta$, let $w \in \Delta$, consider $BP_w = \eta_w \circ BP \circ \eta_z^{-1}$. Let $a_w = BP_w(0) \in \Delta$. Since BP_w is harmonic in Δ and continuous on $\overline{\Delta}$, by the mean value theorem

$$2\pi a_w = \int_{\mathbf{T}} BP_w(\xi)\, d\xi.$$

Then there is a unique $w = BP(z)$ such that $a_w = 0$. Consider $\widetilde{BP} = \eta_w \circ BP \circ \eta_z^{-1}$ for this unique value $w = BP(z)$. It is again a Blaschke product. Furthermore, it fixes 0. By restricting to the unit circle $\mathbf{T}$, we have

$$\mu_{f_{BP}, w, z} = (\eta_w \circ f_{BP} \circ \eta_z^{-1})_* m_0 = m_0.$$

This equality is fairly well-known. For the reader's convenience, we are including the proof (see, for example, [3]). Let $f = \eta_w \circ f_{BP} \circ \eta_z^{-1} \,|\, \mathbf{T}$. Then $f_* m_0 = m_0$ is equivalent to the condition that

$$\int_{\mathbf{T}} \phi \circ f(z)\, dz = \int_{\mathbf{T}} \phi(z)\, dz$$

for all continuous function ϕ on $\mathbf{T}$. Given a continuous function ϕ, let $u(z)$ be the harmonic extension of ϕ into the unit disk. Then $u \circ BP(z)$ is the harmonic extension of $\phi \circ f$ into the unit disk. By the mean value theorem in harmonic analysis,

$$\frac{1}{2\pi} \int_{\mathbf{T}} \phi(z)\, dz = u(0) = u(BP(0)) = \frac{1}{2\pi} \int_{\mathbf{T}} \phi \circ f(z)\, dz.$$

Thus $BC(\mu_{f_{BP}, w, z}) = 0$. This completes the proof. □

If h is a homeomorphism of $\mathbf{T}$, then $Ext_{bc}(h)(z)$ is the formula given in the paper [2]. By [2], we know the following

THEOREM 1 (Douady-Earle). *Suppose h is a homeomorphism of* $\mathbf{T}$. *Then for any $z \in \Delta$, h has a conformal barycentric extension $Ext_{bc}(h)(z)$ at z which defines a homeomorphism $Ext_{bc}(h)$ of $\overline{\Delta}$. Moreover, $Ext_{bc}(h)$ is real analytic on Δ. If h is a quasisymmetric homeomorphism, then $Ext_{bc}(h)$ is a quasiconformal homeomorphism.*

Remark 3. Abikoff, Earle, and Mitra have generalized this theorem to a continuous monotone circle map of $\mathbf{T}$ of degree 1. The reader is referred to the paper [1] for details.

In the next section, we will show that a conformally natural extension to any orientation-preserving circle endomorphism can be easily obtained from Theorem 1 from the point of view of dynamical systems.

2. Circle endomorphisms

Suppose $f : \mathbf{T} \to \mathbf{T}$ is an orientation-preserving circle covering of degree $d > 1$. Then f has a fixed point which we always normalize as 1. It is called a *circle endomorphism*. An example of a circle endomorphism is a Blaschke product BP having a fixed point inside Δ. Such an example is an expanding circle endomorphism (see [3] for the definition). Suppose the degree of BP is also d. Then both sets $f^{-n}(1)$ and $BP^{-n}(1)$ are d^n ordered points on $\mathbf{T}$. Thus we have a one-to-one correspondence h from points in $f^{-n}(1)$ to points in $BP^{-n}(1)$ keeping the order. Since BP is expanding $\bigcup_{n=1}^{\infty} BP^{-n}(1)$ is a dense subset of $\mathbf{T}$. Thus h can be extended to a continuous monotone circle map of degree 1. We add the assumption (usually called a topological expansion condition),

$$\bigcup_{n=1}^{\infty} f^{-n}(1) \text{ is dense subset of } \mathbf{T}. \tag{2}$$

Then h is a homeomorphism of $\mathbf{T}$. This was first proved by Shub in 1967 (see [5]), when f has certain smoothness properties. The reader can find a more general treatment of this by using symbolic dynamical systems in [3]. Thus we have that

PROPOSITION 3. *Suppose $f : \mathbf{T} \to \mathbf{T}$ is an orientation-preserving circle covering of degree $d > 1$ satisfying* (2). *There is an orientation-preserving homeomorphism h of $\mathbf{T}$ of degree* 1 *such that*

$$f = h^{-1} \circ BP \circ h$$

for any expanding Blaschke product BP of degree $d > 1$.

THEOREM 2. *Suppose BP is any Blaschke product of degree $d > 1$* (*in the form* (1)). *Let h and k be two circle homeomorphisms. Then the circle* *endomorphism* $f = k \circ BP \circ h$ *has a conformally natural extension to the closed unit disk* $\overline{\Delta}$ *which is real analytic on the open unit disk* Δ.

Proof. Define

$$Ext_{cn}(f) = Ext_{bc}(k) \circ BP \circ Ext_{bc}(h).$$

Then

$$\begin{aligned} A \circ Ext_{cn}(f) \circ B &= A \circ Ext_{bc}(k) \circ BP \circ Ext_{bc}(h) \circ B \\ &= Ext_{bc}(A \circ k) \circ BP \circ Ext_{bc}(h \circ B) = Ext_{cn}(A \circ k \circ BP \circ h \circ B) \\ &= Ext_{cn}(A \circ f \circ B). \end{aligned}$$

□

COROLLARY 1. *Suppose* f *is an orientation-preserving circle endomorphism of degree* $d > 1$ *satisfying* (2). *Then* f *has a conformally natural extension* $Ext_{cn}(f) = Ext_{bc}(h^{-1}) \circ BP \circ Ext_{bc}(h)$ *for any given expanding Blaschke product* BP *of degree* d.

Proof. It is a direct consequence of Theorem 2 and Proposition 3. □

Remark 4. Note that $Ext_{cn}(f)$ depends on the choice of the Blaschke product BP.

A circle endomorphism f is called *uniformly quasisymmetric* if all inverse branches of f^n are quasisymmetric with a uniformly quasisymmetric constant (see [3, 4] for more precise definition). It was proved that in [3, 4] that f is uniformly quasisymmetric if and only if h in Proposition 3 is quasisymmetric. We therefore have the following

COROLLARY 2. *Suppose* f *is a uniformly quasisymmetric circle endomorphism of degree* $d > 1$. *Then* $Ext_{cn}(f)$ *in Corollary* 1 *is quasiconformal. More precisely,* $Ext_{cn}(f)$ *is quasiregular, which means* $Ext_{cn}(f) = R \circ F$, *where* R *is a rational map preserving the unit disk* (*that is, a Blaschke product*) *and* f *is a quasiconformal homeomorphism of the unit disk.*

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