Y. XU
KODAI MATH. J.
36 (2013), 69–76

ANOTHER IMPROVEMENT OF MONTEL'S CRITERION*

YAN XU

Abstract

Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbf{C}$, let ψ_1, ψ_2 and ψ_3 be three meromorphic functions such that $\psi_i(z) \not\equiv \psi_j(z)$ $(i \neq j)$ in D, one of which may be ∞ identically, and let l_1, l_2 and l_3 be positive integers or ∞ with $1/l_1 + 1/l_2 + 1/l_3 < 1$. Suppose that, for each $f \in \mathscr{F}$ and $z \in D$, (1) all zeros of $f - \psi_i$ have multiplicity at least l_i for $i = 1, 2, 3$; (2) $f(z_0) \neq \psi_i(z_0)$ if there exist $i, j \in \{1, 2, 3\}$ $(i \neq j)$ and $z_0 \in D$ such that $\psi_i(z_0) = \psi_j(z_0)$. Then $\mathscr{F}$ is normal in D. This improves and generalizes Montel's criterion.

1. Introduction

Let D be a domain in the complex plane $\mathbf{C}$, and $\mathscr{F}$ be a family of meromorphic functions defined on D. $\mathscr{F}$ is said to be normal on D, in the sense of Montel, if for any sequence $\{f_n\} \in \mathscr{F}$ there exists a subsequence $\{f_{n_j}\}$, such that $\{f_{n_j}\}$ converges spherically locally uniformly on D, to a meromorphic function or ∞ (see [3, 6, 10]).

The most celebrated theorem in the theory of normal families is the following criterion of Montel [5] (cf [3, 6, 10]), which is the local counterpart of Picard theorem and plays an important role in complex dynamics.

THEOREM A. *Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbf{C}$, and let a_1, a_2 and a_3 be three distinct complex numbers in $\mathbf{C}$. If, for each $f \in \mathscr{F}$, $f(z) \neq a_i$ $(i = 1, 2, 3)$ in D, then $\mathscr{F}$ is normal in D.*

Montel's criterion has undergone various extensions and improvements (see [1, 2, 4, 6, 7, 8, 10], etc.). The next are two extensions of Montel's criterion (see [6, 10]).

THEOREM B. *Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbf{C}$, a_1, a_2 and a_3 be three distinct complex numbers in $\mathbf{C}$, and let l_1, l_2*

2000 *Mathematics Subject Classification*. 30D45.
Key words and phrases. Meromorphic function; Riemann-Hurwitz formula; normal family; Montel's criterion.
*Supported by NNSF of China (Grant No. 11171045).
Received September 27, 2011; revised June 7, 2012.

and l_3 *be positive integers or* ∞ *with* $1/l_1 + 1/l_2 + 1/l_3 < 1$. *If, for each* $f \in \mathscr{F}$, *all zeros of* $f - a_i$ *have multiplicity at least* l_i *for* $i = 1, 2, 3$, *then* $\mathscr{F}$ *is normal in* D.

Theorem C. *Let* $\mathscr{F}$ *be a family of meromorphic functions defined in a domain* $D \subset \mathbf{C}$, *and let* $a_1(z)$, $a_2(z)$ *and* $a_3(z)$ *be three meromorphic functions in* D *such that* $a_i(z) \neq a_j(z)$ $(1 \leq i < j \leq 3)$. *If, for each* $f \in \mathscr{F}$, $f(z) \neq a_i(z)$ $(i = 1, 2, 3)$ *in* D, *then* $\mathscr{F}$ *is normal in* D.

Remark 1. Bonk, Hinkkanen and Martin [1] proved that Montel's criterion is still valid if a, b, c are replaced by three arbitrary continuous functions avoiding each other in D.

A natural problem arise: *What can we say if three distinct complex numbers* a_1, a_2 *and* a_3 *in Theorem* B *are replaced by three meromorphic functions* $\psi_1(z)$, $\psi_2(z)$ *and* $\psi_3(z)$ *such that* $\psi_i(z) \not\equiv \psi_j(z)$ $(1 \leq i < j \leq 3)$ *in* D?

In this paper, we prove the following result.

Theorem 1. *Let* $\mathscr{F}$ *be a family of meromorphic functions defined in a domain* $D \subset \mathbf{C}$, *let* ψ_1, ψ_2 *and* ψ_3 *be three meromorphic functions such that* $\psi_i(z) \not\equiv \psi_j(z)$ $(1 \leq i < j \leq 3)$ *in* D, *one of which may be* ∞ *identically, and let* l_1, l_2 *and* l_3 *be positive integers or* ∞ *with* $1/l_1 + 1/l_2 + 1/l_3 < 1$. *Suppose that, for each* $f \in \mathscr{F}$ *and* $z \in D$, (1) *all zeros of* $f - \psi_i$ *have multiplicity at least* l_i *for* $i = 1, 2, 3$; (2) $f(z_0) \neq \psi_i(z_0)$ *if there exist* $i, j \in \{1, 2, 3\}$ $(i \neq j)$ *and* $z_0 \in D$ *such that* $\psi_i(z_0) = \psi_j(z_0)$. *Then* $\mathscr{F}$ *is normal in* D.

Remark 2. Condition (2) cannot be omitted in Theorem 1, as is shown by the following examples.

Example 1. Let $D = \{z : |z| < 1\}$, $\psi_1(z) = 0$, $\psi_2(z) = \infty$ and $\psi_3(z) = z^k$, where $k \geq 3$ is a positive integer, and

$$\mathscr{F} = \{f_n(z) = nz^3, n = 2, 3, \ldots, z \in D\}.$$

Since $f_n(z) - \psi_3(z) = z^3(n - z^{k-3})$, $l_1 = 3$, $l_2 = \infty$ and $l_3 = 3$, then condition (1) in Theorem 1 is satisfied. But $\mathscr{F}$ is not normal in D. Note that $f_n(0) = \psi_1(0) = \psi_3(0)$.

Example 2. Let $D = \{z : |z| < 1\}$, $\psi_i(z) = i/z^k$ $(i = 1, 2, 3)$, where $k \geq 1$ is a positive integer, and

$$\mathscr{F} = \left\{ f_n(z) = \frac{1}{nz}, n = 2, 3, \ldots, z \in D \right\}.$$

Since

$$f_n(z) - \psi_i(z) = \frac{z^{k-1} - i \cdot n}{nz^k} \neq 0$$

in D, $l_1 = l_2 = l_3 = \infty$, and thus condition (1) in Theorem 1 is satisfied. But $\mathscr{F}$ is not normal in D. Note that $f_n(0) = \psi_1(0) = \psi_2(0) = \psi_3(0)$.

Letting $l_1 = l_2 = l_3 = \infty$ in Theorem 1, we obtain

COROLLARY. *Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbf{C}$, and let $\psi_1(z)$, $\psi_2(z)$ and $\psi_3(z)$ be three meromorphic functions such that $\psi_i(z) \not\equiv \psi_j(z)$ $(1 \le i < j \le 3)$ in D, one of which may be ∞ identically. If, for each $f \in \mathscr{F}$, $f(z) \neq \psi_i(z)$ $(i = 1, 2, 3)$ in D, then $\mathscr{F}$ is normal in D.*

2. Some Lemmas

The following is one local version of Zalcman's lemma due to Xue and Pang [9].

LEMMA 1. *Let $\mathscr{F}$ be a family of functions meromorphic in a domain D such that $f \neq 0$ for each $f \in \mathscr{F}$. If $\mathscr{F}$ is not normal at $z_0 \in D$, then, for each $\alpha \ge 0$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathscr{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^\alpha} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbf{C}$.

LEMMA 2. *Let l_1, l_2 and l_3 be positive integers or ∞ with $1/l_1 + 1/l_2 + 1/l_3 < 1$. Then there does not exist nonconstant rational function f such that all zeros of f have multiplicity at least l_1, all poles of f have multiplicity at least l_2, and all zeros of $f - c$ have multiplicity at least l_3, except for at most one point, where c is nonzero constant.*

Proof. If no exception points exist, then the conclusion follows directly from Nevanlinna's second fundamental theorem since $1/l_1 + 1/l_2 + 1/l_3 < 1$.

Next suppose that there is a nonconstant rational function f satisfying the conditions in Lemma 2, with one exception point. Without loss of generality, we may assume that all zeros of f have multiplicity at least l_1, all poles of f have multiplicity at least l_2, and all 1-points f have multiplicity at least l_3 in $\mathbf{C}$, with the exception of one 1-point $a \in \mathbf{C}$, of multiplicity $s < l_3$. Note that a zero of f of multiplicity N is then a zero of f' of multiplicity $N - 1 (\ge N(1 - 1/l_1))$.

We may assume further that infinity is either a zero of f of multiplicity $r < l_1$, a pole of f of multiplicity $r < l_2$, or a 1-point of f of multiplicity $r < l_3$, because if not then the function $g(z) = f(1/(a - z))$ is a rational function for which all zeros, poles and 1-points in $\mathbf{C}$ have multiplicity at least l_1, l_2, l_3 respectively, and Nevanlinna's second fundamental theorem can be applied to g.

Let q be the degree of f, where the degree of a rational function $f(z)$ is defined as $\max\{\deg P(z), \deg Q(z)\}$, if $f(z) = P(z)/Q(z)$, and $P(z)$, $Q(z)$ are two coprime polynomials. Then Riemann-Hurwitz formula implies that f has $2q-2$ critical points in the extended complex plane, counting multiplicities.

Let f have m zeros, n poles and p 1-points in $\mathbf{C}$, counting multiplicities. Thus the number of critical points, counting multiplicities, of f in the extended complex plane is at least

$$m\left(1-\frac{1}{l_1}\right)+n\left(1-\frac{1}{l_2}\right)+(p-s)\left(1-\frac{1}{l_3}\right)+s-1+r-1.$$

Hence

$$2q-2 \geq m\left(1-\frac{1}{l_1}\right)+n\left(1-\frac{1}{l_2}\right)+(p-s)\left(1-\frac{1}{l_3}\right)+s-1+r-1,$$

that is,

$$m+n+p+r \leq 2q+\frac{m}{l_1}+\frac{n}{l_2}+\frac{p-s}{l_3}. \tag{1}$$

On the other hand, it is not difficult to see that if $m > n$ then $m = n+r = p = q$; if $m < n$ then $m+r = n = p = q$; and finally if $m = n$ then $m = n = p+r = q$. Thus, in all cases, we have

$$m+n+p+r = 3q. \tag{2}$$

From (1) and (2), we obtain

$$3q \leq 2q+\frac{m}{l_1}+\frac{n}{l_2}+\frac{p-s}{l_3} \leq 2q+\left(\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}\right)q,$$

so that we arrive at a contradiction since $1/l_1+1/l_2+1/l_3 < 1$. Lemma 2 is thus proved. □

3. Proof of Theorem 1

Since normality is a local property, it is enough to show that $\mathscr{F}$ is normal at each $z_0 \in D$. We distinguish three cases.

Case 1. $\psi_1(z_0)$, $\psi_2(z_0)$, $\psi_3(z_0)$ are distinct.

Suppose that $\mathscr{F}$ is not normal at z_0. By Lemma 1, there exist functions $f_n \in \mathscr{F}$, points $z_n \to z_0$ and positive numbers $\rho_n \to 0$, such that

$$g_n(\zeta) = f_n(z_n+\rho_n\zeta) \to g(\zeta) \tag{3}$$

converges spherically uniformly on compact subsets of $\mathbf{C}$, where $g(\zeta)$ is a non-constant meromorphic function on $\mathbf{C}$. We from (3) have

$$f_n(z_n+\rho_n\zeta)-\psi_i(z_n+\rho_n\zeta) \to g(\zeta)-\psi_i(z_0),$$

for $i = 1, 2, 3$. By Hurwitz's theorem, all zeros of $g(\zeta) - \psi_i(z_0)$ have multiplicity at least l_i for $i = 1, 2, 3$.

In view of $1/l_1 + 1/l_2 + 1/l_3 < 1$, Nevanlinna's first and second fundamental theorems imply that $g(\zeta)$ is a constant, a contradiction.

Case 2. Exactly two of $\psi_1(z_0)$, $\psi_2(z_0)$, $\psi_3(z_0)$ are equal.

Without loss of generality, we assume that $\psi_1(z_0) = \psi_2(z_0)$, and $\psi_3(z_0) \neq \psi_1(z_0)$ or $\psi_2(z_0)$. Then, $f(z_0) \neq \psi_1(z_0)$ for each $f \in \mathscr{F}$. Now we consider two subcases.

Case 2.1. $\psi_1(z_0)$ and $\psi_2(z_0)$ are finite.

There exists $r > 0$ such that $\psi_i(z) \neq \psi_j(z)$ $(1 \leq i < j \leq 3)$ and $\psi_1(z)$, $\psi_2(z)$ are holomorphic in $D'(z_0, r) = \{z : 0 < |z - z_0| < r\} \subset D$, and $\psi_3(z) \neq \psi_1(z), \psi_2(z)$ in $D(z_0, r) = \{z : |z - z_0| < r\}$.

Then $\mathscr{F}$ is normal in $D'(z_0, r)$ by Case 1.

Set

$$\mathscr{G} = \left\{ g(z) = \frac{f(z) - \psi_1(z)}{\psi_2(z) - \psi_1(z)} : f \in \mathscr{F} \right\}. \tag{4}$$

Note that $\psi_2(z_0) - \psi_1(z_0) = 0$ and $f(z_0) - \psi_1(z_0) \neq 0$ for all $f \in \mathscr{F}$. Thus, for each $g \in \mathscr{G}$, $g(z_0) = \infty$.

Obviously, all zeros of g in $D(z_0, r)$ have multiplicity at least l_1. Since

$$g - 1 = \frac{f - \psi_2}{\psi_2 - \psi_1},$$

$$\frac{1}{g} - \frac{\psi_2 - \psi_1}{\psi_3 - \psi_1} = \frac{(\psi_2 - \psi_1)(\psi_3 - f)}{(\psi_3 - \psi_1)(f - \psi_1)},$$

all zeros of $g - 1$ in $D(z_0, r)$ have multiplicity at least l_2, and all zeros of $1/g - (\psi_2 - \psi_1)/(\psi_3 - \psi_1)$ in $D(z_0, r)$ have multiplicity at least l_3 with the possible exception at z_0.

We first prove that $\mathscr{G}$ is normal at z_0. Suppose not; then by Lemma 1, there exist functions $g_n \in \mathscr{G}$, points $z_n \to z_0$ and positive numbers $\rho_n \to 0$, such that

$$G_n(\zeta) = g_n(z_n + \rho_n \zeta) \to G(\zeta) \tag{5}$$

converges spherically uniformly on compact subsets of $\mathbf{C}$, where $G(\zeta)$ is a non-constant meromorphic function on $\mathbf{C}$.

By Hurwitz's theorem, all zeros of G have multiplicity at least l_1, and all zeros of $G - 1$ have multiplicity at least l_2.

We claim: all poles of G have multiplicity at least l_3, except possibly one pole.

Indeed, for the case $\psi_3 \equiv \infty$, we know that all poles of g in $D(z_0, r)$ have multiplicity at least l_3, then the conclusion follows from Hurwitz's theorem.

For $\psi_3 \not\equiv \infty$, we from (5) have

$$\frac{1}{G_n(\zeta)} - \frac{\psi_2(z_n+\rho_n\zeta) - \psi_1(z_n+\rho_n\zeta)}{\psi_3(z_n+\rho_n\zeta) - \psi_1(z_n+\rho_n\zeta)} = \frac{1}{g_n(z_n+\rho_n\zeta)} - \frac{\psi_2(z_n+\rho_n\zeta) - \psi_1(z_n+\rho_n\zeta)}{\psi_3(z_n+\rho_n\zeta) - \psi_1(z_n+\rho_n\zeta)} \to \frac{1}{G}. \tag{6}$$

Note that all zeros of $1/g_n - (\psi_2 - \psi_1)/(\psi_3 - \psi_1)$ in $D(z_0, r)$ have multiplicity at least l_3 with the possible exception at z_0.

If $(z_n - z_0)/\rho_n \to \infty$, the zero of the right part of (6) corresponding to that of $1/g_n - (\psi_2 - \psi_1)/(\psi_3 - \psi_1)$ at z_0 drifts off to infinity since $1/G_n(-(z_n - z_0)/\rho_n) = 1/g_n(z_0)$. It follows that all zeros of $1/G$ have multiplicity at least l_3. Thus, all poles of G have have multiplicity at least l_3 in $\mathbf{C}$.

If $(z_n - z_0)/\rho_n \not\to \infty$, taking a subsequence and renumbering, we assume that $(z_n - z_0)/\rho_n \to \alpha$ (a finite complex number). By (6), we have

$$\begin{aligned} &\frac{1}{g_n(\rho_n\zeta + z_0)} - \frac{\psi_2(\rho_n\zeta + z_0) - \psi_1(\rho_n\zeta + z_0)}{\psi_3(\rho_n\zeta + z_0) - \psi_1(\rho_n\zeta + z_0)} \\ &= \frac{1}{G_n(\zeta - (z_n - z_0)/\rho_n)} \\ &\quad - \frac{\psi_2(z_n + \rho_n(\zeta - (z_n - z_0)/\rho_n)) - \psi_1(z_n + \rho_n(\zeta - (z_n - z_0)/\rho_n))}{\psi_3(z_n + \rho_n(\zeta - (z_n - z_0)/\rho_n)) - \psi_1(z_n + \rho_n(\zeta - (z_n - z_0)/\rho_n))} \\ &\to \frac{1}{G(\zeta - \alpha)}. \end{aligned}$$

Then all zeros of $1/G$ have multiplicity at least l_3, except the zero at $-\alpha$, and hence all poles of G have have multiplicity at least l_3, except the pole at $-\alpha$.

Therefore, by Nevanlinna's first and second fundamental theorems, $G(\zeta)$ is a rational function. However, such nonconstant rational function does not exist by Lemma 2, a contradiction. We thus have proved that $\mathscr{G}$ is normal at z_0.

Now we turn to prove that $\mathscr{F}$ is normal at z_0. Since $\mathscr{G}$ is normal at z_0, then the family $\mathscr{G}$ is equicontinuous at z_0 with respect to the spherical distance. On the other hand, $g(z_0) = \infty$ for each $g \in \mathscr{G}$. Thus, there exists $0 < r_1 < r$ such that $|g(z)| \geq 1$ for all $g \in \mathscr{G}$ and $z \in D(z_0, r_1)$. It follows that $f(z) - \psi_1 \neq 0$ for all $f \in \mathscr{F}$ and $z \in D(z_0, r_1)$, and then the family $\mathscr{F}_1 = \{1/(f - \psi_1) : f \in \mathscr{F}\}$ is holomorphic in $D(z_0, r_1)$. Suppose that $\mathscr{F}$ is not normal at z_0; then $\mathscr{F}_1$ is normal in $D'(z_0, r_1)$, but not normal at z_0. So, there exists a sequence $\{1/(f_n - \psi_1)\} \subset \mathscr{F}_1$ which converges locally uniformly in $D'(z_0, r_1)$, but none of whose subsequences converges uniformly in a neighborhood of z_0. The maximum modulus principle implies that $1/(f_n - \psi_1) \to \infty$ on compact subsets in $D'(z_0, r_1)$, and hence $f_n - \psi_1 \to 0$ uniformly on compact subsets of $D'(z_0, r_1)$. Note that $g_n = (f_n(z) - \psi_1(z))/(\psi_2(z) - \psi_1(z))$, we see that $g_n \to 0$ uniformly on compact subsets of $D'(z_0, r_1)$. But we already prove that $|g_n(z)| \geq 1$ for $z \in D(z_0, r_1)$ in the above, a contradiction.

Case 2.2. $\psi_1(z_0) = \psi_2(z_0) = \infty$.

There exists $r > 0$ such that $\psi_1(z)$, $\psi_2(z)$ are holomorphic in $D'(z_0, r) \subset D$, and $\psi_i(z) \neq 0$ $(i = 1, 2)$, $\psi_3(z) \neq \infty$ in $D(z_0, r)$.

Set $\varphi_i = 1/\psi_i$ $(1 \leq i \leq 3)$ and $g = 1/f$, where $f \in \mathscr{F}$. Then $\varphi_1(z_0) = \varphi_2(z_0) = 0$, $\varphi_3(z_0) \neq 0$.

Noting that $f(z_0) \neq \psi_1(z_0)$ (or $\psi_2(z_0)$), it is easy to see that all zeros of $g - \varphi_i = (\psi_i - f)/(f\psi_i)$ have multiplicity at least l_i $(1 \leq i \leq 3)$ in $D(z_0, r)$. As in Case 2.1, we can prove that $\mathscr{F}_2 = \{1/f : f \in \mathscr{F}\}$ is normal at z_0, and then $\mathscr{F}$ is normal at z_0.

Case 3. $\psi_1(z_0) = \psi_2(z_0) = \psi_3(z_0)$.

Then $f(z_0) \neq \psi_i(z_0)$ $(1 \leq i \leq 3)$ for $f \in \mathscr{F}$. Similarly, we divide two subcases.

Case 3.1. $\psi_1(z_0)$, $\psi_2(z_0)$ and $\psi_3(z_0)$ are finite.

There exists $r > 0$ such that $\psi_i(z) \neq \psi_j(z)$ $(1 \leq i < j \leq 3)$ in $D'(z_0, r) \subset D$, and $\psi_i(z)$ $(1 \leq i \leq 3)$ is holomorphic in $D(z_0, r)$.

By Case 1, we know that $\mathscr{F}$ is normal in $D'(z_0, r)$.

Let $\mathscr{G}$ be defined as (4). Then $g(z_0) = \infty$ for each $g \in \mathscr{G}$.

Letting $\omega_1 = 0$, $\omega_2 = 1$ and

$$\omega_3 = \frac{\psi_3 - \psi_1}{\psi_2 - \psi_1},$$

as in Case 2.1, all zeros of $g - \omega_1$ in $D(z_0, r)$ have multiplicity at least l_1, and all zeros of $g - \omega_2$ in $D(z_0, r)$ have multiplicity at least l_2.

We have

$$g - \omega_3 = \frac{f - \psi_3}{\psi_2 - \psi_1}.$$

Since all zeros of $f - \psi_3$ have multiplicity at least l_3, $\psi_2 - \psi_1$ is holomorphic in $D(z_0, r)$, and $f - \psi_3$ and $\psi_2 - \psi_1$ have no common zero in $D(z_0, r)$, we conclude that all zeros of $g - \omega_3$ have multiplicity at least l_3 in $D(z_0, r)$. Moreover, $g(z_0) = \infty$ whatever $\omega_3(z_0) = 0$ or 1.

So, by Case 2, $\mathscr{G}$ is normal at z_0.

Hence, by using the same argument as in the latter part of Case 2.1, we can prove that $\mathscr{F}$ is also normal at z_0.

Case 3.2. $\psi_1(z_0) = \psi_2(z_0) = \psi_3(z_0) = \infty$.

Set $\varphi_i = 1/\psi_i$ $(1 \leq i \leq 3)$. Then $\varphi_1(z_0) = \varphi_2(z_0) = \varphi_3(z_0) = 0$. Arguing as in Case 3.1, we can show that $\mathscr{F}_2 = \{1/f : f \in \mathscr{F}\}$ is normal at z_0, and hence $\mathscr{F}$ is normal at z_0.

The proof of Theorem 1 is completed. □

Acknowledgement. We would like to thank the referee for his/her valuable comments and suggestions made to this paper.

REFERENCES

[1] D. BARGMANN, M. BONK, A. HINKKANEN AND G. J. MARTIN, Families of meromorphic functions avoiding continuous functions, J. Anal. Math. **79** (1999), 379–387.
[2] J. M. CHANG, M. L. FANG AND L. ZALCMAN, Composite meromorphic functions and normal families, Proc. Roy. Soc. Edinburgh, Ser. A. **139** (2009), 57–72.
[3] W. K. HAYMAN, Meromorphic functions, Clarendon Press, Oxford, 1964.
[4] B. Q. LI, A joint theorem generalizing the criteria of Montel and Miranda for normal families, Proc. Amer. Math. Soc. **132** (2004), 2639–2646.
[5] P. MONTEL, Leçons sur les familles normales de fonctions analytiques et leurs applications, Gauthier-Villars, Paris, 1927.
[6] J. SCHIFF, Normal families, Springer-Verlag, New York/Berlin, 1993.
[7] Y. XU, On Montel's theorem and Yang's problem, J. Math. Anal. Appl. **305** (2005), 743–751.
[8] Y. XU AND H. L. QIU, An improvement of Montel's criteria, J. Math. Anal. Appl. **343** (2008), 1075–1079.
[9] G. F. XUE AND X. C. PANG, A criterion for normality of a family of meromorphic functions, J. East China Norm. Univ. Natur. Sci. Ed. **2** (1998), 15–22.
[10] L. YANG, Value distribution theory, Springer-Verlag & Science Press, Berlin, 1993.

Yan Xu
INSTITUTE OF MATHEMATICS
SCHOOL OF MATHEMATICS
NANJING NORMAL UNIVERSITY
NANJING 210046
P.R. CHINA
E-mail: xuyan@njnu.edu.cn