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KODAI MATH. J.
**36** (2013), 137–153

# A NOTION OF Δ-MULTIGENUS FOR CERTAIN RANK TWO AMPLE VECTOR BUNDLES

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**Abstract**

A notion of "delta-genus" for ample vector bundles $\mathscr{E}$ of rank two on a smooth projective threefold $X$ is defined as a couple of integers $(\delta_1, \delta_2)$. This extends the classical definition holding for ample line bundles. Then pairs $(X, \mathscr{E})$ with low $\delta_1$ and $\delta_2$ are classified under suitable additional assumptions on $\mathscr{E}$.

## Introduction

Let $X$ be a smooth complex projective variety and let $\mathscr{L}$ be an ample line bundle on $X$. In order to study polarized manifolds $(X, \mathscr{L})$ Fujita [2] introduced the $\Delta$-*genus* of $(X, \mathscr{L})$, which is a nonnegative integer defined by the formula

$$\Delta(X, \mathscr{L}) := \dim X + d(X, \mathscr{L}) - h^0(X, \mathscr{L}),$$

where $d(X, \mathscr{L}) = \mathscr{L}^{\dim X}$. The theory developed around this invariant has been a powerful tool in characterizing polarized varieties with $\Delta$ small enough [2]. As noticed in [2, p. 176] there is not a good vector bundle version of the theory of $\Delta$-genus. This sentence motivated our interest in the subject.

Let $\mathscr{E}$ be an ample vector bundle of rank $r \geq 2$ on $X$. In principle one could conceive a $\Delta$-genus for $(X, \mathscr{E})$ either as a single integer (e.g. see [8]), or as an $r$-tuple of integers, if one wants to give an invariant related to the geometry of the Grassmannian to which $\mathscr{E}$ maps the variety $X$. This last point of view presents several difficulties even for the first non-trivial cases (as explained in Section 1), so that we will restrict ourselves to a very particular case.

Specifically, in this paper we consider ample vector bundles $\mathscr{E}$ of rank 2 on a smooth threefold $X$. We define the $\Delta$-genus of such a pair $(X, \mathscr{E})$ as a couple of integers $(\delta_1, \delta_2)$. While $\delta_1$ is the classical $\Delta$-genus of the scroll associated to $(X, \mathscr{E})$, hence $\delta_1 \geq 0$, $\delta_2$ involves the endomorphisms of $\mathscr{E}$ (see Definition 1.1). Its meaning becomes geometrically clear if we assume that $\mathscr{E}$ has a section

2000 *Mathematics Subject Classification*. Primary 14J60; secondary 14F05, 14M15, 14C20.

*Key words and phrases*. Ample vector bundles, Grassmannian of lines, special varieties, $\Delta$-genus, classification.

Received February 14, 2012; revised August 24, 2012.

vanishing on a smooth curve $Z$ (condition $(*)$ in the paper). Actually, under this assumption, $\delta_2$ turns out to be greater than or equal to the classical $\Delta$-genus of the surface scroll induced by $\mathscr{E}$ on $Z$, which implies that $\delta_2 \geq 0$.

We investigate pairs $(X, \mathscr{E})$ with low $\delta_1$ and $\delta_2$. Of course, the stronger are the properties of the vector bundle $\mathscr{E}$, the larger are the values of $\delta_1$ and $\delta_2$ we can include in our classification. In fact we classify pairs $(X, \mathscr{E})$ with $\delta_1 \leq 2$, results being complete for $\delta_1 = 2$ only when $\mathscr{E}$ is globally generated (Proposition 1.5 and Proposition 1.6). As to the second invariant, under assumption $(*)$ we classify pairs $(X, \mathscr{E})$ with $\delta_2 \leq 1$ (Corollary 2.2 and Proposition 2.3) and with $\delta_2 = 2$ when either the vector bundle $\mathscr{E}$ has curve genus $g = 1$ (Theorem 2.5) or $g \geq 2$, $\mathscr{E}$ being globally generated (Proposition 2.7). Our results are roughly summarized by the following

Theorem. *Let $\mathscr{E}$ be an ample vector bundle of rank 2 on a complex smooth projective threefold $X$.*

(A) *If $\delta_1 \leq 1$, then $\delta_1 = \delta_2 = 0$ and $(X, \mathscr{E}) = (\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1)^{\oplus 2})$.*

(B) *If $\delta_1 = 2$ and in addition $\mathscr{E}$ is spanned, then $\delta_2 = 0$ and $(X, \mathscr{E}) = (\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(1)^{\oplus 2})$.*

(C) *If $\delta_2 = 0$, $\delta_1 \geq 3$ and condition $(*)$ holds, then $(X, \mathscr{E})$ is either $(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2) \oplus \mathscr{O}_{\mathbf{P}^3}(1))$ (here $\delta_1 = 5$), or*

$$(\#) \qquad (\mathbf{P}_{\mathbf{P}^1}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(a_1) \oplus \mathscr{O}_{\mathbf{P}^1}(a_2)), \xi \otimes \pi^*(\mathscr{O}_{\mathbf{P}^1}(b_1) \oplus \mathscr{O}_{\mathbf{P}^1}(b_2))),$$

*where $\pi : X \to \mathbf{P}^1$ is the bundle projection, $\xi$ is the tautological line bundle on $X$, and the integers $a_i$, $b_i$ satisfy $0 \leq a_1 \leq a_2$, $1 \leq b_1 \leq b_2 \leq b_1 + 1$ (here $\delta_1 = 2(a_1 + a_2) + 3(b_1 + b_2) - 2 \geq 4$).*

(D) *If $\delta_2 = 1$ and condition $(*)$ holds, then $(X, \mathscr{E})$ is as in $(\#)$ with $b_2 = b_1 + 2$.*

(E) *If $\delta_2 = 2$, $\delta_1 \geq 3$ and in addition $\mathscr{E}$ is spanned then either $(X, \mathscr{E})$ is as in $(\#)$ with $b_2 = b_1 + 3$, one of the pairs listed in Theorem 2.5, or a general section of $\mathscr{E}$ vanishes along a smooth hyperelliptic curve of genus $\geq 2$ and $\mathscr{E}$ splits on every such curve as $\mathscr{L}^{\oplus 2}$, where $\mathscr{L}$ is the hyperelliptic line bundle.*

In fact we suspect that $\delta_2 \geq 0$ even without assumption $(*)$. In connection with this in Section 3 we discuss the inequality $\delta_2 > \dim(\mathrm{Bs}|\mathscr{E}|)$, proving it is true when $\mathscr{E}$ is generically spanned and the rational map from $X$ to an appropriate Grassmannian of lines defined by $\mathscr{E}$ has one-dimensional image (Proposition 3.2). In Section 4 we discuss some problems that arise in trying to extend our definition to rank 2 ample vector bundles on projective manifolds of higher dimension.

## 1. $\Delta$-genus for ample vector bundles

Before defining the $\Delta$-multigenus of an ample vector bundle, let us analyze closely the definition of the classical $\Delta$-genus of a polarized manifold $(X, \mathscr{L})$.

If $\mathscr{L}$ admits a section vanishing along a smooth hypersurface $Y$ we can easily see that

$$\Delta(X, \mathscr{L}) = \dim X - 1 + d(Y, \mathscr{L}_Y) - \dim \operatorname{Im}[H^0(\mathscr{L}) \to H^0(\mathscr{L}_Y)],$$

the arrow denoting the restriction homomorphism. Recalling the definition of $\Delta(X, \Lambda)$ for any linear system $\Lambda$ on $X$ [2, p. 33], we can consider the linear system $\mathrm{Tr}_Y|\mathscr{L}|$, so that $\Delta(X, \mathscr{L}) = \Delta(Y, \mathrm{Tr}_Y|\mathscr{L}|)$.

It follows that, under suitable assumptions, e.g. the very ampleness of $\mathscr{L}$, the classical $\Delta$-genus can be defined recursively starting from that of a curve. Observe also that this $\Delta$-genus is related to the minimal degree of the embedding of $X$ defined by $\mathscr{L}$.

If we want to imitate this for a pair $(X, \mathscr{E})$, where $\mathscr{E}$ is an ample vector bundle of arbitrary rank, the analogous assumption would be that $\mathscr{E}$ defines an embedding in the Grassmannian. Starting with the case in which $X$ is a curve, there is only one degree, namely the degree of $X$ in the Plücker ambient space of the Grassmannian, which coincides with the degree of the ruled variety $\mathbf{P}_X(\mathscr{E})$ embedded by its tautological line bundle $\xi_{\mathscr{E}}$. Hence if $\dim X = 1$ it is natural to define the $\Delta$-genus of $(X, \mathscr{E})$ as $\delta_1 = \Delta(\mathbf{P}_X(\mathscr{E}), \xi_{\mathscr{E}})$. Note that this is equivalent to $\delta_1 = \operatorname{rk} \mathscr{E} + c_1(\mathscr{E}) - h^0(\mathscr{E})$.

In higher dimension the embedding of $X$ in the Grassmannian has now a multidegree, expressed as the intersection with the corresponding Schubert cycles of the appropriate dimension. According to a natural ordering of these Schubert cycles, the first degree is again the degree of the ruled variety $\mathbf{P}_X(\mathscr{E})$, so that we can still define $\delta_1$ as above.

On the other hand, some of these Schubert cycles consist of linear spaces satisfying, among other conditions, that of being contained in a hyperplane. The pullback on $X$ of the set of linear spaces contained in a hyperplane is the zero locus of a section of $\mathscr{E}$. So to define the part of $\Delta$-genus corresponding to these particular Schubert cycles it looks natural to use the recursive procedure mentioned in the case of line bundles.

So now assume that the ample vector bundle $\mathscr{E}$ satisfies the following condition:

($*$) there exists a section whose zero locus is a smooth subvariety $Z$ of the expected dimension.

For that recursion we need to compute $\dim \operatorname{Im}[H^0(\mathscr{E}) \to H^0(\mathscr{E}_Z)]$. This dimension can be computed by using the Koszul's exact sequence tensored by $\mathscr{E}$. It is difficult to provide the precise expression in general. However for $\operatorname{rk} \mathscr{E} = 2$, the exact sequence becomes

$$0 \to \mathscr{E}^\vee \to \mathscr{E} \otimes \mathscr{E}^\vee \to \mathscr{E} \to \mathscr{E}_Z \to 0,$$

where $\mathscr{E}^\vee$ stands for the dual of $\mathscr{E}$. So, since $h^0(\mathscr{E}^\vee) = h^1(\mathscr{E}^\vee) = 0$, from the Kodaira–Le Potier vanishing theorem, we derive

$$\dim \operatorname{Im}[H^0(\mathscr{E}) \to H^0(\mathscr{E}_Z)] = h^0(\mathscr{E}) - h^0(\mathscr{E} \otimes \mathscr{E}^\vee).$$

This shows the difficulty of that recursive process even for $\mathrm{rk}\,\mathscr{E} = 2$. In fact, if there were a second step in the recursion, we would need to compute $h^0((\mathscr{E} \otimes \mathscr{E}^\vee)_Z)$, and this should be done from the Koszul's exact sequence tensored by $\mathscr{E} \otimes \mathscr{E}^\vee$. On the other hand, for $\dim X = 3$, $Z$ is a smooth curve, so that no further step is needed. This motivates to confine our discussion to the case of ample vector bundles of rank 2 on threefolds and to give the following definition:

Definition 1.1. Let $\mathscr{E}$ be an ample vector bundle of rank 2 on a smooth complex projective variety $X$ of dimension 3. We define the $\Delta$-*genus* of the pair $(X, \mathscr{E})$ as the pair of numbers

$$\Delta(X, \mathscr{E}) := (\delta_1, \delta_2) \tag{1.1.1}$$

defined as follows:

$$\delta_1 := \Delta(\mathbf{P}_X(\mathscr{E}), \xi_{\mathscr{E}}) = 4 + c_1^3 - 2c_1c_2 - h^0(\mathscr{E}) \quad \text{and}$$
$$\delta_2 := 2 + c_1c_2 - h^0(\mathscr{E}) + h^0(\mathscr{E} \otimes \mathscr{E}^\vee),$$

where $\xi_{\mathscr{E}}$ is the tautological line bundle of $\mathscr{E}$ on $\mathbf{P}_X(\mathscr{E})$, $c_i$ denotes the $i$-th Chern class of $\mathscr{E}$, and we used the Chern–Wu formula $\xi_{\mathscr{E}}^4 = c_1^3 - 2c_1c_2$.

*Remark* 1.2. Let $\mathbf{G}$ be the Grassmannian $\mathbf{G}(1, N)$, with $N = h^0(\mathscr{E}) - 1$, and denote by $\Omega(N-4, N)$ and $\Omega(N-3, N-1)$ the two Schubert cycle classes generating $H^6(\mathbf{G}, \mathbf{Z})$. Suppose that $\mathscr{E}$ defines an embedding of $X$ into $\mathbf{G}$. Then $c_1^3 - 2c_1c_2 = a$ and $c_1c_2 = b$, where $a = X \cdot \Omega(N-4, N)$ and $b = X \cdot \Omega(N-3, N-1)$. Note that the class of $X$ can be written in terms of the dual Schubert cycle classes as $X = a\Omega(0,4) + b\Omega(1,3)$.

*Example* 1.3. Let $\mathscr{E} = L \oplus M$, $L$, $M$ being ample line bundles on $X$. Then

$$\delta_1 = \Delta(X, L) + \Delta(X, M) + L^2M + LM^2 - 2 \tag{1.3.1}$$

and

$$\delta_2 = 2 + L^2M + LM^2 - h^0(L) - h^0(M) + h^0(\mathscr{E} \otimes \mathscr{E}^\vee). \tag{1.3.2}$$

Note that $\mathscr{E} \otimes \mathscr{E}^\vee = \mathcal{O}_X^{\oplus 2} \oplus [L - M] \oplus [M - L]$. Suppose that $|L|$ and $|M|$ contain irreducible surfaces, say $S$ and $T$, respectively. Then (1.3.1) takes the form $\delta_1 = \Delta(X, L) + \Delta(X, M) + d(S, M_S) + d(T, L_T) - 2$. Moreover, recalling the definition of $\Delta(X, \Lambda)$ for any linear system $\Lambda$ on $X$ [2, p. 33], and taking into account the exact cohomology sequences induced by

$$0 \to [M - L] \to M \to M_S \to 0 \quad \text{and} \quad 0 \to [L - M] \to L \to L_T \to 0,$$

(1.3.2) becomes $\delta_2 = \Delta(S, \mathrm{Tr}_S|M|) + \Delta(T, \mathrm{Tr}_T|L|)$.

In particular, let $\mathscr{E} = L^{\oplus 2}$, for any ample line bundle $L$ on $X$. Then $\mathscr{E} \otimes \mathscr{E}^\vee = \mathcal{O}_X^{\oplus 4}$, hence $\delta_1 = 2\Delta(X, L) + 2d(X, L) - 2$ and $\delta_2 = 6 + 2L^3 - 2h^0(L)$

$= 2\Delta(X, L)$. For instance, for $(X, \mathscr{E}) = (X, L^{\oplus 2})$, where $(X, L)$ is a del Pezzo threefold of any degree $d$, we have $(\delta_1, \delta_2) = (2d, 2)$.

*Remark* 1.4. The classical theory of $\Delta$-genus implies $\delta_1 \geq 0$ [2, Theorem 4.2]. A priori we cannot claim the same for $\delta_2$. However, assume that $\mathscr{E}$ satisfies $(*)$, so that it has a section whose zero locus is a smooth curve $Z$. Define $\delta_2' := \Delta(\mathbf{P}_Z(\mathscr{E}_Z), (\xi_{\mathscr{E}})_{\mathbf{P}_Z(\mathscr{E}_Z)})$. Then $\delta_2' = 2 + c_1(\mathscr{E})c_2(\mathscr{E}) - h^0(\mathscr{E}_Z)$. According to Definition 1.1, $\delta_2 = \Delta(\mathbf{P}_Z(\mathscr{E}_Z), \Lambda)$, where $\Lambda = \mathrm{Tr}_{\mathbf{P}_Z(\mathscr{E}_Z)}|\xi_{\mathscr{E}}|$, i.e. the linear system corresponding to the image of the restriction homomorphism $H^0(\mathscr{E}) \to H^0(\mathscr{E}_Z)$. Hence $\delta_2 \geq \delta_2'$, equality holding when the homomorphism $H^0(\mathscr{E}) \to H^0(\mathscr{E}_Z)$ is surjective. In particular, $\delta_2 \geq 0$ under assumption $(*)$.

In the following Proposition we give the characterization of pairs $(X, \mathscr{E})$ whose $\Delta$-genus is $(0, 0)$.

Proposition 1.5. *Let $X$ and $\mathscr{E}$ be as in Definition* 1.1. *Then the following facts are equivalent*:

(1) $\Delta(X, \mathscr{E}) = (0, \delta_2)$;
(2) $\Delta(X, \mathscr{E}) = (0, 0)$;
(3) $(X, \mathscr{E}) = (\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1)^{\oplus 2})$.

*Proof.* Assume that $\delta_1 = 0$, i.e. $\Delta(\mathbf{P}_X(\mathscr{E}), \xi_{\mathscr{E}}) = 0$. By [8, Theorem 3.6], this is equivalent to $(X, \mathscr{E}) = (\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1)^{\oplus 2})$. Then we conclude by Example 1.3. □

The previous proposition shows that $\Delta(X, \mathscr{E}) = (0, 0)$ is equivalent to $\delta_1 = 0$. Notice that it cannot be $\delta_1 = 1$ in view of [8, Lemma 1.4]. Then

$$\Delta(X, \mathscr{E}) \neq (1, \delta_2) \quad \text{for any } \delta_2.$$

So the next case to analyze is $\delta_1 = 2$. The only example we know fitting in Fujita's partial classification of polarized manifolds with $\Delta$-genus 2 [2, Ch. I, §10] is $(X, \mathscr{E}) = (\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(1)^{\oplus 2})$. We will meet this pair in Section 2 again. Moreover, we have

Proposition 1.6. *Let $X$ and $\mathscr{E}$ be as in Definition* 1.1 *and assume that $\mathscr{E}$ is spanned. Then the following facts are equivalent*:

(1) $\Delta(X, \mathscr{E}) = (2, \delta_2)$;
(2) $\Delta(X, \mathscr{E}) = (2, 0)$;
(3) $(X, \mathscr{E}) = (\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(1)^{\oplus 2})$.

*Proof.* Assume that $\delta_1 = 2$, i.e. $\Delta(\mathbf{P}_X(\mathscr{E}), \xi_{\mathscr{E}}) = 2$. Then [8, §4, pp. 684–687] implies $(X, \mathscr{E}) = (\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(1)^{\oplus 2})$. Then we conclude by Example 1.3. □

In connection with our goal of classifying pairs $(X, \mathscr{E})$ according to $\delta_2$ in Section 2, it is useful to spend some words on the $\Delta$-genus of surface scrolls.

Let $\mathscr{E}$ be an ample vector bundle of rank $r \geq 2$ on a smooth curve $C$ of genus $q$. Set $Y := \mathbf{P}_C(\mathscr{E})$ and let $L$ be the tautological line bundle. Note that $L$ is ample, so being $\mathscr{E}$. We say that $(Y, L)$ is the scroll associated with $\mathscr{E}$. Looking at it as a polarized variety, we have $\Delta(Y, L) = r + d - h^0(\mathscr{E})$, where $d = L^r = \deg \mathscr{E}$. Hence, recalling the Riemann–Roch theorem, $\Delta(Y, L) = rq - h^1(\mathscr{E})$. In particular,

*Remark* 1.7. If $\mathscr{E}$ is non-special (i.e. $h^1(\mathscr{E}) = 0$), then $\Delta(Y, L) = rq$.

Note that if $q \leq 1$ then any ample vector bundle $\mathscr{E}$ is non-special. This is obvious if $q = 0$ ($\mathscr{E}$ is a direct sum of $r$ ample line bundles and $h^1 = 0$ for all of them) and is due to Atiyah for $q = 1$ (e.g. see [4, Lemma 1.1]). Moreover, we know that $\Delta(Y, L) \geq 0$ with equality characterizing the case $q = 0$ [2, Theorem 5.10, p. 41]. On the other hand, for $q = 1$ we have $\Delta(Y, L) = r$. This allows us to assume $q \geq 2$ in the following. Moreover, for our need in Section 2 from now on we will confine our discussion to the case $r = 2$, i.e.

(1.7.1) $(Y, L)$ will be a surface scroll of genus $q \geq 2$.

According to Remark 1.7, we know that $\Delta(Y, L) \geq 2q \geq 4$, provided that $\mathscr{E}$ is non-special. Nevertheless, in general $\Delta(Y, L)$ can be smaller. It cannot be 0 by what we said, but, in principle it could be 1. However, this is not the case. This follows from [8, Lemma 1.4], but for surfaces the argument is very easy and we include it for the convenience of the reader.

Proposition 1.8. *Let $C$ be any smooth curve of genus $q \geq 2$. Then $\Delta(Y, L) \geq 2$.*

*Proof.* As we said, $\Delta(Y, L) \geq 1$. Suppose this equality holds. According to Fujita's classification of polarized manifolds of $\Delta$-genus one [2, Ch. I, Section 6], we have the following possibilities: (i) $d \geq 3$ and $Y$ is a del Pezzo surface with $L = -K_Y$; (ii) $d = 2$ and there is a finite morphism $p : Y \to \mathbf{P}^2$ of degree 2 with $L = p^*\mathscr{O}_{\mathbf{P}^2}(1)$; (iii) $d = 1$. Cases (i) and (ii) are not compatible with (1.7.1). This is obvious in case (i) and it follows easily from the ramification formula in case (ii) (to see this, let $D \in |\mathscr{O}_{\mathbf{P}^2}(2b)|$ be the branch locus of $p$; then $0 > 8(1 - q) = K_Y^2 = (p^*\mathscr{O}_{\mathbf{P}^2}(b - 3))^2 = 2(b - 3)^2 \geq 0$, a contradiction). In case (iii) we have $h^0(L) = 3 - \Delta(Y, L) = 2$, hence $|L|$ is a pencil with a single base point, say $y$. Let $f_0$ be the fiber of $Y$ containing $y$. Since $|L| = |L - y|$ is a pencil, there is an element $D \in |L|$ tangent to $f_0$ at $y$. This implies that $D = \Gamma + f_0$, with $\Gamma$ consisting of a section plus possibly some fibers. We thus get

$$1 = L^2 = LD = Lf_0 + L\Gamma = 1 + L\Gamma.$$

But this would imply that $L\Gamma = 0$, which contradicts the ampleness of $L$. Therefore case (iii) cannot occur. $\square$

Next we focus on the $\Delta$-genus 2 case, which, as already noted, includes all elliptic surface scrolls. Here is an interesting example showing that for any $q \geq 2$ there are surface scrolls with $\Delta = 2$.

*Example* 1.9. Let $C$ be a smooth hyperelliptic curve of genus $q$ and let $\mathscr{E} = \mathscr{L}^{\oplus 2}$, where $\mathscr{L}$ is the hyperelliptic line bundle of $C$ (i.e. $|\mathscr{L}|$ is the $g_2^1$ of $C$). Then $d = 2 \deg \mathscr{L} = 4$ and $h^0(\mathscr{E}) = 2h^0(\mathscr{L}) = 4$, so that $\Delta(Y, L) = 2$.

Note that in the example above $\mathscr{E}$ is ample and spanned, but not very ample. In fact, according to [5, Corollary 1], any very ample vector bundle on a smooth hyperelliptic curve with $q \geq 2$ is non-special. Then, for the corresponding surface scroll we have $\Delta(Y, L) = 2q \geq 4$ by Remark 1.7.

Assuming that $\mathscr{E}$ is also spanned, we have the following characterization.

PROPOSITION 1.10. *Let $\mathscr{E}$ be an ample and spanned vector bundle of rank* 2 *on a smooth curve $C$ of genus $q \geq 2$. If $\Delta(Y, L) = 2$, then $(C, \mathscr{E})$ is as in the example above.*

*Proof.* $L$ is spanned, since $\mathscr{E}$ is so, hence $h^0(L) \geq 3$. From $2 = \Delta(Y, L) = 2 + d - h^0(L)$ we thus see that $d \geq 3$. If $d = 3$, then the morphism $\varphi_L : Y \to \mathbf{P}^2$ defined by $L$ expresses $Y$ as a triple plane. Then by using Miranda's formula for triple covers as in [8, Proposition 4.4, proof of case (a)] we see that this is not compatible with the scroll structure of $(Y, L)$. Let $d = 4$. Then, arguing as in [8, Theorem 4.3, proof of case $d = 4$] we conclude that $(C, \mathscr{E})$ is as in the example above. Finally, if $d \geq 5$ then $L$ is very ample, by [2, Theorem 3.5, p. 30]. Hence $\mathscr{E}$ itself is very ample, but this is impossible. Otherwise $\mathscr{E}$ would be non-special by [5, Corollary 1], contradicting $\Delta(Y, L) = 2$, as noted before. $\square$

## 2. Classification results under condition $(*)$

In this section we continue the classification of polarized pairs with small values of the $\Delta$-genus. We will need for this to assume condition $(*)$. We start with $\delta_2' = 0$. We set $(Y, L) = (\mathbf{P}_Z(\mathscr{E}_Z), (\xi_{\mathscr{E}})_{\mathbf{P}_Z(\mathscr{E}_Z)})$.

THEOREM 2.1. *Let $X$ and $\mathscr{E}$ be as in Definition* 1.1 *and assume that $\mathscr{E}$ satisfies condition $(*)$. If $\Delta(Y, L) = 0$, then one of the following holds*:

(1) $(X, \mathscr{E}) = (\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1)^{\oplus 2})$;
(2) $(X, \mathscr{E}) = (\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2) \oplus \mathscr{O}_{\mathbf{P}^3}(1))$;
(3) $(X, \mathscr{E}) = (\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(1)^{\oplus 2})$;
(4) $(X, \mathscr{E}) = (\mathbf{P}_{\mathbf{P}^1}(\mathscr{V}), [\xi_{\mathscr{V}} + b_1 f] \oplus [\xi_{\mathscr{V}} + b_2 f])$, *where* $\mathscr{V} = \bigoplus_{i=0}^{2} \mathscr{O}_{\mathbf{P}^1}(a_i)$ *with* $0 = a_0 \leq a_1 \leq a_2$, *$\xi_{\mathscr{V}}$ is the tautological line bundle of $\mathscr{V}$, $f$ stands for a fiber of the scroll projection, $b_1$, $b_2$ are positive integers.*

*Moreover,* $\Delta(X, \mathscr{E}) = (0,0)$ *in case* (1), $\Delta(X, \mathscr{E}) = (5,0)$ *in case* (2), $\Delta(X, \mathscr{E}) = (2,0)$ *in case* (3), *while* $\Delta(X, \mathscr{E}) = \begin{cases} (2a+3b-2, 0) & \text{if } b_1 = b_2 \\ (2a+3b-2, b_2 - b_1 - 1) & \text{if } b_2 > b_1 \end{cases}$, *where* $a := a_1 + a_2$ *and* $b := b_1 + b_2$ *in case* (4).

*Proof.* Since $\Delta(Y, L) = 0$ and the Picard number is $\rho(Y) > 1$, [2, Theorem 5.10] implies that $(Y, L)$ is a scroll over $\mathbf{P}^1$; in particular $Z = \mathbf{P}^1$, hence $g(X, \mathscr{E}) = g(Z) = 0$. Therefore, by [6, Theorem A] we have the following possibilities for $(X, \mathscr{E})$:

(a) $(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1)^{\oplus 2})$;
(b) $(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2) \oplus \mathscr{O}_{\mathbf{P}^3}(1))$;
(c) $(\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(1)^{\oplus 2})$;
(d) $(\mathbf{P}_{\mathbf{P}^1}(\mathscr{V}), [\xi_{\mathscr{V}} + b_1 f] \oplus [\xi_{\mathscr{V}} + b_2 f])$, where $\mathscr{V}$ is a rank-3 vector bundle on $\mathbf{P}^1$ normalized in the form $\mathscr{V} = \mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(a_1) \oplus \mathscr{O}_{\mathbf{P}^1}(a_2)$ with $0 \le a_1 \le a_2$, $\xi_{\mathscr{V}}$ is the tautological line bundle of $\mathscr{V}$, $f$ stands for a fiber of the scroll projection and $b_1$, $b_2$ are positive integers, due to the ampleness of $\mathscr{E}$ [1, Lemma 3.2.4].

As to the last assertion, Example 1.3 gives

$$\Delta(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1)^{\oplus 2}) = (0,0), \quad \Delta(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2) \oplus \mathscr{O}_{\mathbf{P}^3}(1)) = (5,0),$$
$$\Delta(\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(1)^{\oplus 2}) = (2,0),$$

which proves the assertion in cases (1)–(3) of the statement. Moreover, in case (d), we get $\delta_1 = 3b + 2a - 2$, where $a = a_1 + a_2 = \deg \mathscr{V}$ and $b = b_1 + b_2$. Of course we can assume that $b_2 \ge b_1$, up to exchanging the summands of $\mathscr{E}$. Then

$$h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = 2 + h^0((b_1 - b_2)f) + h^0((b_2 - b_1)f) = \begin{cases} 4 & \text{if } b_1 = b_2, \\ 3 + b_2 - b_1 & \text{if } b_2 > b_1, \end{cases}$$

which leads to the value of $\delta_2$ in case (4) of the statement. □

As a corollary we get the characterization of pairs $(X, \mathscr{E})$ with $\delta_2 = 0$.

Corollary 2.2. *Let* $X$ *and* $\mathscr{E}$ *be as in Definition* 1.1 *and assume that* $\mathscr{E}$ *satisfies condition* $(*)$. *Then* $\delta_2 = 0$ *if and only if* $(X, \mathscr{E})$ *is as in Theorem* 2.1, *cases* (1)–(3) *and case* (4) *with* $b_2 = b_1 \ge 1$ *or* $b_2 = b_1 + 1 \ge 2$.

*Proof.* Assume that $\delta_2 = 0$, consider the smooth curve $Z$ and the polarized surface $(Y, L)$. It follows from Remark 1.4 that $\Delta(Y, L) = 0$, so we are in the assumption of Theorem 2.1. Clearly $\delta_2 = 0$ in cases (1)–(3) of that proposition. Assume now that $(X, \mathscr{E})$ is as in case (4) of Theorem 2.1, then $\delta_2 = 0$ if and only if $b_1 = b_2$ or $b_2 = b_1 + 1$. □

As to the next values of $\delta_2$ we have the following.

PROPOSITION 2.3. *Let $X$ and $\mathscr{E}$ be as in Definition* 1.1 *and assume that $\mathscr{E}$ satisfies condition* $(*)$. *If $\delta_2 = 1$ then $(X, \mathscr{E})$ is as in case* (4) *of Theorem* 2.1 *with $b_2 = b_1 + 2$.*

*Proof.* Let $\delta_2 = 1$ and consider the polarized surface $(Y, L)$. According to the discussion in Remark 1.4, we have $1 = \delta_2 \geq \delta_2'$, so that either $(X, \mathscr{E})$ is as in Theorem 2.1 or $\Delta(Y, L) = 1$. The former case gives the assertion in view of the second part of Theorem 2.1, and the converse is obvious. The latter case is ruled out by Remark 1.7 and Proposition 1.8. □

PROPOSITION 2.4. *Let $X$ and $\mathscr{E}$ be as in Definition* 1.1 *and assume that $\mathscr{E}$ satisfies condition* $(*)$. *If $\delta_2 = 2$ then either*:

(1) $(X, \mathscr{E})$ *is as in case* (4) *of Theorem* 2.1 *with $b_2 = b_1 + 3$, or*
(2) $\delta_2' = 2$.

*Proof.* Let $\delta_2 = 2$ and consider the polarized surface $(Y, L)$. According to the discussion in Remark 1.4, we have $2 = \delta_2 \geq \delta_2'$, so that there are three possibilities, according to whether $\delta_2' = 0, 1$ or $2$. The first case leads to (1) in view of the second part of Theorem 2.1, while the third gives (2). On the other hand, the second possibility is ruled out by Remark 1.7 and Proposition 1.8. □

Let $g := g(X, \mathscr{E})$ be the curve genus of $(X, \mathscr{E})$. Concerning case (2), condition $(*)$ allows us to get a complete classification result for $g = 1$.

THEOREM 2.5. *Let $X$ and $\mathscr{E}$ be as in Definition* 1.1, *assume that $\mathscr{E}$ satisfies condition* $(*)$ *and let $g = 1$. Then $\delta_2 = 2$ if and only if $(X, \mathscr{E})$ is one of the following pairs*:

(1) $(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2)^{\oplus 2})$;
(2) $(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(3) \oplus \mathscr{O}_{\mathbf{P}^3}(1))$;
(3) $(\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(2) \oplus \mathscr{O}_{\mathbf{Q}^3}(1))$;
(4) $(X, H^{\oplus 2})$ *where $(X, H)$ is a del Pezzo threefold*;
(5) $(\mathbf{P}^2 \times \mathbf{P}^1, p^* T_{\mathbf{P}^2} \otimes \mathscr{O}_{\mathbf{P}^2 \times \mathbf{P}^1}(0, 1))$, *where $p$ stands for the projection onto the first factor*;
(6) $(\mathbf{P}^2 \times \mathbf{P}^1, \mathscr{O}_{\mathbf{P}^2 \times \mathbf{P}^1}(2, 1) \otimes \mathscr{O}_{\mathbf{P}^2 \times \mathbf{P}^1}(1, 1))$;
(7) $(\mathbf{Q}^3, \mathscr{S}(2))$, *where $\mathscr{S}$ is the spinor bundle on $\mathbf{Q}^3$* (*see* [11, Definition 1.3]).

*Proof.* Pairs $(X, \mathscr{E})$ satisfying assumption $(*)$ with $g = 1$ are listed in [7, Theorem 1]. As $n = \dim X = 3$, we have the following possibilities:

(a) $X$ is a $\mathbf{P}^2$-bundle on a smooth curve $B$ isomorphic to $Z$ and $\mathscr{E}_F = \mathscr{O}_{\mathbf{P}^2}(1)^{\oplus 2}$ for every fiber $F$ of the bundle projection $\pi : X \to B$;
(b) $(X, \mathscr{E})$ is as in cases (1)–(7) of the statement;
(c) $(X, \mathscr{E}) = (\mathbf{P}^3, \mathscr{N}(2))$, where $\mathscr{N}$ is a null correlation bundle on $\mathbf{P}^3$ (see [10, p. 76]).

We have to select those with $\delta_2 = 2$. As to (b), we can easily see that $\delta_2 = 2$ in cases (1)–(4). This follows immediately from what we said at the end of Example 1.3 in cases (1) and (4), and from a straightforward direct computation in cases (2) and (3). In the next two cases let $F \cong \mathbf{P}^2$ be any fiber of the second projection. In case (5) we get $c_1c_2 = 15$ and $h^0(\mathscr{E}) = 2h^0(T_{\mathbf{P}^2}) = 16$, as we see with the help of the exact sequence

$$0 \to p^*T_{\mathbf{P}^2} \to \mathscr{E} \to \mathscr{E}_F \to 0.$$

Moreover, $h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = h^0(p^*T_{\mathbf{P}^2} \otimes p^*T^\vee_{\mathbf{P}^2}) = h^0(T_{\mathbf{P}^2} \otimes T^\vee_{\mathbf{P}^2}) = 1$, since $T_{\mathbf{P}^2}$ is simple [10, p. 74]. Therefore $\delta_2 = 2$, by definition. In case (6) we have $c_1c_2 = 13$ and the exact sequence

$$0 \to M - F \to M \to M_F \to 0$$

applied to each summand $M$ of $\mathscr{E}$ shows that $h^0(\mathscr{E}) = 2(h^0(\mathscr{O}_{\mathbf{P}^2}(2)) + h^0(\mathscr{O}_{\mathbf{P}^2}(1)))$ $= 18$. Finally, $h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = h^0(\mathscr{O}_X^{\oplus 2}) + h^0(\mathscr{O}_X(1,0)) = 5$. All this gives $\delta_2 = 2$.

In case (7) [11, Theorem 2.8 (i)] shows that $c_1(\mathscr{S}) = -h$, $c_2(\mathscr{S}) = \dfrac{h^2}{2}$, where $h$ is the ample generator of the Picard group. Then $c_1(\mathscr{S}(2))c_2(\mathscr{S}(2)) = 15\dfrac{h^3}{2} =$ 15. Furthermore, $h^0(\mathscr{S}(2)) = 16$ (cf. [8, Proof of Theorem 5.1 Case (v)]) and $\mathscr{E} \otimes \mathscr{E}^\vee = \mathscr{S}(2) \otimes \mathscr{S}^\vee(-2) = \mathscr{S} \otimes \mathscr{S}^\vee$. Notice that $\mathscr{S}$ is stable [11, Theorem 2.1]. It thus follows from [11, Lemma 2.7] that $h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = 1$. Thus, by definition, we get $\delta_2 = 2$.

Next we rule out case (c). From the exact sequence (e.g. see [10, p. 77])

$$0 \to \mathscr{N}(2) \to T_{\mathbf{P}^3}(1) \to \mathscr{O}_{\mathbf{P}^3}(3) \to 0$$

one easily gets $c_1(\mathscr{N}(2)) = 4$ and $c_2(\mathscr{N}(2)) = 5$. Recalling that $\mathscr{N}^\vee = (\det \mathscr{N})^{-1} \otimes \mathscr{N} = \mathscr{N}$, dualizing the above exact sequence and taking into account that $h^0(\Omega^1_{\mathbf{P}^3}(3)) = 20$ by Bott's formula [10, p. 8], we get $h^0(\mathscr{E}) = 16$. Moreover, $\mathscr{E} \otimes \mathscr{E}^\vee = \mathscr{N}(2) \otimes \mathscr{N}^\vee(-2) = \mathscr{N} \otimes \mathscr{N}^\vee$, so that $h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = 1$, since $\mathscr{N}$ is simple [10, p. 77]. Then by definition we get $\delta_2 = 7$.

The proof is completed by the following lemma.

Lemma 2.6. *In case* (a) *it cannot be* $\delta_2 = 2$.

*Proof.* Since $n = 3$, there exists a vector bundle $\mathscr{V}$ of rank 3 on the smooth curve $B$ of genus 1 such that $X = \mathbf{P}_B(\mathscr{V})$, and up to a twist we can suppose that $\mathscr{V}$ is ample. Let $h$ be the tautological line bundle of $\mathscr{V}$ on $X$. Since $\mathscr{E}_F = \mathscr{O}_{\mathbf{P}^2}(1)^{\oplus 2}$, $\mathscr{E} \otimes h^{-1}$ restricts trivially to every fiber $F$ of $\pi$, hence there exists a vector bundle $\mathscr{G}$ of rank 2 on $B$ such that $\mathscr{E} = h \otimes \pi^*\mathscr{G}$. Notice that the equality

$$H^0(\mathscr{E}) = H^0(\pi_*\mathscr{E}) = H^0(\pi_*(h \otimes \pi^*\mathscr{G})) = H^0(\mathscr{V} \otimes \mathscr{G}),$$

coming from the projection formula, can be interpreted as saying that any morphism from $\pi^*\mathscr{G}^\vee \to h$ factorizes as $\pi^*\mathscr{G}^\vee \to \pi^*\mathscr{V} \to h$, where the first map comes from a morphism $\mathscr{G}^\vee \to \mathscr{V}$ and the second one is the surjection of the relative Euler sequence on $\mathbf{P}_B(\mathscr{V})$. In particular, this implies that, if the morphism $\mathscr{G}^\vee \to \mathscr{V}$ vanishes at some point $b \in B$, then the corresponding morphism $\pi^*\mathscr{G}^\vee \to h$ vanishes at the whole fiber $\pi^{-1}(b)$. Since $\mathscr{E}$ satisfies condition $(*)$, there is some morphism $\mathscr{G}^\vee \to \mathscr{V}$ not vanishing at any point of $B$, so that there is an exact sequence

$$0 \to \mathscr{G}^\vee \to \mathscr{V} \to L \to 0 \tag{2.6.1}$$

where $L$ is a line bundle on $B$.

Note that $c_1 = 2h + \pi^* \det \mathscr{G}$ and $c_2 = h^2 + h\pi^* \det \mathscr{G}$, where $c_i = c_i(\mathscr{E})$. As a consequence,

$$c_1c_2 = 2h^3 + 3h^2\pi^* \det \mathscr{G} = 2 \deg \mathscr{V} + 3 \deg \mathscr{G} = \deg(\mathscr{V} \otimes \mathscr{G}),$$

the last equality coming from the splitting principle. Since $B$ has genus 1 we have $\chi(\mathscr{V} \otimes \mathscr{G}) = \deg(\mathscr{V} \otimes \mathscr{G})$, by the Riemann–Roch theorem. Hence we get

$$h^0(\mathscr{E}) - c_1c_2 = h^0(\mathscr{V} \otimes \mathscr{G}) - \deg(\mathscr{V} \otimes \mathscr{G}) = h^1(\mathscr{V} \otimes \mathscr{G}) = h^0(\mathscr{V}^\vee \otimes \mathscr{G}^\vee),$$

the last equality coming from Serre's duality. Finally, $h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = h^0(\pi_*(\mathscr{E} \otimes \mathscr{E}^\vee)) = h^0(\mathscr{G} \otimes \mathscr{G}^\vee)$, by the projection formula again. Recalling the definition of $\delta_2$, all this gives

$$\delta_2 = 2 - h^0(\mathscr{V}^\vee \otimes \mathscr{G}^\vee) + h^0(\mathscr{G} \otimes \mathscr{G}^\vee).$$

Assume by contradiction that $\delta_2 = 2$, i.e. $h^0(\mathscr{V}^\vee \otimes \mathscr{G}^\vee) = h^0(\mathscr{G} \otimes \mathscr{G}^\vee)$. Suppose for a while that we also have $H^0(L^\vee \otimes \mathscr{G}^\vee) = 0$. Then, applying the functor $\mathrm{Hom}(\_, \mathscr{G}^\vee)$ to (2.6.1) we get that the natural map $\mathrm{Hom}(\mathscr{V}, \mathscr{G}^\vee) \to \mathrm{Hom}(\mathscr{G}^\vee, \mathscr{G}^\vee)$ is an isomorphism because we are assuming that both spaces have the same dimension and the kernel of this map is zero. In particular the identity map on $\mathscr{G}^\vee$ would lift to a morphism $\mathscr{V} \to \mathscr{G}^\vee$, yielding the splitting of (2.6.1). But then we would have a natural inclusion $\mathbf{P}_B(\mathscr{G}^\vee) \subset \mathbf{P}_B(\mathscr{V})$. This is absurd, since restricting $\mathscr{E}$ to $S := \mathbf{P}_B(\mathscr{G}^\vee)$ we would get a vector bundle $\mathscr{E}_S$ with $c_1(\mathscr{E}_S)^2 = c_2(\mathscr{E}_S) = 0$, so that $\mathscr{E}_S$, hence $\mathscr{E}$, could not be ample.

We thus arrive to the conclusion that $H^0(L^\vee \otimes \mathscr{G}^\vee) \neq 0$. Since $\deg(\mathscr{V} \otimes \mathscr{G}) = \deg(L \otimes \mathscr{G})$, as we see tensoring (2.6.1) with $\mathscr{G}$, it follows that $c_1c_2 = \deg(L \otimes \mathscr{G})$, which is positive because $\mathscr{E}$ is ample. Hence [4, Lemma 1.1] implies that $L \otimes \mathscr{G}$ decomposes, so the same is true for $\mathscr{G}$. Write $\mathscr{G} = L_1 \oplus L_2$. Hence $\mathscr{E} = (h \otimes \pi^*L_1) \oplus (h \otimes \pi^*L_2)$, so that each $h \otimes \pi^*L_i$ must be ample. But then an easy calculation gives $0 < c_1(h \otimes \pi^*L_i)^2 \cdot c_1(h \otimes \pi^*L_j) = \deg(L \otimes L_i)$ if $i \neq j$, so that each $L \otimes L_i$ is ample and hence $H^0(L^\vee \otimes L_i^\vee) = 0$. This implies again $H^0(L^\vee \otimes \mathscr{G}^\vee) = 0$, which we proved to be false. □

To say more on case (2) of Proposition 2.4, when $g \geq 2$, we will assume that $\mathscr{E}$ is spanned. Note that this assumption implies condition $(*)$. This leads to a very restricted situation.

Proposition 2.7. *Let $X$ and $\mathscr{E}$ be as in Definition* 1.1, *assume that $\mathscr{E}$ is spanned and let $g \geq 2$. If $\delta_2 = 2$, then the general section of $\mathscr{E}$ vanishes along a smooth hyperelliptic curve of genus $g$ and $\mathscr{E}$ splits on every such curve as $\mathscr{L}^{\oplus 2}$, $\mathscr{L}$ being the hyperelliptic line bundle.*

*Proof.* As $\mathscr{E}$ is spanned, its general section vanishes along a smooth curve $Z$, whose genus is $g \geq 2$ by assumption. We know that $2 = \delta_2 \geq \delta_2' = \Delta(Y, L) = 2$ by Proposition 2.4, because $g = 0$ in case (4) of Theorem 2.1. Then the assertion follows from Proposition 1.10 since $\mathscr{E}$ is spanned. □

An obvious nice pair $(X, \mathscr{E})$ as in the proposition above is given by the following.

*Example* 2.8. Let $p : X \to \mathbf{P}^3$ be a double cover branched along a smooth surface $D \in |\mathcal{O}_{\mathbf{P}^3}(2b)|$ with $b \geq 3$, and let $\mathscr{E} = \mathscr{A}^{\oplus 2}$, where $\mathscr{A} := p^*\mathcal{O}_{\mathbf{P}^3}(1)$. Then $\mathscr{E}$ is ample and spanned and its general section vanishes along a smooth hyperelliptic curve $Z$ of genus $g = b - 1$. Moreover, $\mathscr{E}_Z = \mathscr{L}^{\oplus 2}$, $\mathscr{L} := \mathscr{A}_Z$ being the hyperelliptic line bundle. According to (1.3.2), we have

$$\delta_2 = 2 + 2\mathscr{A}^3 - 2h^0(\mathscr{A}) + h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = 2.$$

Note also that $\delta_1 = 4$ by (1.3.1).

## 3. An approach without condition (*)

We would like to show that $\delta_2 \geq 0$ without the assumption (*). This is in line with the non-negativity of the $\Delta$-genus of an ample line bundle $L$ on a smooth projective variety $X$. In fact this property follows from the inequality

$$\Delta(X, L) > \dim(\mathrm{Bs}|L|) \tag{3.0.1}$$

[2, Theorem 4.2]. This is obvious if $h^0(L) = 0$, while for $h^0(L) > 0$, Fujita uses the rational map $X \dashrightarrow \mathbf{P}^N$ defined by $L$.

Coming back to our ample vector bundle $\mathscr{E}$ of rank 2 on a threefold $X$ we define the base locus of $|\mathscr{E}|$, $\mathrm{Bs}|\mathscr{E}|$ as the locus of points $x \in X$ where $\mathscr{E}$ is not spanned, i.e. where the evaluation homomorphism

$$\mathrm{ev}_x : H^0(\mathscr{E}) \otimes \mathcal{O}_X \to \mathscr{E}_x \tag{3.0.2}$$

fails to be surjective. We put the following

Question 3.1. Let $\mathscr{E}$ be any ample vector bundle of rank 2 on a smooth projective threefold. Is it true that

$$\delta_2 > \dim(\mathrm{Bs}|\mathscr{E}|)? \tag{3.1.1}$$

If $h^0(\mathscr{E}) = 0$, we have $\dim(\mathrm{Bs}|\mathscr{E}|) = 3$. On the other hand, by Definition 1.1 we get $\delta_2 := 2 + c_1c_2 + h^0(\mathscr{E} \otimes \mathscr{E}^\vee) \geq 4$, due to the positivity of $c_1c_2$ [3, Theorem I] and of the last summand. Hence (3.1.1) holds in this case.

If $\mathscr{E}$ is spanned, then obviously $\mathrm{Bs}|\mathscr{E}| = \emptyset$ and condition $(*)$ holds. Therefore (3.1.1) follows from Remark 1.4.

If $h^0(\mathscr{E}) > 0$ and $\mathscr{E}$ is not spanned the first object we need to adapt Fujita's argument to our setting is a rational map

$$\varphi : X \dashrightarrow \mathbf{G}(1, N), \tag{3.1.2}$$

from $X$ to the Grassmannian of lines of $\mathbf{P}(H^0(\mathscr{E})^\vee)$. To have such a map, which is defined via the evaluation homomorphism (3.0.2), we need to assume that $\mathscr{E}$ is generically spanned, so that $\dim(\mathrm{Bs}|\mathscr{E}|) \leq 2$. Assuming this, we can provide a first evidence for (3.1.1) to be true in general.

Proposition 3.2. *Let $X$ be a threefold and let $\mathscr{E}$ be an ample and generically spanned vector bundle of rank* 2 *on $X$. Assume that the image $W$ of the map $\varphi$ in* (3.1.2) *has dimension* 1. *Then*

(i) $\delta_2 \geq 2$ *if* $\dim(\mathrm{Bs}|\mathscr{E}|) = 2$, *and*

(ii) $\delta_2 \geq 1$ *if* $\dim(\mathrm{Bs}|\mathscr{E}|) \leq 1$.

*Proof.* As $\mathscr{E}$ is generically spanned, there is a rational map $\varphi : X \dashrightarrow \mathbf{G}(1, N)$ as in (3.1.2). Observe that $W = \overline{\varphi(X)}$ is nondegenerate in $\mathbf{G}(1, N)$ in the sense that there is no hyperplane in $\mathbf{P}^N$ containing all lines parameterized by $W$.

Note that $B := \mathrm{Bs}|\mathscr{E}|$ has a natural scheme structure given by the inclusion of the ideal sheaf $\mathscr{I}_B \hookrightarrow \mathscr{O}_X$ factoring the natural map $\bigwedge^2 H^0(\mathscr{E}) \otimes (\det \mathscr{E})^\vee \to \mathscr{O}_X$.

Let $\tilde{X}$ be the normalization of the blowing-up of $X$ along the scheme $B$, and let $\sigma : \tilde{X} \to X$ be the composed morphism. Then the pull-back of the ideal sheaf of $B$ is $\sigma^*\mathscr{I}_B = \mathscr{O}_{\tilde{X}}(-E)$, where $E$ is an effective Cartier divisor such that $\sigma(E) = B$ as a set.

Consider the following diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\sigma"] \arrow[d, "\rho"'] & X \arrow[dl, dashed, "\varphi"] \\ \mathbf{G}(1, N). & \end{tikzcd} \tag{3.2.1}$$

Let $\tilde{\mathscr{E}}$ be the pull-back $\rho^*\mathscr{Q}$, where $\mathscr{Q}$ is the universal rank 2 quotient bundle. Since the sheaf homomorphism $H^0(\mathscr{E}) \otimes \mathscr{O}_{\tilde{X}} \to \tilde{\mathscr{E}}$ is surjective, so is $\bigwedge^2 H^0(\mathscr{E}) \otimes \mathscr{O}_{\tilde{X}} \to \det \tilde{\mathscr{E}}$, which is the pull-back via $\sigma$ of $\bigwedge^2 H^0(\mathscr{E}) \otimes \mathscr{O}_X \to \det \mathscr{E} \otimes \mathscr{I}_B$. Hence $\tilde{c}_1 := c_1(\tilde{\mathscr{E}}) = \sigma^*c_1 - E$. Therefore

$$c_1c_2 = \sigma^*c_1\sigma^*c_2 = (\tilde{c}_1 + E)\sigma^*c_2. \tag{3.2.2}$$

Note that $W = \rho(\tilde{X})$ and recall that $\dim W = 1$ by assumption. Then, letting $F$ denote the general fiber of $\rho$, we have $\tilde{c}_1\sigma^*c_2 = \deg W\sigma^*c_2|_F$. As $\dim \sigma(F) = 2$ and $\mathscr{E}$ is ample, it follows that $c_2\sigma(F) > 0$ [3, p. 58], so that $\tilde{c}_1\sigma^*c_2 \geq \deg W$. Similarly we get $E\sigma^*c_2 \geq 0$, with strict inequality if $\dim B = 2$ (note that $\dim B \leq 2$, since $\mathscr{E}$ is generically spanned).

Denote by $\overline{W} \subset \mathbf{P}^N$ the ruled surface corresponding to $W$. Since it is not degenerate because $W$ is not degenerate in $\mathbf{G}(1, N)$, it follows that

$$h^0(\mathscr{E}) \leq h^0(\mathscr{O}_{\overline{W}}(1)) \leq 2 + \deg \overline{W} = 2 + \deg W \tag{3.2.3}$$
$$\leq 2 + \tilde{c}_1\sigma^*c_2 = 2 + c_1c_2 - E\sigma^*c_2,$$

the last equality coming from (3.2.2). As $h^0(\mathscr{E} \otimes \mathscr{E}^\vee) \geq 1$, it follows from the definition of $\delta_2$ that

$$\delta_2 \geq 2 + c_1c_2 - h^0(\mathscr{E}) + h^0(\mathscr{E} \otimes \mathscr{E}^\vee) \geq 3 + c_1c_2 - h^0(\mathscr{E}) \geq 1 + E\sigma^*c_2. \tag{3.2.4}$$

Hence $\delta_2 \geq 1$; moreover $\delta_2 \geq 2$ if $\dim B = 2$. □

*Remark* 3.3. Taking into account (3.2.3) and (3.2.4), we observe that in each case (i) and (ii) of Proposition 3.2 equality implies that $h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = 1$, so that $\mathscr{E}$ is simple, and also that $h^0(\overline{W}, \mathscr{O}_{\overline{W}}(1)) = 2 + \deg \overline{W}$, so that $\overline{W}$ is a rational normal scroll or a cone.

We would like to understand the actual necessity of $\mathscr{E}$ being generically spanned. This seems difficult even in the decomposable case. In fact the generic spannedness of $\mathscr{E} = L \oplus M$ implies that both $h^0(L) > 0$ and $h^0(M) > 0$; otherwise the homomorphism (3.0.2) could not be surjective at the general point. So, one should look at the case $h^0(L) > 0$ and $h^0(M) = 0$, in which, by (1.3.2),

$$\delta_2 = 4 + LM(L + M) - h^0(L) + h^0(L - M).$$

## 4. Final remarks

The question of what happens in dimension $n \neq 3$ is probably in the reader's mind. Here are some comments.

Let $X$ be a smooth projective variety of dimension $n \geq 2$ and let $\mathscr{E}$ be an ample vector bundle of rank 2 on $X$ satisfying condition $(*)$.

For instance, let us look at $\delta_2$. We would like to define $\delta_2 = \Delta(\mathbf{P}_Z(\mathscr{E}_Z), \Lambda)$, with $\Lambda = \mathrm{Tr}_{\mathbf{P}_Z(\mathscr{E}_Z)}|\xi_{\mathscr{E}}|$ where $\xi_{\mathscr{E}}$ is the tautological line bundle of $\mathscr{E}$ on $\mathbf{P}_X(\mathscr{E})$. Set $P := \mathbf{P}_Z(\mathscr{E}_Z)$, $\zeta = (\xi_{\mathscr{E}})_P$ and denote by $\pi$ be the projection $\mathbf{P}_X(\mathscr{E}) \to X$. Then $\dim P = n - 1$,

$$\deg(P, \zeta) = \zeta^{n-1} = (\xi_{\mathscr{E}})^{n-1} \cdot P = (\xi_{\mathscr{E}})^{n-1} \cdot \pi^*Z = (\xi_{\mathscr{E}})^{n-1} \cdot \pi^*c_2(\mathscr{E}) = p(c_1, c_2)c_2,$$

where $p(c_1, c_2)$ is a polynomial in $c_1$ and $c_2$ thanks to the Chern–Wu relation. Then $\delta_2 = n - 1 + p(c_1, c_2)c_2 - (h^0(\mathscr{E}) - h^0(\mathscr{E} \otimes \mathscr{I}_Z))$.

Since $r = 2$, we have the following diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathscr{E}^\vee \arrow[r] & \mathscr{E}\otimes\mathscr{E}^\vee \arrow[rr] \arrow[dr] & & \mathscr{E} \arrow[r] & \mathscr{E}_Z \arrow[r] & 0 \\
& & & \mathscr{E}\otimes\mathscr{I}_Z \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0. & &
\end{tikzcd}$$

It follows that $h^0(\mathscr{E}\otimes\mathscr{I}_Z) = h^0(\mathscr{E}\otimes\mathscr{E}^\vee) - h^0(\mathscr{E}^\vee) + \sigma = h^0(\mathscr{E}\otimes\mathscr{E}^\vee) + \sigma$, with $\sigma \le h^1(\mathscr{E}^\vee) = h^{n-1}(\Omega_X^n \otimes \mathscr{E})$ by Serre duality, which is equal to zero according to the Kodaira–Le Potier vanishing theorem if $n \ge 3$.

So, if $n \ge 3$, we get $\delta_2 = n - 1 + p(c_1, c_2)c_2 - h^0(\mathscr{E}) + h^0(\mathscr{E}\otimes\mathscr{E}^\vee)$.

Here is the expression of $p(c_1, c_2)$ in some instances:

| | |
|---|---|
| $(n = 2)$ | $p(c_1, c_2) = 1$ |
| $n = 3$ | $p(c_1, c_2) = c_1$ |
| $n = 4$ | $p(c_1, c_2) = c_1^2 - c_2$ |
| $n = 5$ | $p(c_1, c_2) = c_1(c_1^2 - 2c_2)$ |
| $n = 6$ | $p(c_1, c_2) = c_1^4 - 3c_1^2c_2 + c_2^2$ |
| $n = 7$ | $p(c_1, c_2) = c_1^5 - 4c_1^3c_2 + 3c_1c_2^2$ |
| $n = 8$ | $p(c_1, c_2) = c_1^6 - 5c_1^4c_2 + 6c_1^2c_2^2 - c_2^3$ |
| $n = 9$ | $p(c_1, c_2) = c_1^7 - 6c_1^5c_2 + 10c_1^3c_2^2 - 4c_1c_2^3$ |

For $n = 2$ we have no control on the term $\sigma$. Mimicking the 3-dimensional case, one could be tempted to define $\delta_2 = 1 + c_2 - h^0(\mathscr{E}) + h^0(\mathscr{E}\otimes\mathscr{E}^\vee)$, while the other natural choice would be $\delta_2 = 1 + c_2 - h^0(\mathscr{E}) + h^0(\mathscr{E}\otimes\mathscr{E}^\vee) + h^1(\mathscr{E}^\vee)$. In either case, for instance when $X = \mathbf{P}^2$ and $\mathscr{E} = T_{\mathbf{P}^2}$ (which is even very ample), we get $\delta_2 = -3, -2$, respectively. This prevents from expecting a reasonable positivity result. This is clearly due to the fact that $Z$ is reducible. On the other hand, the analogue of $\delta_2'$ for surfaces would be $\delta_2' = 1 + c_2(\mathscr{E}) - h^0(\mathscr{E}_Z) = 1 - c_2(\mathscr{E}_Z)$, where $Z$, the zero set of a section of $\mathscr{E}$, according to $(*)$ is a finite set consisting of $c_2(\mathscr{E})$ points. So, even $\delta_2' \le 0$, since the ampleness of $\mathscr{E}$ implies that $c_2(\mathscr{E}) \ge 1$. Moreover, we get equality if and only if $c_2(\mathscr{E}) = 1$. Note that when $\mathscr{E}$ is spanned, this happens if and only if $(X, \mathscr{E}) = (\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(1)^{\oplus 2})$ by [9].

If $X$ is a curve, the general section $s \in \Gamma(\mathscr{E})$ has no zeroes, i.e. $Z = (s)_0 = \emptyset$. Therefore $\delta_2'$ is not defined; however, when $X$ is a curve, we can define $\Delta(X, \mathscr{E}) = \delta_1(X, \mathscr{E})$, and this is equal to the definition given in [8].

On the other hand, suppose that $\dim X = n \ge 3$ is odd, say $n = 2h + 1$, and $\mathscr{E}$ admits $h$ sections $s_1, \dots, s_h \in \Gamma(\mathscr{E})$ whose zero loci $Z_i := (s_i)_0$ satisfy $\dim \bigcap_{i=1}^t Z_i = n - 2t$ and $Y_t := \bigcap_{i=1}^t Z_i$ is smooth for every $t = 1, \dots, h$. Letting

$P_t = \mathbf{P}_{Y_t}(\mathscr{E}_{Y_t})$ and $\zeta_t = (\xi_{\mathscr{E}})_{P_t}$, then we can define

$$\delta'_{t+1} := \Delta(P_t, \zeta_t), \quad t = 1, \dots, h.$$

When $\dim X = 5$, this says that $\mathscr{E}$ admits 2 sections $s_1, s_2 \in \Gamma(\mathscr{E})$ whose zero loci $Z_i := (s_i)_0$ satisfy $\dim Z_1 = 5 - 2 = 3$, $\dim Z_1 \cap Z_2 = 5 - 4 = 1$ and $Y_1 := Z_1$ $Y_2 := Z_1 \cap Z_2$ are smooth; then we can define

$$\delta'_2 := \Delta(P_1, \zeta_1), \quad \text{and} \quad \delta'_3 := \Delta(P_2, \zeta_2).$$

Note that $\dim P_1 = 4$, while $\dim P_2 = 2$.

*Acknowledgements.* We would like to thank Universidad Complutense de Madrid, Università degli Studi di Genova and Università degli Studi di Milano for support received during the preparation of this paper and for making this collaboration possible.

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