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EMDEN EQUATION INVOLVING THE CRITICAL SOBOLEV EXPONENT WITH THE THIRD-KIND BOUNDARY CONDITION IN $\mathbf{S}^3$

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Abstract

We consider a positive solution of the Emden equation with the critical Sobolev exponent on a geodesic ball in $\mathbf{S}^3$. In the case of the Dirichlet boundary condition, Bandle and Peletier [2] proved the precise result on the existence of a positive radial solution. We investigate the same equation with the third kind boundary condition and obtain a more general result. Namely we prove that the existence and the nonexistence of solutions depend on the geodesic radius and the boundary condition. Moreover the set of solutions consists of a unique radial classical solution and a continuum of singular solutions.

1. Introduction

Our aim is to investigate the structure of positive solutions to the Emden equation having the critical Sobolev exponent on a geodesic ball

$$\begin{cases} \Delta_{\mathbf{S}^N} u + u^{(N+2)/(N-2)} = 0 & \text{in } B_{\theta_0}, \\ u > 0 & \text{in } B_{\theta_0}, \\ u + \kappa \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B_{\theta_0}, \end{cases} \tag{1.1}$$

where $N \geq 3$, $\mathbf{S}^N = \{x \in \mathbf{R}^{N+1} \mid |x| = 1\}$, $\Delta_{\mathbf{S}^N}$ is the Laplace-Beltrami operator on $\mathbf{S}^N$, n is the outer unit normal vector to ∂B_{θ_0} and $\kappa \geq 0$. Here B_{θ_0} is a geodesic ball in $\mathbf{S}^N$ with its geodesic radius $\theta_0 \in (0, \pi)$, and its center is located at the north pole $P_n = (x_1, x_2, \dots, x_{N+1}) = (0, 0, \dots, 1)$. In this paper we consider a classical solution and a singular solution $u \in C^2(B_{\theta_0} \setminus \{P_n\})$ to (1.1) with $\lim_{x \to P_n} u(x) = +\infty$.

First, for the sake of comparison with results on (1.1), we refer to known results on the Emden equation in $B^* := \{x \in \mathbf{R}^N \mid |x| < 1\}$, that is, a ball in the Euclidean space with the same dimensions as $\mathbf{S}^N$:

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$$
\begin{cases} \Delta u + u^p = 0 & \text{in } B^*, \\ u > 0 & \text{in } B^*, \\ u + \kappa \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B^*. \end{cases} \tag{1.2}
$$

If $\kappa = 0$, then we can immediately show that the problem (1.2) with $p = (N+2)/(N-2)$ has no solution by applying the well-known Pohozaev identity. In fact, for a solution u to (1.2), there holds the Pohozaev identity

$$
\left(\frac{N}{p+1} - \frac{N-2}{2}\right) \int_{B^*} u^{p+1}\, dx = \frac{1}{2} \int_{\partial B^*} \left|\frac{\partial u}{\partial n}\right|^2 x \cdot n \, d\sigma. \tag{1.3}
$$

Set $p = (N+2)/(N-2)$. Then the left hand side of (1.3) vanishes, and hence we obtain $\partial u/\partial n = 0$ on ∂B^* from $x \cdot n = 1$. Therefore it follows that $u \equiv 0$ in B^* from the Hopf boundary lemma (e.g., see Lemma 3.4 in [4]). Moreover, in the case of $\kappa \geq 0$ and $N = 3$, Kabeya, Yanagida and Yotsutani [6] proved that if $0 \leq \kappa \leq 1$, then (1.2) has no radial classical or singular solution. On the other hand, if $\kappa > 1$, then (1.2) has a unique classical solution and a continuum of singular solutions. Namely the existence of a solution to (1.2) depends on κ, and $\kappa = 1$ is the critical value of the existence or the nonexistence of a solution. Concerning a higher dimensional case, e.g., see [5].

For (1.1) with $N = 3$ and $\kappa = 0$, Bandle, Brillard and Flucher [1] proved the existence of a radial solution from the viewpoint of the Sobolev imbedding. Here a solution to (1.1) depending only on the geodesic distance from P_n is said to be a radial solution. They proved that there exists some constant θ_c such that

(a) a radial classical solution to (1.1) exists if $\theta_0 \in (\theta_c, \pi)$;

(b) (1.1) has no radial classical solution if $\theta_0 \in (0, \theta_c)$.

To obtain θ_c exactly, Bandle and Peletier [2] investigated (1.1) with $N = 3$ more precisely. They used the stereographic projection from the south pole $(0,0,0,-1)$ onto the plane $x_4 = 0$ and investigated the existence of a positive radial solution in a ball on $\mathbf{R}^3$. Namely let θ be the geodesic distance from P_n, and define $r = \tan(\theta/2)$. Then a radial solution to (1.1) satisfies

$$
\begin{cases} \left(\dfrac{r^2}{1+r^2} u_r\right)_r + \dfrac{r^2}{(1+r^2)^3} u^5 = 0 & \text{for } r \in (0, R), \\ u(r) > 0 & \text{for } r \in (0, R), \\ u(R) = 0, \end{cases} \tag{1.4}
$$

where $R = \tan(\theta_0/2)$. Let $u(r) \in C^2((0, R))$ be a solution to (1.4). If $u(r)$ converges to a positive constant as $r \to 0$, then $u_r(r) \to 0$ as $r \to 0$ (see Lemma 1 in [2]). Hence $u(\tan(\theta/2))$ is a classical solution to (1.1). Bandle and Peletier proved the following theorem.

Theorem A (Theorem 1 in [2]). *For* (a) *and* (b), *it holds that* $\theta_c = \pi/2$. *Moreover if* $\theta_0 = \pi/2$, *then* (1.1) *has no radial classical solution.*

From Theorem A, it seems that the structure of solutions to (1.1) with $\kappa \geq 0$ also has a difference between the case $B_{\theta_0} \subset B_{\pi/2}$ and the case $B_{\theta_0} \supset B_{\pi/2}$. Furthermore Kumaresan and Prajapat [8] proved that if there exists a solution $u \in C^2(\overline{B_{\theta_0}})$ to (1.1) with $\kappa = 0$ and $0 < \theta_0 < \pi/2$, then u is radially symmetric. The Kumaresan–Prajapat result is the analogue of the Gidas–Ni–Nirenberg result [3], and thus we see that there exists no positive solution in the case of $0 < \theta_0 < \pi/2$. In contrast it is not yet known whether or not the analogue of the Gidas–Ni–Nirenberg result holds for $\pi/2 \leq \theta_0 < \pi$. The above result on $N = 3$ is different from that on $N \geq 4$ in [1]: for any $\theta_0 \in (0, \pi)$, there exists a radial solution to (1.1) with $N \geq 4$ and $\kappa = 0$. Namely it holds that $\theta_c = 0$ if $N \geq 4$. In addition the Kumaresan–Prajapat result also holds with $N \geq 4$ and $0 < \theta_0 < \pi/2$.

On the other hand, suppose that u is a classical or singular solution to (1.1) with the Neumann boundary condition ($\kappa = +\infty$) and $N = 3$. Then u is a solution to (1.4) with $u_r(R) = 0$ instead of $u(R) = 0$. By integrating (1.4) from $r \in (0, R)$ to R, it holds that

$$u_r(r) = \frac{1 + r^2}{r^2} \int_r^R \frac{s^2}{(1 + s^2)^3} u(s)^5 \, ds > 0 \quad \text{for } r \in (0, R). \tag{1.5}$$

The relation (1.5) implies $\lim_{r \to 0} u_r(r) = +\infty$. If u is a classical solution to (1.1), then $u_r(0) = 0$, which is a contradiction to $\lim_{r \to 0} u_r(r) = +\infty$. Thus u is not a classical solution. Similarly we see that u is not a singular solution. In fact if u is a singular solution to (1.1), then $\lim_{r \to 0} u_r(r) = -\infty$. Therefore there exists no classical or singular solution to (1.1) with the Neumann boundary condition.

Our main purpose is to investigate the structure of radial solutions to (1.1) in the case of $N = 3$ and $0 \leq \kappa < +\infty$. First we adopt the polar coordinates instead of the stereographic projection used in Bandle and Peletier's investigation. It seems that the polar coordinates is more natural than the stereographic projection to investigate (1.1) from the standpoint of ODE. For $x = (x_1, x_2, x_3, x_4) \in \mathbf{S}^3$, let

$$\begin{cases} x_1 = \sin\theta \sin\varphi_1 \sin\varphi_2 \\ x_2 = \sin\theta \sin\varphi_1 \cos\varphi_2 \\ x_3 = \sin\theta \cos\varphi_1 \\ x_4 = \cos\theta \end{cases}$$

with $\theta, \varphi_1 \in [0, \pi]$ $(i = 1, 2)$ and $\varphi_2 \in [0, 2\pi]$. Then the Laplace-Beltrami operator $\Delta_{\mathbf{S}^3}$ is described by

$$\begin{aligned} \Delta_{\mathbf{S}^3} u = {} & \frac{1}{\sin^2\theta} \frac{\partial}{\partial\theta}\left(\sin^2\theta \frac{\partial u}{\partial\theta}\right) \\ & + \frac{1}{\sin^2\theta \sin\varphi_1} \frac{\partial}{\partial\varphi_1}\left(\sin\varphi_1 \frac{\partial u}{\partial\varphi_1}\right) + \frac{1}{\sin^2\theta \sin^2\varphi_1} \frac{\partial^2 u}{\partial\varphi_2^2}. \end{aligned}$$

Therefore a radial solution to (1.1) satisfies

$$\begin{cases} \dfrac{1}{\sin^2\theta}(u_\theta \sin^2\theta)_\theta + u^5 = 0 & \text{for } \theta \in (0, \theta_0), \\ u(\theta) > 0 & \text{for } \theta \in (0, \theta_0), \\ u(\theta_0) + \kappa u_\theta(\theta_0) = 0. \end{cases} \tag{1.6}$$

Next we define some notations for a solution to (1.6). In this paper we consider the following two types of solutions to (1.6).

Definition 1.1. (i) A solution $u \in C^2((0, \theta_0))$ to (1.6) is said to be a regular solution if u converges to some positive constant as $\theta \to 0$.

(ii) A solution $u \in C^2((0, \theta_0))$ to (1.6) is said to be a singular solution if u has singularity at $\theta = 0$.

Here we remark that a regular solution to (1.6) defined in Definition 1.1 is a classical solution to (1.1). Similarly a singular solution to (1.6) is corresponding to a singular solution to (1.1). In addition let $u(\theta; \alpha)$ be a solution to (1.6) satisfying $u_\theta(\theta_0) = -\alpha$. Moreover the principal value of $\arcsin x$ is denoted by $\operatorname{Arcsin} x$. Our main result is the following theorem for (1.6).

Theorem 1.1. *For the problem* (1.6), *the following statements hold.*

(i) *Suppose that* $0 \le \kappa \le 1/2$.

(a) *If* θ_0 *satisfies* $\theta_0 \ne 0$ *and*

$$\frac{1}{2} \operatorname{Arcsin} 2\kappa \le \theta_0 \le \frac{1}{2}(\pi - \operatorname{Arcsin} 2\kappa),$$

then (1.6) *does not have a regular or singular solution.*

(b) *On the other hand, if* θ_0 *satisfies*

$$0 < \theta_0 < \frac{1}{2} \operatorname{Arcsin} 2\kappa \quad \textit{or} \quad \frac{1}{2}(\pi - \operatorname{Arcsin} 2\kappa) < \theta_0,$$

then there exists a constant $\alpha_* > 0$ *such that the solution* $u(\theta; \alpha_*)$ *is a regular solution. Moreover* $u(\theta; \alpha)$ *is a singular solution for* $\alpha \in (0, \alpha_*)$. *In addition, for* $\alpha \in (\alpha_*, +\infty)$, *the problem* (1.6) *does not have either a regular solution or a singular solution.*

(ii) *Suppose that* $\kappa > 1/2$. *Then there exists a constant* $\alpha_* > 0$ *such that the solution* $u(\theta; \alpha_*)$ *is a regular solution. Moreover* $u(\theta; \alpha)$ *is a singular solution for* $\alpha \in (0, \alpha_*)$. *In addition, for* $\alpha \in (\alpha_*, +\infty)$, *the problem* (1.6) *does not have either a regular solution or a singular solution.*

Remark 1.1. Compare Theorem A and Theorem 1.1. Since a regular solution to (1.6) is a classical solution to (1.4), Theorem 1.1 with $\kappa = 0$ implies the same result as in Theorem A. Thus Theorem 1.1 is the extension of Theorem A and this provides us a comprehensive view to (1.1).

The proof of Theorem 1.1 is due to the method used in [6]. Namely we change the variable and consider the corresponding exterior problem

$$\begin{cases} \dfrac{1}{\tau^2}(\tau^2 w_\tau)_\tau + K(\tau)w^5(\tau) = 0 \quad \text{for } \tau \in (\rho, +\infty), \\ w(\rho) = \beta, \\ w_\tau(\rho) = 0, \end{cases}$$

where $\beta > 0$ and $\rho \geq 0$ depend on κ and θ_0. Next we investigate the structure of solutions to the exterior problem by using results proved in [7], [9] and [10].

Our paper is organized as follows: we precisely define the change of the variable to transform (1.6) to the exterior problem in Section 2. The structure theorem concerning this exterior problem is stated in Section 3. Theorem 1.1 is shown in Section 4 by using the structure theorem.

2. Transforming to the exterior problem

In this section, we transform (1.6) to an exterior problem. Hereafter let u be a regular solution or a singular solution to (1.6). First set

$$\rho := \frac{\kappa}{\sin^2 \theta_0}, \tag{2.1}$$

$$\tau := \int_\theta^{\theta_0} \frac{d\psi}{\sin^2 \psi} + \rho = \cot \theta - \cot \theta_0 + \rho. \tag{2.2}$$

From (2.1), $\rho = 0$ if and only if $\kappa = 0$. Moreover, from (2.2), τ attains ρ as $\theta = \theta_0$, and $\tau \to +\infty$ as $\theta \to 0$. Next we define

$$w(\tau) := \frac{u(\theta)}{\tau}. \tag{2.3}$$

By using τ and w defined above, we can transform (1.6) to the following exterior problem.

Lemma 2.1. *The function* $w = u/\tau$ *satisfies*

$$\begin{cases} \dfrac{1}{\tau^2}(\tau^2 w_\tau)_\tau + K(\tau)w^5(\tau) = 0 \quad \textit{for } \tau \in (\rho, +\infty), \\ w(\rho) = \beta, \\ w_\tau(\rho) = 0, \end{cases} \tag{2.4}$$

where $\beta := \alpha \sin^2 \theta_0$ *with* $\alpha := -u_\theta(\theta_0)$. *Here* $K(\tau)$ *is defined as*

$$K(\tau) := \tau^4 \sin^4 \theta = [\cos \theta + (\rho - \cot \theta_0) \sin \theta]^4. \tag{2.5}$$

Conversely if $w \in C^2(\rho, +\infty)$ *is a positive solution to* (2.4), *then* $u = \tau w$ *is a solution to* (1.6).

Proof. First, from (2.2), it holds that $d/d\theta = -(\sin\theta)^{-2} d/d\tau$. Hence it follows that

$$(u_\theta \sin^2\theta)_\theta = \frac{1}{\sin^2\theta}[(\tau w)_\tau]_\tau = \frac{1}{\tau\sin^2\theta}(\tau^2 w_\tau)_\tau.$$

From (1.6), we obtain

$$\frac{1}{\tau^2}(\tau^2 w_\tau)_\tau + K(\tau)w^5 = 0. \tag{2.6}$$

Next we consider the boundary condition. We prove that $w_\tau(\tau) \to 0$ as $\tau \to \rho$. If $\rho > 0$ $(\kappa > 0)$, then, from

$$\tau^2 w_\tau(\tau) = -\tau u_\theta(\theta)\sin^2\theta - u(\theta), \tag{2.7}$$

it follows that

$$\rho^2 w_\tau(\rho) = -\rho u_\theta(\theta_0)\sin^2\theta_0 - u(\theta_0) = -\kappa u_\theta(\theta_0) - u(\theta_0) = 0,$$

and hence we obtain $w_\tau(\rho) = 0$. Moreover, from (1.6), (2.1) and (2.3), it follows that

$$w(\rho) = -\frac{\kappa u_\theta(\theta_0)}{\rho} = -u_\theta(\theta_0)\sin^2\theta_0. \tag{2.8}$$

From the above arguments, the function w defined in (2.3) satisfies (2.4) with $\rho > 0$.

On the other hand, suppose that $\rho = 0$ $(\kappa = 0)$. First we show that $\tau^2 w_\tau(\tau) \to 0$ as $\tau \to 0$. Multiplying (1.6) by $\sin^2\theta$ and integrating it over (θ, θ_0), we obtain

$$u_\theta(\theta_0)\sin^2\theta_0 = u_\theta(\theta)\sin^2\theta - \int_\theta^{\theta_0} u(\psi)^5 \sin^2\psi\, d\psi \quad \text{for } \theta \in (0, \theta_0). \tag{2.9}$$

From $u \in C^2((0, \theta_0))$, the right hand side of (2.9) is finite. Hence it follows that

$$|u_\theta(\theta_0)| < +\infty. \tag{2.10}$$

Therefore, from (2.7), (2.10) and $u(\theta_0) = 0$, it follows that

$$\tau^2 w_\tau(\tau) = -\tau u_\theta(\theta)\sin^2\theta - u(\theta) \to 0 \quad \text{as } \tau \to 0. \tag{2.11}$$

Next, to prove $w_\tau(0) = 0$, we show that $w(0)$ is finite. From (2.2) and (2.3), it holds that

$$w(0) = \lim_{\theta\to\theta_0} \frac{u(\theta) - 0}{\theta - \theta_0}\,\frac{\theta - \theta_0}{\tau - 0} = -u_\theta(\theta_0)\sin^2\theta_0. \tag{2.12}$$

From (2.10) and (2.12), it follows that $w(0)$ is finite. Therefore, from (2.6) and (2.11), $w_\tau(\tau)$ is described by

$$w_\tau(\tau) = -\int_0^\tau \frac{s^2}{\tau^2} K(s) w(s)^5\, ds.$$

Since $w(0)$ is finite and $K(0) = 0$ (see (2.5)), we obtain $w_\tau(0) = 0$. From the above arguments, the function w defined in (2.3) satisfies (2.4) with $\rho = 0$.

Conversely if $w \in C^2((\rho, +\infty))$ be a positive solution to (2.4), then the function $u := \tau w$ is a solution to (1.6), which is confirmed by direct calculation. ∎

3. Structure theorem for the exterior problem

From the argument in Section 2, a solution u to (1.6) is associated with a solution w to (2.4). In this section we state the structure theorem for

$$\begin{cases} \dfrac{1}{\tau^2}(\tau^2 w_\tau)_\tau + L(\tau) w_+^p(\tau) = 0 \quad \text{for } \tau \in (\rho, +\infty), \\ w(\rho) = \beta, \\ w_\tau(\rho) = 0, \end{cases} \tag{3.1}$$

where $w_+ := \max\{w, 0\}$ and β is a positive constant. Here the function L satisfies

$$\begin{cases} L(\tau) \in C^1((\rho, +\infty)), \\ L(\tau) \geq 0 \quad \text{and} \quad L(\tau) \not\equiv 0 \quad \text{on } (\rho, +\infty), \\ \tau L(\tau) \in L^1(\rho, \rho_*), \\ \tau^{1-p} L(\tau) \in L^1(\rho_*, +\infty), \end{cases} \tag{L}$$

where $\rho_* \in (\rho, +\infty)$ is an arbitrary constant. Hereafter the solution to (3.1) with an initial data β is denoted by $w(\tau; \beta)$.

The problem (3.1) is more general than (2.4). In fact it is obvious that $K(\tau)$ defined in (2.5) satisfies $K(\tau) \in C^1((\rho, +\infty))$, $K(\tau) \geq 0$ on $(\rho, +\infty)$ and $K(\tau) \not\equiv 0$. Moreover, for any $\tau \in (\rho, +\infty)$, it holds that

$$\int_\rho^\tau |sK(s)|\, ds = \int_\theta^{\theta_0} \sin^2 \psi (\cot \psi + \rho - \cot \theta_0)^5\, d\psi < +\infty,$$

and

$$\int_\tau^{+\infty} |s^{-4} K(s)|\, ds = \int_0^\theta \sin^2 \psi\, d\psi < +\infty.$$

Thus K satisfies condition (L), and therefore we consider (3.1) in this section.

We classify solutions to (3.1) into one of the following three types. First, for a solution to (3.1), the following statement holds.

Lemma 3.1. *If a solution w to* (3.1) *satisfies $w > 0$ on $(\rho, +\infty)$, then $\tau w(\tau)$ is non-decreasing for $\tau \in (\rho, +\infty)$.*

Proof. From

$$w_{\tau\tau} + \frac{2}{\tau} w_\tau + L(\tau) w^p = 0,$$

it follows that

$$(\tau w)_{\tau\tau}(\tau) = 2w_\tau(\tau) + \tau w_{\tau\tau}(\tau) = -\tau L(\tau) w^p(\tau) \le 0.$$

Hence if $(\tau w)_\tau(\tau_1) < 0$ for some $\tau_1 \in (\rho, +\infty)$, then it holds that

$$(\tau w)_\tau(\tau) \le (\tau w)_\tau(\tau_1) < 0 \quad \text{for } \tau > \tau_1.$$

Therefore there exists some $\tau_2 > \tau_1$ such that $w(\tau_2) = 0$, and it is contradiction to $w > 0$ on $(\rho, +\infty)$. This lemma is proved. ■

By Lemma 3.1, $\lim_{\tau \to +\infty} \tau w(\tau)$ is a positive value or $+\infty$ if $w > 0$ in $(\rho, +\infty)$. Thus every solution to (3.1) is classified into one of the following three types.

Definition 3.1. (i) A solution w to (3.1) is said to be a rapidly decaying solution if $w > 0$ on $[\rho, +\infty)$ and $\tau w(\tau)$ converges to some positive constant as $\tau \to +\infty$.

(ii) A solution w to (3.1) is said to be a slowly decaying solution if $w > 0$ on $[\rho, +\infty)$ and $\tau w(\tau) \to +\infty$ as $\tau \to +\infty$.

(iii) A solution w to (3.1) is said to be a crossing solution if w has a zero in $(\rho, +\infty)$.

Remark 3.1. If w is a rapidly decaying solution, then the solution $u := \tau w$ to (1.6) is a regular solution; if w is a slowly decaying solution, then the solution $u = \tau w$ to (1.6) is a singular solution; if w is a crossing solution, then there exists no regular or singular solution to (1.6) corresponding to w.

Next we state the structure theorem which classifies solutions to (3.1) into one of the above three types. To classify solutions to (3.1), it is effective to use the Pohozaev identity. Define

$$P(\tau; w) := \frac{1}{2} \tau^2 w_\tau \{\tau w_\tau + w\} + \frac{\tau^3}{p+1} L(\tau) w_+^{p+1}. \tag{3.2}$$

The following proposition implies that three types of solutions defined in Definition 3.1 are characterized by using (3.2).

Proposition 3.1. *Under the condition* (L), *let* w *be a solution to* (3.1). *If* w *satisfies* $\liminf_{\tau \to +\infty} P(\tau; w) > 0$, *then* w *is a crossing solution. In contrast if* w *satisfies* $\limsup_{\tau \to +\infty} P(\tau; w) < 0$, *then* w *is a slowly decaying solution.*

If $\rho = 0$, then Proposition 3.1 is identical to Propositions 3.1 and 3.2 in [10]. Moreover we can prove that with $\rho > 0$ by minor modifications.

Before we state the structure theorem, some preliminaries are needed. First we define

$$G(\tau) := \frac{1}{p+1}\left\{\tau^3 L(\tau) - \frac{1}{2}(p+1)\int_\rho^\tau s^2 L(s)\, ds\right\},$$
$$H(\tau) := \frac{1}{p+1}\left\{\tau^{2-p} L(\tau) - \frac{1}{2}(p+1)\int_\tau^{+\infty} s^{1-p} L(s)\, ds\right\}.$$

The above functions are well-defined on $(\rho, +\infty)$ under the condition (L). For G and H, it holds that

$$G_\tau(\tau) = \frac{\tau^{(p+1)/2}}{p+1}(\tau^{-\xi}L)_\tau = \tau^{p+1}H_\tau(\tau) \tag{3.3}$$

with

$$\xi = \frac{p-5}{2}.$$

Hence the functions G and H attain a critical value at the same point. In addition P is associated with G and H by the following lemma.

Lemma 3.2 (Lemma 3.2 in [6]). *Any solution w to* (3.1) *satisfies the identity*

$$\frac{d}{d\tau}P(\tau; w) = G_\tau(\tau) w_+^{p+1}(\tau) \tag{3.4}$$

and its integral form

$$P(\tau; w) = G(\tau) w_+^{p+1}(\tau) - (p+1)\int_\rho^\tau G(s) w_+^p w_s(s)\, ds.$$

By using G and H, we define

$$\tau_G := \inf\{\tau \in [\rho, +\infty) \mid G(\tau) < 0\}, \tag{3.5}$$
$$\tau_H := \sup\{\tau \in [\rho, +\infty) \mid H(\tau) < 0\}. \tag{3.6}$$

Here we define $\tau_G = +\infty$ if $G(\tau) \geq 0$ on $(\rho, +\infty)$ and $\tau_H = \rho$ if $H(\tau) \geq 0$ on $(\rho, +\infty)$. Next we introduce the following condition

$$\text{(G)} \qquad \begin{cases} \text{there exists } \eta_1 \in [\rho, +\infty) \text{ such that} \\ G(\tau) \geq 0 \text{ for } (\rho, \eta_1) \text{ and } G_\tau(\tau) \leq 0 \text{ for } (\eta_1, +\infty). \end{cases}$$

Now we are ready to state the structure theorem of solutions to (3.1).

Proposition 3.2. *Assume* (L) *and* $G \not\equiv 0$ *on* $(\rho, +\infty)$. *Then the following four statements hold.*

(i) *If* $\tau_G = +\infty$, *then the structure of solutions to* (3.1) *is of type C*: $w(\tau;\beta)$ *is a crossing solution for any* $\beta > 0$.

(ii) *If* $\rho = 0$ *and* $\tau_H = 0$, *then the structure of solutions to* (3.1) *is of type S*: $w(\tau;\beta)$ *is a slowly decaying solution for any* $\beta > 0$.

(iii) *If* $\rho < \tau_H \le \tau_G < +\infty$, *then the structure of solutions to* (3.1) *is of type M*: *there exists a constant* $\beta_* > 0$ *such that* $w(\tau;\beta)$ *is a slowly decaying solution for* $\beta \in (0,\beta_*)$, $w(\tau;\beta_*)$ *is a rapidly decaying solution, and* $w(\tau;\beta)$ *is a crossing solution for* $\beta \in (\beta_*,+\infty)$.

(iv) *If* $0 < \rho = \tau_H \le \tau_G < +\infty$ *and G satisfies* (G), *then the structure of solutions to* (3.1) *is of type M.*

Propositions 3.2 (i)–(iii) are already proved, e.g., see Theorem 3.3 in [6]. However Proposition 3.2 (iv) is not shown yet, and thus we prove that below.

Before beginning the proof of Proposition 3.2, we mention some lemmas required to prove Proposition 3.2.

LEMMA 3.3. *Define* $\tau_0 := \inf\{\tau \mid L(\tau) > 0\}$. *Then a solution w to* (3.1) *satisfies* $w_\tau \equiv 0$ *for* $\tau \in (\rho,\tau_0]$ *and* $w_\tau < 0$ *for* $\tau \in (\tau_0,+\infty)$.

Proof. Since w_τ is written by

$$w_\tau(\tau) = -\int_\rho^\tau \left(\frac{s}{\tau}\right)^2 L(s) w_+^p \, ds,$$

this lemma follows. ■

By the following two lemmas, if we find a rapidly decaying solution to (3.1), then we see that the structure of solutions to (3.1) is of type M.

LEMMA 3.4. *Assume* (L) *and* (G). *If w is a rapidly decaying solution to* (3.1), *then it holds that*

$$P(\tau;w) \ge 0 \quad \text{and} \quad P(\tau;w) \not\equiv 0 \quad \text{on } (\rho,+\infty).$$

LEMMA 3.5. *Assume* (L). *If there exists a rapidly decaying solution w to* (3.1) *satisfying*

$$P(\tau;w) \ge 0 \quad \text{and} \quad P(\tau;w) \not\equiv 0 \quad \text{on } (\rho,+\infty),$$

then the structure of solutions to (3.1) *is of type M.*

The next lemma describes behaviors of w and $P(\tau;w)$ as $\beta \to 0$.

LEMMA 3.6. *Assume* (L). *If* $w(\tau;\beta)$ *is a solution to* (3.1), *then there hold* $\lim_{\beta\to 0} \beta^{-1} w(\tau;\beta) = 1$ *and* $\lim_{\beta\to 0} \beta^{-p-1} P(\tau;w) = G(\tau)$ *uniformly in* $[\rho,\eta_2]$, *where* $\eta_2 \in (\rho,+\infty)$ *is an arbitrarily fixed number.*

In the case of $\rho = 0$, Lemmas 3.4–3.6 are identical to Propositions 4.1, 4.2 in [7] and Lemma 2.5 in [10], respectively. Moreover we can show the above lemmas by simple modifications, and we omit precise arguments. While behaviors of w and $P(\tau; w)$ for a small β are described by Lemma 3.6, the following lemma describes the behavior of w for a large β.

Lemma 3.7 (Lemma 3.4 in [6]). *Suppose that $\rho > 0$. If $\beta > 0$ is sufficiently large, then the unique solution $w(\tau; \beta)$ to* (3.1) *is a crossing solution.*

Next, for three types of solutions to (3.1) defined in Definition 3.1, we define three sets of initial data.

Definition 3.2. We define sets of initial data of (3.1) as follows:

$$A_c := \{\beta > 0 \mid w(\tau; \beta) \text{ is a crossing solution to (3.1)}\};$$

$$A_s := \{\beta > 0 \mid w(\tau; \beta) \text{ is a slowly decaying solution to (3.1)}\};$$

$$A_r := \{\beta > 0 \mid w(\tau; \beta) \text{ is a rapidly decaying solution to (3.1)}\}.$$

Remark 3.2. Since every solution to (3.1) is classified into one of three types defined in Definition 3.1, there holds $A_c \cup A_s \cup A_r = (0, +\infty)$.

For sets A_c and A_s, the following lemma holds.

Lemma 3.8. *The set A_c is open in $(0, +\infty)$. Moreover if* (G) *is satisfied, then A_s is open in $(0, +\infty)$.*

In the case of $\rho = 0$, Lemma 3.8 is equivalent to Lemmas 2.6 and 2.7 in [7], and we can show that in the case of $\rho > 0$ by simple modifications. Hence we omit the proof of Lemma 3.8. Now we show Proposition 3.2 (iv).

Proof of Proposition 3.2 (iv). Assume (G) and $0 < \rho \le \tau_H \le \tau_G < +\infty$. Then, from Lemma 3.7, $w(\tau; \beta)$ is a crossing solution for a sufficiently large $\beta > 0$, and hence $A_c \neq \emptyset$.

Next we prove $A_s \neq \emptyset$. From $\tau_G < +\infty$, it holds that $G(\mu) < 0$ for sufficiently large $\mu > \tau_G$. Since G satisfies (G), it follows that $\mu \in (\eta_1, +\infty)$. Hence there holds $G(\mu) < 0$ and $G_\tau(\mu) \le 0$. From Lemma 3.6, there exist a constant $\delta_1 > 0$ and a sufficiently small $\beta_1 > 0$ such that, for any $\beta \in (0, \beta_1)$, it holds that

$$P(\mu; w(\mu; \beta)) \le \beta^{p+1}(G(\mu) + \delta_1) < 0. \tag{3.7}$$

From (3.4) and $G_\tau(\tau) \le 0$ on $(\mu, +\infty)$, it follows that

$$\frac{dP(\tau; w)}{d\tau} = G_\tau(\tau) w_+^{p+1} \le 0 \quad \text{on } [\mu, +\infty). \tag{3.8}$$

From (3.7) and (3.8), it follows that, for any $\beta < \beta_1$,

$$P(\tau; w(\tau;\beta)) \le -\delta_2 < 0 \quad \text{on } [\mu, +\infty),$$

where some $\delta_2 > 0$. By Proposition 3.1, $w(\tau;\beta)$ is a slowly decaying solution for any $\beta < \beta_1$, and hence $A_s \neq \emptyset$.

By Lemma 3.8, it holds that $A_c \cup A_s \neq (0, +\infty)$. Hence, from $A_c \neq \emptyset$, $A_s \neq \emptyset$ and Remark 3.2, we obtain $A_r \neq \emptyset$. Thus, by Lemma 3.4, it follows that, for $w(\tau;\beta)$ with $\beta \in A_r$,

$$P(\tau; w) \ge 0 \quad \text{and} \quad P(\tau; w) \not\equiv 0 \quad \text{on } (\rho, +\infty).$$

Therefore, from Lemma 3.5, the structure of solutions to (3.1) is of type M. The proof is completely finished. ■

Remark 3.3. By similar arguments above, we can confirm that Proposition 3.2 is valid for more general problems

$$\begin{cases} \dfrac{1}{\tau^{N-1}}(\tau^{N-1} w_\tau)_\tau + L(\tau) w_+^p(\tau) = 0 \quad \text{for } \tau \in (\rho, +\infty), \\ w(\rho) = \beta, \\ w_\tau(\rho) = 0, \end{cases}$$

where $N \ge 4$ and $L(\tau)$ satisfies (L) with $\tau^{1-(N-2)p} L(\tau) \in L^1(\rho_*, +\infty)$ instead of $\tau^{1-p} L(\tau) \in L^1(\rho_*, +\infty)$.

4. Proof of Theorem 1.1

In this section we will show Theorem 1.1. Before we start the proof of Theorem 1.1, we prove several lemmas as its preliminary.

First, for our problem (2.4), G and H are written by

$$G(\tau) = \frac{1}{6}\tau^3 K(\tau) - \frac{1}{2}\int_\rho^\tau s^2 K(s)\, ds, \tag{4.1}$$

$$H(\tau) = \frac{1}{6}\tau^{-3} K(\tau) - \frac{1}{2}\int_\tau^{+\infty} s^{-4} K(s)\, ds. \tag{4.2}$$

Here we define

$$k(\theta) := \cos\theta + (\rho - \cot\theta_0)\sin\theta. \tag{4.3}$$

and $K(\tau)$ is written by $K(\tau) = k(\theta)^4$ (see (2.5)). Moreover, from (3.3), it follows that

$$G_\tau(\tau) = \frac{1}{6}\tau^3 K_\tau(\tau) = \tau^6 H_\tau(\tau), \tag{4.4}$$

where

$$K_\tau(\tau) = -4k(\theta)^3 k_\theta(\theta)\sin^2\theta = -4\sin^6\theta(\cot\theta+\rho-\cot\theta_0)^3[-1+(\rho-\cot\theta_0)\cot\theta]. \tag{4.5}$$

Furthermore, as a critical point of G and H, the following lemma holds.

LEMMA 4.1. *If $\theta_0 \in (0, \pi/2]$, then G and H have at most one critical point in $(\rho, +\infty)$. On the other hand, if $\theta_0 \in (\pi/2, \pi)$, then G and H have one critical point or two critical points in $(\rho, +\infty)$.*

Proof. From (4.4), G_τ and H_τ attain a zero at the same point as K_τ. Moreover, to investigate a zero of K_τ, it suffices to consider a zero of kk_θ in $(0, \theta_0)$ (see (4.5)). By direct computation, it follows that

$$(kk_\theta)(\theta) = (A^2+B^2)^{1/2}\sin(2\theta+\nu),$$

where

$$A = \frac{1}{2}\{(\rho-\cot\theta_0)^2-1\}, \quad B = \rho-\cot\theta_0, \quad \tan\nu = \frac{B}{A}. \tag{4.6}$$

If $\theta_0 \in (0, \pi/2]$, then kk_θ attains a zero once or does not attain a zero in $(0, \theta_0)$. On the other hand, if $\theta_0 \in (\pi/2, \pi)$, then kk_θ attains a zero once or twice. Therefore the lemma follows. ∎

Second we show the result on the behavior of G as $\tau \to +\infty$.

LEMMA 4.2. *Under $\theta_0 \in (0, \pi)$, if $\kappa > 2^{-1}\sin 2\theta_0$, then $G(\tau) \to -\infty$ as $\tau \to +\infty$. In contrast if $\kappa \le 2^{-1}\sin 2\theta_0$, then $G(\tau) \to +\infty$ as $\tau \to +\infty$.*

Proof. From (2.2), (4.4) and (4.5), we obtain

$$G_\tau(\tau) = \frac{1}{6}\tau^3 K_\tau(\tau) = -\frac{2}{3}(\cos\theta + B\sin\theta)^6(-1+B\cot\theta).$$

Here we used B defined in (4.6). If $\kappa > 2^{-1}\sin 2\theta_0$, then $B = \rho - \cot\theta_0 > 0$ follows from (2.1). Thus $G_\tau(\tau) \to -\infty$ as $\theta \to 0$ ($\tau \to +\infty$), and therefore $G(\tau) \to -\infty$ as $\tau \to +\infty$. Similarly if $\kappa \le 2^{-1}\sin 2\theta_0$, then $G_\tau(\tau) \to +\infty$ or $2/3$ as $\theta \to 0$. Therefore $G(\tau) \to +\infty$ as $\tau \to +\infty$. ∎

Finally we prove the result on behaviors of G and H near $\tau = \rho$.

LEMMA 4.3. *Assume $\theta_0 \in (0, \pi/2)$. If $0 \le \kappa < \tan\theta_0$, then G and H increase near $\tau = \rho$. On the contrary if $\kappa \ge \tan\theta_0$, then G and H decrease near $\tau = \rho$.*

Proof. From (4.4), it suffices to investigate K_τ near $\tau = \rho$. First suppose that $\kappa = 0$ (then $\rho = 0$). From $\rho = 0$ and (4.5), we obtain

$$K_\tau(\tau) = 4\sin^6\theta(\cot\theta - \cot\theta_0)^3(1 + \cot\theta_0 \cot\theta). \tag{4.7}$$

Since $\theta_0 \in (0, \pi/2)$, it follows that $K_\tau(\tau) > 0$ near $\theta = \theta_0$, that is, $\tau = 0$.

Next suppose that $\kappa > 0$ (then $\rho > 0$). Since $\tau = \rho$ is equivalent to $\theta = \theta_0$, it follows that

$$K_\tau(\rho) = -4\rho^3 \sin^6\theta_0[-1 + (\rho - \cot\theta_0)\cot\theta_0] \tag{4.8}$$

and

$$\begin{aligned} -1 + (\rho - \cot\theta_0)\cot\theta_0 &= \frac{-\sin^3\theta_0 + \kappa\cos\theta_0 - \cos^2\theta_0 \sin\theta_0}{\sin^3\theta_0} \\ &= \frac{\cos\theta_0}{\sin^3\theta_0}(\kappa - \tan\theta_0). \end{aligned} \tag{4.9}$$

If $0 < \kappa < \tan\theta_0$, then we obtain $K_\tau(\rho) > 0$. Therefore, by the continuity of $K(\tau)$, there holds $K_\tau(\tau) > 0$ near $\tau = \rho$. Similarly if $\kappa > \tan\theta_0$, then it holds that $K_\tau(\tau) < 0$ near $\tau = \rho$.

Finally we prove that if $\kappa = \tan\theta_0$, then $K_\tau(\tau) < 0$ near $\tau = \rho$. From (4.9), it follows that

$$-1 + (\rho - \cot\theta_0)\cot\theta_0 = 0. \tag{4.10}$$

In addition (4.10) implies $\rho - \cot\theta_0 > 0$. Thus, since $\cot\theta$ is monotone decreasing in $(0, \theta_0)$, it holds that

$$-1 + (\rho - \cot\theta_0)\cot\theta > 0 \quad \text{for } \theta \in (0, \theta_0). \tag{4.11}$$

Therefore, from (4.5) and (4.11), we see that $K_\tau(\tau) < 0$ near $\theta = \theta_0$ $(\tau = \rho)$, that is, near $\tau = \rho$. This lemma is shown. ■

Now we are ready to show Theorem 1.1 with $0 < \theta_0 \le \pi/2$.

Proof of Theorem 1.1 *with* $0 < \theta_0 \le \pi/2$. First we assume $0 \le \kappa \le 2^{-1}\sin 2\theta_0$. From Lemmas 4.2 and 4.3, it holds that $G(\tau)$ increase near $\tau = \rho$ and $G(\tau) \to +\infty$ as $\tau \to +\infty$. Moreover, from Lemma 4.1, we see that G has no critical point in $(\rho, +\infty)$, that is, G is monotone increasing in $(\rho, +\infty)$. In addition, from (2.5) and (4.1), it holds that

$$G(\rho) = \frac{1}{6}\rho^3 K(\rho) = \frac{1}{6}\rho^7 \sin^4\theta_0 \ge 0. \tag{4.12}$$

Thus we obtain $\tau_G = +\infty$ by (3.5). By Proposition 3.2 (i), the structure of solutions is of type C. Namely, for any β, a solution $w(\tau;\beta)$ to (2.4) is a crossing solution, and hence (1.6) has no regular and no singular solution (see

Definition 3.1 and Remark 3.1). Furthermore we rewrite the above result as κ. Namely $0 \le \kappa \le 2^{-1} \sin 2\theta_0$ with $0 < \theta_0 \le \pi/2$ is equivalent to $\theta_0 \ne 0$ and 2^{-1} Arcsin $2\kappa \le \theta_0 \le 2^{-1}(\pi - \text{Arcsin } 2\kappa)$ with $0 \le \kappa \le 2^{-1}$. Therefore Theorem 1.1 (i–a) with $0 < \theta_0 \le \pi/2$ is proved.

Next we assume $\kappa > 2^{-1} \sin 2\theta_0$. From Lemma 4.2, it follows that $G_\tau(\tau) < 0$ for sufficiently large τ and $\tau_G < +\infty$. In addition, since $G(\rho) > 0$ from (4.12) and G has at most one critical point by (4.1), the function G satisfies (G). Moreover, from (2.5) and (4.2), it follows that, as $\tau \to +\infty$,

$$H(\tau) = \frac{1}{6}\tau^{-3}K(\tau) + o(1) = \frac{1}{6}(\cot\theta + \rho - \cot\theta_0)\sin^4\theta + o(1) \to 0. \tag{4.13}$$

From (4.4) and $G_\tau(\tau) < 0$ for sufficiently large τ, it holds that $H_\tau(\tau) < 0$ for sufficiently large τ, and hence $H(\tau) > 0$ for sufficiently large τ. Thus we obtain $\rho \le \tau_H < +\infty$ from (3.6). Furthermore, since G has at most one critical point and H attains its critical value at the same τ as G, it holds that $0 < \rho \le \tau_H \le \tau_G < +\infty$. Thus we can apply Propositions 3.2 (iii) and (iv), and it is proved that the structure of solutions is of type M. From (2.8), we define $\alpha = \beta(\sin\theta_0)^{-2}$ and $\alpha_* = \beta_*(\sin\theta_0)^{-2}$. Recall that $u(\theta;\alpha)$ is a solution to (1.6) satisfying $u_\theta(\theta_0) = -\alpha$. If $\alpha \in (0, \alpha_*)$, then $u(\theta;\alpha)$ is a singular solution. Moreover $u(r,\alpha)$ is a regular solution for $\alpha = \alpha_*$. On the other hand, if $\alpha \in (\alpha_*, +\infty)$, then a regular or singular solution $u(\theta;\alpha)$ to (1.6) does not exist.

We also rewrite this result as κ. Namely $2^{-1}\sin 2\theta_0 < \kappa \le 2^{-1}$ with $0 < \theta_0 \le \pi/2$ is equivalent to $0 < \theta_0 < 2^{-1}$ Arcsin 2κ or $2^{-1}(\pi - \text{Arcsin } 2\kappa) < \theta_0$ with $0 \le \kappa \le 2^{-1}$. Therefore Theorem 1.1 (i–b) with $0 < \theta_0 \le \pi/2$ is proved. Furthermore, since the structure of solutions is of type M in the case of $\kappa > 2^{-1}$ with $0 < \theta_0 \le \pi/2$, Theorem 1.1 (ii) with $0 < \theta_0 \le \pi/2$ is shown. ■

Next we will prove Theorem 1.1 with $\pi/2 < \theta_0 < \pi$. Instead of Lemma 4.3, the following lemma holds.

LEMMA 4.4. *If $\theta_0 \in (\pi/2, \pi)$, then G and H increase near $\tau = \rho$.*

Proof. If $\kappa > 0$ ($\rho > 0$), then, from (4.8) and (4.9), we obtain $K_\tau(\rho) > 0$. Therefore, by the continuity of $K(\tau)$, it holds that $K_\tau(\tau) > 0$ near $\tau = \rho$. On the other hand, If $\kappa = 0$ ($\rho = 0$), then, from (4.7), $K_\tau(\tau) > 0$ near $\theta = \theta_0$ ($\tau = 0$). ■

Now we prove Theorem 1.1 with $\pi/2 < \theta_0 < \pi$.

Proof of Theorem 1.1 *with* $\pi/2 < \theta_0 < \pi$. From Lemma 4.1, functions G and H have one critical point or two critical points. In addition, by Lemmas 4.2 and 4.4, it holds that $G(\tau) \to -\infty$ as $\tau \to +\infty$ and G and H increase near $\tau = \rho$. Hence G and H have one local maximum point τ_0. Therefore it holds that $\tau_G \in (\tau_0, +\infty)$ and G satisfies (G).

If $\rho > 0$ ($\kappa > 0$), then we obtain $\tau_H \in [\rho, \tau_0)$ from the above arguments. By Propositions 3.2 (iii) and (iv), it is proved that the structure of solutions is of type M.

On the other hand, if $\rho = 0$ ($\kappa = 0$), then, from (2.2), (2.5) and (4.2), it follows that

$$H(0) = \tau \sin^4 \theta|_{\tau=0} - \frac{1}{2}\int_0^{\theta_0} \sin^2 \theta \, d\theta < 0.$$

Since $H(\tau) \to 0$ as $\tau \to +\infty$ (see (4.13)), there holds $H(\tau) > 0$ for $\tau \geq \tau_0$. Hence $\tau_H \in (0, \tau_0)$, and we obtain $0 < \tau_H < \tau_0 < \tau_G < +\infty$. Therefore, by Proposition 3.2 (iii), it is proved that the structure of solutions is of type M. The proof of Theorem 1.1 is completely finished. ■

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