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KODAI MATH. J.
35 (2012), 642–651

ON THE LINEAR INDEPENDENCE OF THE SET OF DIRICHLET EXPONENTS

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Abstract

Given $k \geq 2$ let $\alpha_1, \dots, \alpha_k$ be transcendental numbers such that $\alpha_1, \dots, \alpha_{k-1}$ are algebraically independent over $\mathbf{Q}$ and $\alpha_k \in \mathbf{Q}(\alpha_1, \dots, \alpha_{k-1})$, but $\alpha_k \neq (a\alpha_i + c)/b$ for some $i \in \{1, \dots, k-1\}$ and some $a, b \in \mathbf{N}$, $c \in \mathbf{Z}$ satisfying $\gcd(a, b) = 1$. We prove that then there exists a nonnegative integer q such that the set of so-called Dirichlet exponents $\log(n + \alpha_j)$, where n runs through the set of all nonnegative integers for $j = 1, \dots, k-1$ and $n = q, q+1, q+2, \dots$ for $j = k$, is linearly independent over $\mathbf{Q}$. As an application we obtain a joint universality theorem for corresponding Hurwitz zeta functions $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_k)$ in the strip $\{s \in \mathbf{C} : 1/2 < \Re(s) < 1\}$. In our approach we follow a recent result of Mishou who analyzed the case $k = 2$.

1. Introduction

For any given complex number $\alpha \notin \{0, -1, -2, -3, \dots\}$ we consider the set

$$\mathscr{D}(\alpha) := \{\log \alpha, \log(1 + \alpha), \log(2 + \alpha), \dots\},$$

where log stands for the principal branch of the natural logarithm. The set $\mathscr{D}(\alpha)$ is known as the set of *Dirichlet exponents* of the Hurwitz zeta function

$$\zeta(s, \alpha) := \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s} = \sum_{n=0}^{\infty} e^{-s \log(n+\alpha)},$$

where α is a real number in the interval $(0, 1)$. More generally, for each integer $q \geq 0$ let us denote

$$\mathscr{D}_q(\alpha) := \{\log(q + \alpha), \log(q + 1 + \alpha), \log(q + 2 + \alpha), \dots\},$$

so that $\mathscr{D}_0(\alpha) = \mathscr{D}(\alpha)$.

Recall that a (finite or infinite) set of complex numbers V is *linearly dependent* over $\mathbf{Q}$ if there exist some $m \in \mathbf{N}$, distinct $v_1, \dots, v_m \in V$ and nonzero

2000 *Mathematics Subject Classification*. 11M35, 11J72.

Key words and phrases. Hurwitz zeta function, Dirichlet exponents, universality, linear independence.

Received March 6, 2012; revised May 16, 2012.

$r_1, \dots, r_m \in \mathbf{Q}$ such that $\sum_{j=1}^{m} r_j v_j = 0$ and *linearly independent* otherwise. Obviously, if α is a transcendental number then the set $\mathscr{D}(\alpha)$ is linearly independent over $\mathbf{Q}$. The set $\mathscr{D}(\alpha)$ for algebraic α have been studied by Cassels [3] (see also [4] and [5]). The question if there is an algebraic number α for which the set $\mathscr{D}(\alpha)$ is linearly independent over $\mathbf{Q}$ is still open (see [4], [5] and also [7], [12]). A finite set of distinct complex numbers $v_1, \dots, v_m$ is *algebraically dependent* over $\mathbf{Q}$ if there is a nonzero polynomial $P(z_1, \dots, z_m) \in \mathbf{Q}[z_1, \dots, z_m]$ such that $P(v_1, \dots, v_m) = 0$ and *algebraically independent* otherwise.

The main result of this note is the following:

Theorem 1. *Let $k \geq 2$ be an integer and let $\alpha_1, \dots, \alpha_{k-1}, \alpha_k$ be some transcendental numbers. Suppose that the numbers $\alpha_1, \dots, \alpha_{k-1}$ are algebraically independent over $\mathbf{Q}$ and $\alpha_k \in \mathbf{Q}(\alpha_1, \dots, \alpha_{k-1})$, and suppose for each $i = 1, \dots, k-1$ we have $\alpha_k \neq (a\alpha_i + c)/b$ for $a, b \in \mathbf{N}$, $c \in \mathbf{Z}$ satisfying $\gcd(a,b) = 1$. Then there is an integer $q \geq 0$ such that set of Dirichlet exponents*

$$\mathscr{D}(\alpha_1) \cup \cdots \cup \mathscr{D}(\alpha_{k-1}) \cup \mathscr{D}_q(\alpha_k)$$

is linearly independent over $\mathbf{Q}$.

Following the result of Nesterenko [15], the numbers π and e^{π} are algebraically independent over $\mathbf{Q}$, so Theorem 1 can be applied to the numbers

$$\alpha_1 := \pi = 3.14159\dots, \quad \alpha_2 := e^{\pi} = 23.14069\dots,$$

$$\alpha_3 := \alpha_1^2 + \alpha_2 = \pi^2 + e^{\pi} = 33.01029\dots.$$

Note that the condition $\alpha_k \neq (a\alpha_i + c)/b$ for integers $a > 0$, $b > 0$ and c satisfying $\gcd(a,b) = 1$ cannot be removed from Theorem 1. Indeed, if $\alpha_k = (a\alpha_i + c)/b$ with some $i \in \{1, \dots, k-1\}$ and a, b, c as above then there exists $d \in \mathbf{N}$ for which $u := (bd + c)/a$ is a positive integer. Thus for each $N \in \mathbf{N}$ we have the identity

$$\frac{\alpha_k + d + aN}{\alpha_k + d + a(N-1)} = \frac{a\alpha_i + c + b(d + aN)}{a\alpha_i + c + b(d + aN - a)} = \frac{\alpha_i + u + bN}{\alpha_i + u + b(N-1)}.$$

Consequently, the four logarithms $\log(\alpha_k + d + aN)$, $\log(\alpha_k + d + a(N-1))$, $\log(\alpha_i + u + bN)$, $\log(\alpha_i + u + b(N-1))$ are linearly dependent over $\mathbf{Q}$, and hence the set $\mathscr{D}_q(\alpha_i) \cup \mathscr{D}_q(\alpha_k)$ is linearly dependent over $\mathbf{Q}$ for any $q \in \mathbf{N}$.

As an application of Theorem 1 we shall prove the following joint universality theorem for Hurwitz zeta functions. (Throughout, $\mu(A)$ stands for the Lebesgue measure of the set $A \subseteq \mathbf{R}$.)

Theorem 2. *Let $\alpha_1, \alpha_2, \dots, \alpha_k$, $k \geq 2$, be real transcendental numbers in the interval $(0,1)$ such that for some integers $q_1, q_2, \dots, q_k \geq 0$ the set of Dirichlet exponents*

$$\mathscr{D}_{q_1}(\alpha_1) \cup \mathscr{D}_{q_2}(\alpha_2) \cup \cdots \cup \mathscr{D}_{q_k}(\alpha_k)$$

is linearly independent over $\mathbf{Q}$. *For each* j *in the range* $1 \le j \le k$ *let* K_j *be a compact subset of the strip* $\{s \in \mathbf{C} : 1/2 < \Re(s) < 1\}$ *with connected complement and let* $f_j(s)$ *be a continuous function on* K_j *which is analytic in the interior of* K_j. *Then for any* $\varepsilon > 0$ *we have*

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{1\le j\le k} \max_{s\in K_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon\right\} > 0. \tag{1}$$

The subject of "universality" for Dirichlet L-functions started with the paper of Voronin [16], where he proved that for every positive number ε and every continuous non-vanishing function $f(s)$ in the disc $|s| \le r$, where $0 < r < 1/4$, which is analytic in $|s| < r$ there exists a number $\tau = \tau(\varepsilon)$ for which $\max_{|s|\le r} |\zeta(s + 3/4 + i\tau) - f(s)| < \varepsilon$. So certain shifts of zeta function are arbitrarily close to every analytic function. Later, this result have been extended to other L-functions and it was shown that the set of those τ for which the shift of the L-function by $i\tau$ approximates $f(s)$ has positive density; see, e.g., [8], [9] for some references on this. In particular, for the Hurwitz zeta function $\zeta(s,\alpha)$ it was shown that if $\alpha \in (0,1/2) \cup (1/2,1)$ is either rational or transcendental number then for any function $f(s)$ which is continuous in a compact set $K \subset \{s \in \mathbf{C} : 1/2 < \Re(s) < 1\}$ with connected complement and analytic in the interior of K we have

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0$$

for any given $\varepsilon > 0$ (see [1], [6]).

Later, certain joint universality theorems when instead of one function f we have several analytic functions $f_1, \dots, f_k$ and approximate them with some shifts of $\zeta(s,\alpha_j)$, $j = 1,\dots,k$, were obtained in [2], [11], etc. In particular, the joint universality theorem which asserts the conclusion (1) of Theorem 1 under assumption that all k transcendental numbers $\alpha_1,\dots,\alpha_k$ are algebraically independent follows from the results of Nakamura in [14]. Laurinčikas proved the same statement under weaker assumption that the set of Dirichlet exponents $\mathscr{D}(\alpha_1) \cup \cdots \cup \mathscr{D}(\alpha_k)$ is linearly independent over $\mathbf{Q}$ (see [10]). This corresponds to the case $q_1 = \cdots = q_k = 0$ in Theorem 2.

The above mentioned result of Nakamura covers the case when the *transcendence degree* $\operatorname{trdeg}(\mathbf{Q}(\alpha_1,\dots,\alpha_k)/\mathbf{Q})$ of the field extension $\mathbf{Q}(\alpha_1,\dots,\alpha_k)/\mathbf{Q}$ (i.e., the largest cardinality of an algebraically independent subset of $\mathbf{Q}(\alpha_1,\dots,\alpha_k)$ over $\mathbf{Q}$) is equal to k. On the other hand, when $k \ge 3$ and $2 \le r \le k-1$ the next simple example

$$\alpha_1 := \alpha, \quad \alpha_2 := \alpha/r, \quad \alpha_3 := (\alpha+1)/r, \dots, \alpha_{r+1} := (\alpha+r-1)/r,$$

where α and α_j, $j = r+1,\dots,k$, are algebraically independent transcendental numbers in the interval $(0,1)$ (so that $\operatorname{trdeg}(\mathbf{Q}(\alpha_1,\dots,\alpha_k)/\mathbf{Q}) = k - r \le k-2$), shows that the first $r+1$ Hurwitz zeta functions are linearly dependent

$$r^s\zeta(s,\alpha_1) = \zeta(s,\alpha_2) + \cdots + \zeta(s,\alpha_{r+1}).$$

Therefore, no joint universality theorem holds for these k Hurwitz zeta functions $\zeta(s, \alpha_j)$, $j = 1, \dots, k$. Theorem 2 deals with the remaining case when the transcendence degree of the field extension $\mathbf{Q}(\alpha_1, \dots, \alpha_k)/\mathbf{Q}$ is equal to $k-1$. The case $k = 2$ was recently analyzed by Mishou [13]. We will follow his approach. It seems likely that the conclusion (1) is true for any distinct transcendental numbers $\alpha_1, \dots, \alpha_k \in (0,1)$ for which $\operatorname{trdeg}(\mathbf{Q}(\alpha_1, \dots, \alpha_k)/\mathbf{Q}) = k-1$.

2. Proof of Theorem 1

Recall that if $P(z_1, \dots, z_m) = \sum_{\mathbf{i}} p_{\mathbf{i}} z_1^{i_1} \cdots z_m^{i_m} \in \mathbf{C}[z_1, \dots, z_m]$, where $\mathbf{i} = (i_1, \dots, i_m)$ and $p_{\mathbf{i}} \in \mathbf{C} \setminus \{0\}$, is a nonzero polynomial then its *leading coefficient* is the coefficient $p_{\mathbf{j}}$ for $z_1^{j_1} \cdots z_m^{j_m}$ such that the vector $\mathbf{j} = (j_1, \dots, j_m)$ is the largest lexicographically among all vectors $\mathbf{i} = (i_1, \dots, i_m)$ with maximal sum $i_1 + \cdots + i_m = \deg P$. For instance, the leading coefficient of the polynomial $P(z_1, z_2) = z_1^4 + 2z_1z_2^4 + 3z_2^5 - z_1z_2$ is equal to 2.

LEMMA 3. *Suppose that for $m \in \mathbf{N}$ two nonzero polynomials with integer coefficients $P(z_1, \dots, z_m)$ with positive leading coefficient and $Q(z_1, \dots, z_m)$, not both constants, are relatively prime. Then there exist infinitely many positive integers t for which*

$$P(z_1, \dots, z_m) + tQ(z_1, \dots, z_m) = A \prod_{i \in I} \prod_j (z_i + a_{ij}), \tag{2}$$

where I is a nonempty subset of the set $\{1, \dots, m\}$, A is a nonzero integer and $a_{ij} \in \mathbf{N} \cup \{0\}$ (where a_{ij} are not necessarily distinct), if and only if there are $i \in \{1, \dots, m\}$, $a, b \in \mathbf{N}$, $c \in \mathbf{Z}$, $\gcd(a, b) = 1$ for which $P(z_1, \dots, z_m) = az_i + c$ and $Q(z_1, \dots, z_m) = b$.

Proof. For $m = 1$ the lemma was proved by Mishou in [13]. Our proof is different from that given in [13] and works for any $m \in \mathbf{N}$.

The lemma is trivial in one direction. If $P(z_1, \dots, z_m) = az_i + c$ and $Q(z_1, \dots, z_m) = b$ with a, b, c as above then there are infinitely many $t \in \mathbf{N}$ for which $c + bt \geq 0$ and $a \mid (c + bt)$. For each of those t the representation (2) for the polynomial

$$P(z_1, \dots, z_m) + tQ(z_1, \dots, z_m) = az_i + c + bt = a(z_i + (c + bt)/a)$$

holds with $A = a$, $I = \{i\}$ and $\prod_j (z_i + a_{ij}) = z_i + (c + bt)/a$.

Assume now that $P, Q \in \mathbf{Z}[z_1, \dots, z_m]$, not both constants, are relatively prime, and the leading coefficient of P is positive. Assume that there exist infinitely many positive integers t for which (2) holds with $A = A(t) \in \mathbf{Z} \setminus \{0\}$ and $a_{ij} = a_{ij}(t) \in \mathbf{N} \cup \{0\}$. It is clear that the coefficients of the polynomial

$$R_t(z_1, \dots, z_m) := A(t) \prod_{i \in I} \prod_j (z_i + a_{ij}(t)) \tag{3}$$

on the right hand side of (2) all have the form $ut + v$ with some integers u, v lying in a finite set V. By the condition of the lemma, the nonzero coefficients of $R_t/A(t)$ are all positive. So if two nonzero coefficients, say $r_{\mathbf{i}}(t)$ for $z_1^{i_1} \cdots z_m^{i_m}$ and $r_{\mathbf{j}}(t)$ for $z_1^{j_1} \cdots z_m^{j_m}$, of the polynomial R_t are unbounded then $r_{\mathbf{i}}(t) = ut + v$ and $r_{\mathbf{j}}(t) = u't + v'$ with some integers $u, u' \neq 0$. It follows that the modulus of their quotient $|r_{\mathbf{i}}(t)/r_{\mathbf{j}}(t)|$ is bounded in terms of t. The fact that the quotient of two unbounded coefficients of R_t must be bounded will be used below several times.

Now we shall prove that all the zeros $-a_{ij}(t)$ of the polynomial R_t given in (3) are unbounded in terms of t. For a contradiction assume that $a_{ij}(t)$ for some fixed pair i, j is bounded and assume without restriction of generality that $i = m$. Then $0 \leq a_{mj}(t) \leq K$ for certain $K \in \mathbf{N}$. Since $a_{mj}(t)$ can only take $K + 1$ values, we must have $a_{mj}(t) = a^*$ for some fixed $a^* \in \{0, 1, \ldots, K\}$ and infinitely many $t \in \mathbf{N}$. Thus the factor $z_m + a^*$ occurs in all those polynomials $R_t = P + tQ$ defined in (2) corresponding to those t. Then the polynomial

$$R_t(z_1, \ldots, z_{m-1}, -a^*) = P(z_1, \ldots, z_{m-1}, -a^*) + tQ(z_1, \ldots, z_{m-1}, -a^*)$$

is zero identically. Thus $Q(z_1, \ldots, z_{m-1}, -a^*)$ must be the zero polynomial. It follows that $P(z_1, \ldots, z_{m-1}, -a^*)$ is also the zero polynomial. Hence $Q(z_1, \ldots, z_m)$ and $P(z_1, \ldots, z_m)$ are both divisible by the same factor $z_m + a^*$, a contradiction. This proves that all the zeros $-a_{ij}(t)$ of R_t in (3) are unbounded, i.e. $a_{ij}(t) \to \infty$ as $t \to \infty$. Since $A(t) \in \mathbf{Z}\backslash\{0\}$, in view of (3) it follows that all the nonzero coefficients of R_t are also unbounded except possibly for the leading coefficient $A(t)$.

Next, if the leading coefficient $A(t)$ is unbounded then $A(t)$ and $A(t) \prod_{i \in I} \prod_j a_{ij}(t)$ are two unbounded coefficients of R_t, which is impossible, because their quotient $\prod_{i \in I} \prod_j a_{ij}(t)$ tends to infinity as $t \to \infty$. (Recall that, by the fact established above, the quotient of two unbounded coefficients of R_t must be bounded.) So $A(t)$ is bounded. Hence the leading coefficient $A(t)$ of $R_t = P + tQ$ must be that of P. This yields $A(t) = a$, where $a > 0$ is the leading coefficient of P.

Suppose next that for infinitely many $t \in \mathbf{N}$ the product

$$R_t(z_1, \ldots, z_m) = a \prod_{i \in I} \prod_j (z_i + a_{ij}(t))$$

contains exactly $r \geq 2$ not necessarily distinct factors with the same i, say $z_i + a_{i1}(t), \ldots, z_i + a_{ir}(t)$. Put $B = B(t)$ for the constant term of the polynomial $R_t(z_1, \ldots, z_r)/\prod_{j=1}^r (z_i + a_{ij}(t))$. Then both $B(t) \prod_{j=1}^r a_{ij}(t)$ and $B(t) \sum_{j=1}^r a_{ij}(t)$ are the coefficients of the polynomial R_t corresponding to its constant term and the term for z_1^{r-1}, respectively. They are both unbounded, so their quotient $\prod_{j=1}^r a_{ij}(t)/\sum_{j=1}^r a_{ij}(t)$ must be bounded. This is not the case, because all $a_{ij}(t)$ are unbounded, so the product of $r \geq 2$ terms $\prod_{j=1}^r a_{ij}(t)$ divided by their sum $\sum_{j=1}^r a_{ij}(t)$ tends to infinity as $t \to \infty$.

The only remaining possibility is that $R_t(z_1, \dots, z_m) = a\prod_{i \in I}(z_i + a_i(t))$ for infinitely many $t \in \mathbf{N}$. In case $|I| \geq 2$ we see that the constant coefficient of R_t is equal to $a\prod_{i \in I} a_i(t)$ and the coefficient for z_l, where $l \in I$, is equal to $a\prod_{i \in I \setminus \{l\}} a_i(t)$. They both are unbounded, because $|I| \geq 2$. But their quotient $a_l(t)$ is also unbounded, a contradiction.

It follows that $|I| = 1$ and thus $R_t(z_1, \dots, z_m) = a(z_i + a_i(t))$ for some $i \in \{1, \dots, m\}$ and infinitely many $t \in \mathbf{N}$. From $P + tQ = R_t = az_i + aa_i(t)$ we conclude that $P(z_1, \dots, z_m) = az_i + c$, where $a \in \mathbf{N}$, $c \in \mathbf{Z}$ and $Q(z_1, \dots, z_m) = b \neq 0$. Then

$$P + tQ = az_i + c + tb = a(z_i + (c + tb)/a)$$

has the required form only when $b > 0$ and a divides $c + bt$ for infinitely many $t \in \mathbf{N}$. From the equality $at_1 - bt = c$, where $t_1 \in \mathbf{Z}$, we see that such positive integers t exist if and only if $\gcd(a, b)$ divides c. However, if $\gcd(a, b) > 1$ and $\gcd(a, b) \mid c$ then the polynomials $P = az_i + c$ and $Q = b$ are divisible by $\gcd(a, b) > 1$, and so they are not relatively prime. Consequently, we must have $\gcd(a, b) = 1$. Hence $P(z_1, \dots, z_m) = az_i + c$ for some $i \in \{1, \dots, m\}$ and $Q(z_1, \dots, z_m) = b$ with $a, b \in \mathbf{N}$, $c \in \mathbf{Z}$ and $\gcd(a, b) = 1$, as claimed in the statement of the lemma. $\square$

Now we can give the proof of Theorem 1. Assume that the set

$$\mathscr{D}(\alpha_1) \cup \cdots \cup \mathscr{D}(\alpha_{k-1}) \cup \mathscr{D}_q(\alpha_k)$$

is linearly dependent over $\mathbf{Q}$. Since the sets $\mathscr{D}(\alpha_1) \cup \cdots \cup \mathscr{D}(\alpha_{k-1})$ and $\mathscr{D}_q(\alpha_k)$ are both linearly independent over $\mathbf{Q}$, writing $\alpha_k = P(\alpha_1, \dots, \alpha_{k-1})/Q(\alpha_1, \dots, \alpha_{k-1})$ with two relatively prime polynomials P, Q in $\mathbf{Z}[z_1, \dots, z_{k-1}]$ we must have

$$\prod_{i \in I} \prod_j (\alpha_i + n_{ij})^{u_{ij}} = \prod_j (P(\alpha_1, \dots, \alpha_{k-1})/Q(\alpha_1, \dots, \alpha_{k-1}) + n_j)^{u_j} \tag{4}$$

for some $I \subseteq \{1, \dots, k-1\}$, $n_{ij}, n_j \in \mathbf{N} \cup \{0\}$, $n_j \geq q$ and $u_{ij}, u_j \in \mathbf{Z} \setminus \{0\}$. Of course, P and Q are not both constants, because α_k is transcendental. Also, without restriction of generality, by multiplying both P and Q by -1 if necessary, we may assume that the leading coefficient of P is positive.

Since the numbers $\alpha_1, \dots, \alpha_{k-1}$ are algebraically independent, the equality (4) must be the identity, namely,

$$\prod_{i \in I} \prod_j (z_i + n_{ij})^{u_{ij}} \equiv \prod_j (P(z_1, \dots, z_{k-1})/Q(z_1, \dots, z_{k-1}) + n_j)^{u_j}. \tag{5}$$

Note that the polynomials $P + n_j Q$ and $P + n_l Q$ with $n_j \neq n_l$ can have only constant common factor, since P and Q are relatively prime. Hence selecting any $n_j \geq q$ on the right hand side of (5) we see that the corresponding polynomial $P(z_1, \dots, z_{k-1}) + n_j Q(z_1, \dots, z_{k-1})$ must be a constant multiplied by certain product $\prod_{i \in I_1} (z_i + n_{is})^{v_{is}}$, where $I_1 \subseteq I$, $n_{is} \in \mathbf{N} \cup \{0\}$ and $v_{is} \in \mathbf{N}$. However, by Lemma 3, this is impossible for q large enough whenever $(P, Q) \neq (az_i + c, b)$

with a, b, c as in Lemma 3. This completes the proof of Theorem 1, since the condition of the theorem and that of the lemma which exclude the case $P(z_1, \dots, z_{k-1}) = az_i + c$, $Q(z_1, \dots, z_{k-1}) = b$, where $i \in \{1, \dots, k-1\}$, $a, b \in \mathbf{N}$, $c \in \mathbf{Z}$ and $\gcd(a, b) = 1$, are the same.

3. Proof of Theorem 2

Assume that the set of Dirichlet exponents

$$\mathscr{D}_{q_1}(\alpha_1) \cup \dots \cup \mathscr{D}_{q_k}(\alpha_k)$$

is linearly independent over $\mathbf{Q}$. Evidently, its subset

$$\mathscr{D}_q(\alpha_1) \cup \dots \cup \mathscr{D}_q(\alpha_k),$$

where $q := \max_{1 \le j \le k} q_j$, is linearly independent over $\mathbf{Q}$ too. Take a maximal subset M_1 of the finite set $\bigcup_{j=1}^{k} (\mathscr{D}(\alpha_j) \backslash \mathscr{D}_q(\alpha_j))$ for which the set

$$\mathscr{D}_1 := M_1 \cup \mathscr{D}_q(\alpha_1) \cup \dots \cup \mathscr{D}_q(\alpha_k)$$

is linearly independent over $\mathbf{Q}$. This means that each of the $qk - |M_1|$ remaining logarithms $\log(n + \alpha_j) \notin \mathscr{D}_1$, where $0 \le n \le q - 1$ and $1 \le j \le k$, is a linear combination with rational coefficients of some elements of $\mathscr{D}_1$. (Of course, the choice of the set M_1 is not necessarily unique.)

Fix an integer $m \ge q$ such that each of the logarithms $\log(n + \alpha_j) \notin \mathscr{D}_1$ is expressible in the form

$$\log(n + \alpha_j) = \sum_{r=1}^{k} \sum_{i=0}^{m-1} c_{j,n,r,i} \log(i + \alpha_r)$$

with $c_{j,n,r,i} \in \mathbf{Q}$. (Some of the coefficients $c_{j,n,r,i}$ can be zeros.) Therefore, by increasing q to m if necessary and adding more logarithms to the set M_1 we may assume that each $\log(n + \alpha_j)$ which is not in the set

$$\mathscr{D} := M \cup \mathscr{D}_m(\alpha_1) \cup \dots \cup \mathscr{D}_m(\alpha_k),$$

where

$$M := M_1 \cup \{\log(q + \alpha_1), \dots, \log(m - 1 + \alpha_1)\} \cup \cdots \\ \cup \{\log(q + \alpha_k), \dots, \log(m - 1 + \alpha_k)\},$$

is a linear combination of at most km logarithms of the set M. Obviously, there exists a positive integer ℓ such that for each $\log(n + \alpha_j) \notin \mathscr{D}$ we have the representation

$$\ell \log(n + \alpha_j) = \sum_{\log(i+\alpha_r) \in M} c_{i,r} \log(i + \alpha_r) \tag{6}$$

with $c_{i,r} \in \mathbf{Z}$.

Let K_j be the sets and let $f_j(s)$ be the functions described in Theorem 2. Fix $\varepsilon > 0$. Let K be a simply connected compact subset of the strip $\{s \in \mathbf{C} : 1/2 < \Re(s) < 1\}$ such that the union $\bigcup_{j=1}^{k} K_j$ is included in the interior of K. By Mergelyan's theorem (see Lemma 5 in [13]), there exist polynomials with complex coefficients $p_j(s)$, $j = 1, \dots, k$, such that

$$\max_{1 \le j \le k} \max_{s \in K_j} |f_j(s) - p_j(s)| < \varepsilon. \tag{7}$$

By Gonek's lemma (see Lemma 7, (29) and (30) in [13]), there is a large positive integer $v > m$ such that for each sufficiently large integer t and each $j = 1, \dots, k$ we have

$$\max_{s \in K} \left| p_j(s) - \sum_{0 \le n < v} \frac{1}{(n + \alpha_j)^s} - \sum_{v \le n \le t} \frac{\exp(2\pi i \theta_{n,j})}{(n + \alpha_j)^s} \right| < \varepsilon$$

with some $\theta_{n,j} \in \mathbf{R}$. Selecting $\theta_{n,j} = 0$ for $n = m, \dots, v$, we can rewrite the above inequality in the form

$$\max_{s \in K} \left| p_j(s) - \sum_{0 \le n < m} \frac{1}{(n + \alpha_j)^s} - \sum_{m \le n \le t} \frac{\exp(2\pi i \theta_{n,j})}{(n + \alpha_j)^s} \right| < \varepsilon. \tag{8}$$

For $\delta > 0$ let $B_T(\delta)$ be a set of those $\tau \in [T, 2T]$ for which

$$\|-(\tau/2\pi) \log(n + \alpha_j) - \theta_{n,j}\| \le \delta \quad \text{when } m \le n \le t,\ 1 \le j \le k$$

and

$$\|-(\tau/2\pi) \log(n + \alpha_j)\| \le \delta \quad \text{when } \log(n + \alpha_j) \in M.$$

Observe that in view of (6) the second inequality implies that for each sufficiently small δ there is a positive constant c_0 (which depends on ℓ, M and the coefficients $c_{i,r}$ in $qm - |M|$ equalities (6)) such that

$$\|-(\tau/2\pi) \log(n + \alpha_j)\| \le c_0 \delta \tag{9}$$

for each $n = 0, 1, \dots, m - 1$ and each $j = 1, \dots, k$. Since the logarithms involved in the definition of $B_T(\delta)$ are linearly independent over $\mathbf{Q}$, by Kronecker's theorem (see Lemma 6 in [13]), the Lebesgue measure of the set $B_T(\delta)$ satisfies

$$\mu(B_T(\delta)) \sim \varepsilon_1 T \quad \text{as } T \to \infty, \text{ where } \varepsilon_1 := (2\delta)^{k(t-m+1)+|M|}. \tag{10}$$

For each $j = 1, \dots, k$ and each $\tau \in B_T(\delta)$ we have

$$\max_{s \in K} \left| \sum_{m \le n \le t} \frac{\exp(2\pi i \theta_{n,j})}{(n + \alpha_j)^s} - \sum_{m \le n \le t} \frac{1}{(n + \alpha_j)^{s+i\tau}} \right| < \varepsilon$$

whenever δ is small enough. Similarly, by (9), we obtain

$$\max_{s \in K} \left| \sum_{0 \le n < m} \frac{1}{(n + \alpha_j)^s} - \sum_{0 \le n < m} \frac{1}{(n + \alpha_j)^{s+i\tau}} \right| < \varepsilon$$

when δ is small enough. Combined with (8) this gives

$$\max_{1\le j\le k}\max_{s\in K}\left|p_j(s)-\sum_{0\le n\le t}\frac{1}{(n+\alpha_j)^{s+i\tau}}\right|<3\varepsilon. \tag{11}$$

The next two inequalities are standard and can be obtained by considering the second moments of the involved functions. Firstly, for any pair of positive numbers ε, ε_2 and a set

$$A_T(\varepsilon,z):=\left\{\tau\in[T,2T]:\max_{1\le j\le k}\max_{s\in K_j}\left|\zeta(s+i\tau,\alpha_j)-\sum_{0\le n\le z}\frac{1}{(n+\alpha_j)^{s+i\tau}}\right|<\varepsilon\right\} \tag{12}$$

we have

$$\liminf_{T\to\infty}\frac{\mu(A_T(\varepsilon,z))}{T}>1-\varepsilon_2 \tag{13}$$

for each sufficiently large z (see Lemma 9 in [13]). Secondly, let $C_T(\delta)$ be a subset of $B_T(\delta)$ for which the inequality

$$\max_{1\le j\le k}\max_{s\in K}\left|\sum_{t<n\le z}\frac{1}{(n+\alpha_j)^{s+i\tau}}\right|<\varepsilon \tag{14}$$

holds uniformly for $z>t$. Then (see Lemma 11 in [13] and (10)) for each sufficiently large t we have

$$\liminf_{T\to\infty}\frac{\mu(C_T(\delta))}{T}>\frac{1}{2}\lim_{T\to\infty}\frac{\mu(B_T(\delta))}{T}=\frac{\varepsilon_1}{2}.$$

Hence selecting $\varepsilon_2=\varepsilon_1/4$ in (13) we obtain

$$\liminf_{T\to\infty}\frac{\mu(A_T(\varepsilon,z)\cap C_T(\delta))}{T}>\frac{\varepsilon_1}{4}$$

for each sufficiently large z. Finally, for $\tau\in A_T(\varepsilon,z)\cap C_T(\delta)$ combining (7), (11), (12), (14) we find that

$$\max_{1\le j\le k}\max_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<7\varepsilon.$$

This completes the proof of Theorem 2.

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