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A MATHEMATICAL THEORY FOR DOUBLE-SLIT EXPERIMENTS OF WALBORN ET AL

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1. Introduction

Walborn et al (2002) performed remarkable experiments on Young's double-slit problem. In their experiments a pair of photons are employed. Photons from an argon ion pump laser are used as a source. A source photon is transformed by a special nonlinear crystal to a pair of photons, the polarizations of which are orthogonal of each other. They use this pair of photons in their experiments which consist of four different experimental setups.

To realize the four experiments they choose, in terms of statistics, four different sub-populations from the total population which consists of all photons from the source. This point has, however, never been clarified anywhere.

We will give a mathematical model for the double-slit problem, in particular for the experiments of Walborn et al, based on a theory of stochastic processes originated by Schrödinger (1931) and developed by Nagasawa (cf. e.g. (1993), (2000), (2002), (2007)).

2. Experiments of Walborn et al

We first explain briefly the experiment performed by Walborn et al (2002).

A photon (351.1 nm) from an argon ion pump laser is used as a source. A special nonlinear crystal called BBO (beta-barium borate) transforms the source photon to a couple of photons of 702.2 nm, the polarizations of which are orthogonal of each other. One photon will be called p-photon and the other one s-photon. They will play different roles in the experiments.

In the figure below explaining their experimental setup, p-photons will go along the upper path, and s-photons will go along the lower path on which the double slit is placed.

If the detector D_p catches a p-photon, it sends a click to the coincidence counter. If the detector D_s then catches a s-photon, it sends a click to the coincidence counter. Once both clicks are detected, we resister "count 1". Such counts will be done 400 seconds. Then detector D_s is moved a millimeter and the number of counts in a 400 second interval is recorded for the new detector

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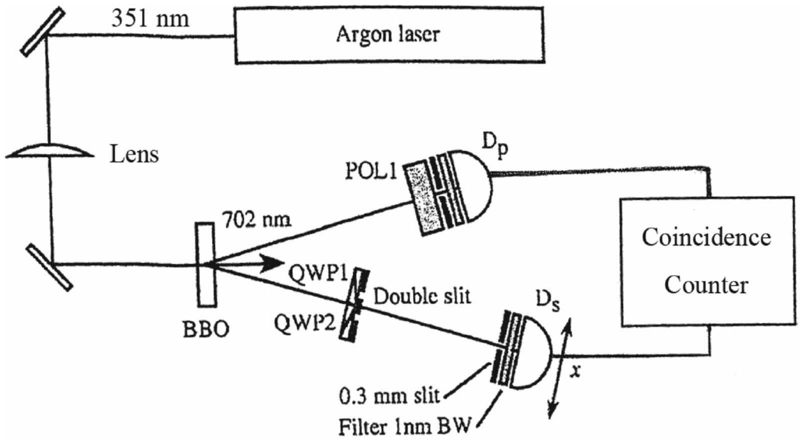
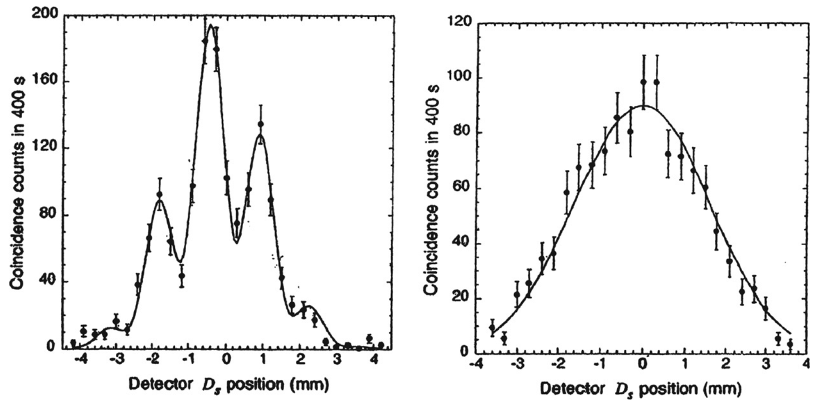
position. This is repeated until D_s has scanned across a region equivalent to the figure below.



In the first experiment, the linear polarizer POL1 in front of detector D_p and the quarter wave plates QWP1 and QWP2 in front of double slit are not placed. Then a stripe-like distribution pattern was observed, as shown at the left-hand side of the following figure.



In the second experiment, the quarter-wave plates QWP1 and QWP2 are placed in front of double slit in the above experimental setup, but the linear polarizer POL1 in front of detector D_p is not placed. Then no stripe-like pattern was observed, as shown at the right-hand side in the above figure.

In the third experiment, the quarter-wave plates QWP1 and QWP2 are placed in front of double slit in the above experimental setup as in the second experiment, and in addition the linear polarizer POL1 in front of detector D_p is inserted. Then stripe-like pattern was observed similar to the figure shown at the left-hand side in the above figure.

In the forth experiment, the experimental setup is the same as in the third experiment, but the polarizer POL1 and detector D_p are placed farther away from BBO crystal so that the path of the p-photon is lengthened. In this experiment if the detector D_s catches an s-photon first and the detector D_p catches a p-photon, we resister "count 1". Then stripe-like pattern was observed as in the third experiment.

3. A theory of stochastic processes

We will give a mathematical model for the double-slit problem, in particular for the experiments of Walborn et al, based on a theory of stochastic processes of Schrödinger (1931) and Nagasawa (1993), (2000), (2002), (2007). We will explain main ideas of the theory quickly.

Let S be d-dimendional Euclidian space. We assume

Postulate. There is a non-negative function called a *transit function*

$$p(s,x,t,y), \quad s,t \in [a,b], \ x,y \in S,$$

which satisfies Chapman-Kolmogorov equation, but in general

$$\int p(a,x,b,y)\,dy \neq 1.$$

Therefore, the transit function $p(s,x,t,y)$ is not a transition probability density.

As will be explained in Appendix, a transit function is the fundamental solution of the equation of motion of stochastic processes, which consists of a pair of evolution equations.

Definition 3.1. We take a pair of nonnegative functions

$$\hat{\phi}_a(x),\ x \in S \quad \text{and} \quad \phi_b(y),\ y \in S,$$

normalized as

$$\int dx\hat{\phi}_a(x)p(a,x,b,y)\phi_b(y)\,dy = 1. \tag{3.1}$$

We call (3.1) the *normality condition* of a triplet $\{p(s,x,t,y), \hat{\phi}_a(x), \phi_b(y)\}$, in which $\hat{\phi}_a(x)$ is an *entrance function* and $\phi_b(y)$ is an *exit function*.

As will be shown, a triplet $\{p(s,x,t,y), \hat{\phi}_a(x), \phi_b(y)\}$ determines a stochastic process. We remark that $\hat{\phi}_a(x)$ is not the initial distribution density of the stochastic process. The initial distribution density $\mu_a(x)$ is given by

$$\mu_a(x) = \hat{\phi}_a(x)\phi(a,x),$$

where

$$\phi(a,x) = \int p(a,x,b,y)\phi_b(y)\,dy$$

is determined by the exit function $\phi_b(y)$.

THEOREM 3.1. *Under the normality condition* (3.1) *of a triplet* $\{p(s,x,t,y), \hat{\phi}_a(x), \phi_b(y)\}$, *there exists a stochastic process* $\{X_t(\omega); t \in [a,b], \mathbf{Q}\}$, *which has the finite dimensional distribution given below*:

For $a \le t_1 \le t_2 \le \cdots \le t_n \le b$ *and for subsets* $\Gamma_1, \Gamma_2, \dots, \Gamma_n$

$$\begin{aligned} &\mathbf{Q}[\{\omega : \omega(t_1) \in \Gamma_1, \omega(t_2) \in \Gamma_2, \dots, \omega(t_n) \in \Gamma_n\}] \\ &\quad = \int dx_0 \hat{\phi}_a(x_0) p(a,x_0,t_1,x_1) 1_{\Gamma_1}(x_1)\,dx_1 p(t_1,x_1,t_2,x_2) 1_{\Gamma_2}(x_2)\,dx_2 \\ &\qquad \cdots p(t_{n-1},x_{n-1},t_n,x_n) 1_{\Gamma_n}(x_n)\,dx_n p(t_n,x_n,b,y)\phi_b(y)\,dy, \end{aligned} \tag{3.2}$$

where $\hat{\phi}_a(x)$ *is an entrance function and* $\phi_b(y)$ *is an exit function.*

Proof. Under the normality condition (3.1), equation (3.2) defines a probability measure $\mathbf{Q}$ on the product space S^n. Therefore, by the extension theorem (cf. e.g. Parthasarathy (1967)), we get a probability measure $\mathbf{Q}$ on the sample space $\Omega = \{\omega(t), t \in [a,b]\}$. We set $X_t(\omega) = \omega(t)$. Then we get a stochastic process $\{X_t(\omega); t \in [a,b], \mathbf{Q}\}$ satisfying (3.2). This completes the proof.

We remark that the normality condition in (3.1) and the finite dimensional distribution in (3.2) are intrinsic properties of the dynamic theory of random motion of Schrödinger (1931) and Nagasawa (1993), (2000), (2002), (2007), and different from those in the conventional theory of Kolmogoroff (1931).

DEFINITION 3.2. We set

$$\hat{\phi}(t,x) = \int dz \hat{\phi}_a(z) p(a,z,t,x),$$

$$\phi(t,x) = \int p(t,x,b,y)\phi_b(y)\,dy,$$

where $\hat{\phi}_a(x)$ is an entrance function and $\phi_b(y)$ is an exit function, and we will call $\phi(t,x)$ *evolution function* and $\hat{\phi}(t,x)$ *backward evolution function.*

THEOREM 3.2. *Let* $\{\hat{\phi}(t,x), \phi(t,x)\}$ *be the pair of evolution functions defined above. Then the distribution density* $\mu_t(x)$ *of the stochastic process* $\{X_t(\omega); t \in [a,b], \mathbf{Q}\}$ *is given by*

$$\mu_t(x) = \hat{\phi}(t,x)\phi(t,x). \tag{3.3}$$

Proof. From equation (3.2) of the finite dimensional distribution, we get

$$\mathbf{Q}[X(t) \in \Gamma] = \int dx_0 \hat{\phi}_a(x_0) p(a, x_0, t, x) 1_\Gamma(x)\, dx p(t, x, b, y) \phi_b(y)\, dy$$
$$= \int \hat{\phi}(t, x) 1_\Gamma(x)\, dx \phi(t, x),$$

which implies (3.3). This completes the proof.

Definition 3.3. With the evolution functions $\phi(t, x)$ and $\hat{\phi}(t, x)$ we define

$$R(t, x) = \frac{1}{2} \log \phi(t, x) \hat{\phi}(t, x), \tag{3.4}$$

$$S(t, x) = \frac{1}{2} \log \frac{\phi(t, x)}{\hat{\phi}(t, x)}. \tag{3.5}$$

We call the pair $\{R(t, x), S(t, x)\}$ the *generating function* of random motion.

Theorem 3.3. *The pair of evolution functions can be written as*

$$\phi(t, x) = e^{R(t,x)+S(t,x)}, \quad \hat{\phi}(t, x) = e^{R(t,x)-S(t,x)}, \tag{3.6}$$

with the generating function $\{R(t, x), S(t, x)\}$ *defined by* (3.4) *and* (3.5).

We will call (3.6) the "*exponential form*" of the evolution functions. We then define a complex-valued function in an exponential form.

Definition 3.4. We define

$$\psi(t, x) = e^{R(t,x)+iS(t,x)}, \tag{3.7}$$

with the generating function $\{R(t, x), S(t, x)\}$ given in Definition 3.3, and call it the *complex evolution function* corresponding to the evolution functions in (3.6).

We remark that the same generating function $\{R(t, x), S(t, x)\}$ determines both of the pair of evolution functions $\{\phi(t, x), \hat{\phi}(t, x)\}$ in (3.6) and the complex evolution function $\psi(t, x)$ in (3.7). Hence they are equivalent. (See Theorem 7.3 in Appendix.)

There is, however, a decisive difference of ability between the complex evolution function $\psi(t, x)$ and the pair of evolution functions $\{\phi(t, x), \hat{\phi}(t, x)\}$. Namely, if a complex evolution function $\psi(t, x)$ is given, one get a generating function $\{R(t, x), S(t, x)\}$ from its exponent, but one can't construct stochastic process from the complex evolution function. To get a stochastic process we need a pair of evolution functions $\{\phi(t, x), \hat{\phi}(t, x)\}$, in particular, the transit function $p(s, x, t, y)$ as shown in Theorem 3.1.

THEOREM 3.4. *Let* $\{\hat{\phi}(t,x), \phi(t,x)\}$ *be the pair of evolution functions in* (3.6) *and* $\psi(t,x)$ *be the complex evolution function in* (3.7). *Then the distribution density* $\mu_t(x)$ *of the stochastic process* $\{X_t(\omega); t \in [a,b], \mathbf{Q}\}$ *is given by*

$$\mu_t(x) = \hat{\phi}(t,x)\phi(t,x) = \psi(t,x)\overline{\psi(t,x)}, \tag{3.8}$$

where $\overline{\psi(t,x)}$ *is the complex conjugate of* $\psi(t,x)$.

Proof. This is clear because $\hat{\phi}(t,x)\phi(t,x) = \psi(t,x)\overline{\psi(t,x)} = e^{2R(t,x)}$.

4. Entangled random motion

The entangled random motion will be applied to the double-slit problem in the next section. We define an "*entangled random motion*" as follows. We take a pair of stochastic processes, and call them "*stochastic process* 1" and "*stochastic process* 2", respectively.

Let

$$\phi_1(t,x) = e^{R_1(t,x)+S_1(t,x)}, \quad \hat{\phi}_1(t,x) = e^{R_1(t,x)-S_1(t,x)}, \tag{4.1}$$

be the evolution functions of the "*stochastic process* 1".

Let

$$\phi_2(t,x) = e^{R_2(t,x)+S_2(t,x)}, \quad \hat{\phi}_2(t,x) = e^{R_2(t,x)-S_2(t,x)}, \tag{4.2}$$

be the evolution functions of the "*stochastic process* 2".

Corresponding to the pairs of evolution functions in (4.1) and (4.2), we introduce a "*complex evolution function* 1" defined with $\{R_1, S_1\}$

$$\psi_1(t,x) = e^{R_1(t,x)+iS_1(t,x)},$$

and also a "*complex evolution function* 2" with $\{R_2, S_2\}$

$$\psi_2(t,x) = e^{R_2(t,x)+iS_2(t,x)}.$$

We then define a linear combination:

$$\psi^*(t,x) = \beta(\psi_1(t,x) + \psi_2(t,x)), \tag{4.3}$$

where β is a normalizing constant so that $\int |\psi^*(t,x)|^2\,dx = 1$, and represent $\psi^*(t,x)$ in an exponential form

$$\psi^*(t,x) = e^{R^*(t,x)+iS^*(t,x)}. \tag{4.4}$$

To write $\psi^*(t,x)$ in the exponential form is of fundamental importance in our discussion of the entanglement.

DEFINITION 4.1. (i) We will call $\psi^*(t,x) = e^{R^*(t,x)+iS^*(t,x)}$ in (4.4) *entangled complex evolution function*, and the pair $\{R^*(t,x), S^*(t,x)\}$ in the exponent of $\psi^*(t,x)$ in (4.4) *entangled generating function*.

(ii) With the entangled generating function $\{R^*(t,x), S^*(t,x)\}$ we define a pair of functions by

$$\phi^*(t,x) = e^{R^*(t,x)+S^*(t,x)}, \quad \hat{\phi}^*(t,x) = e^{R^*(t,x)-S^*(t,x)}, \tag{4.5}$$

which will be called *entangled evolution functions*.

The entangled evolution functions in (4.5) determine an entangled random motion $\{X_t(\omega); t \in [a,b], \mathbf{Q}^*\}$. For this we apply Theorem 7.4 to the entangled generating function $\{R^*(t,x), S^*(t,x)\}$. Then we get a transit function $p^*(s,x,t,y)$, and a stochastic process $\{X_t(\omega); t \in [a,b], \mathbf{Q}^*\}$ satisfying

$$\hat{\phi}^*(t,x) = \int dz \hat{\phi}_a^*(z) p^*(a,z,t,x),$$

$$\phi^*(t,x) = \int p^*(t,x,b,y) \phi_b^*(y)\, dy,$$

where $\hat{\phi}_a^*(z)$ is an entrance function and $\phi_b^*(y)$ is an exit function for the entangled motion.

Definition 4.2. We will call the random motion $\{X_t(\omega); t \in [a,b], \mathbf{Q}^*\}$ obtained above which has the transit function $p^*(s,x,t,y)$, the entrance function $\hat{\phi}_a^*(x)$ and the exit function $\phi_b^*(y)$ *entangled stochastic process*.

In applications later on, various entangled motion will appear. This means that we will adopt various pairs of stochastic processes for the entanglement, or equivalently different pairs of complex evolution functions $\{\psi_1(t,x), \psi_2(t,x)\}$.

We note that these entangled stochastic processes can be distinguished by indicating the difference of their exit functions.

Theorem 4.1. *Define an entangled complex evolution function* $\psi^*(t,x)$ *by* (4.4) *and a pair of entangled evolution functions* $\phi^*(t,x)$ *and* $\hat{\phi}^*(t,x)$ *by* (4.5). *Then*

$$\phi^*(t,x)\hat{\phi}^*(t,x) = \psi^*(t,x)\overline{\psi^*(t,x)}, \tag{4.6}$$

where $\overline{\psi^*(t,x)}$ *is the complex conjugate of* $\psi^*(t,x)$.

Proof. This is clear because both sides of (4.6) equal to a distribution density $e^{2R^*(t,x)}$.

5. Entanglement by the double-slit

We now regard the "stochastic process 1" and the "stochastic process 2" as the motion of a particle (s-photon) going through slit 1 and slit 2, respectively.

At the double slit, a particle chooses slit 1 *or slit* 2 *at random, and hence the entanglement occurs.*

Then the motion of a particle after the double slit is described by the entangled stochastic process $\{X_t(\omega); t \in [a,b], \mathbf{Q}^*\}$ defined by Definition 4.2.

We apply the formula in (3.2) of the finite dimensional distribution to the entangled stochastic process $\{X_t(\omega); t \in [a,b], \mathbf{Q}^*\}$. We then get

THEOREM 5.1. *The distribution density of the stochastic process* $\{X_t(\omega); t \in [a,b], \mathbf{Q}^*\}$ *entangled by the double slit is*

$$\mu_t^*(x) = \hat{\phi}^*(t,x)\phi^*(t,x). \tag{5.1}$$

Proof. This is nothing but Theorem 3.2 applied to the entangled stochastic process.

Since $\phi^*(b,x) = \phi_b^*(x)$, the distribution density of a particle at $t = b$ (in applications, an s-photon counted at the detector D_s) is

$$\mu_b^*(x) = \hat{\phi}^*(b,x)\phi_b^*(x). \tag{5.2}$$

As will be explained in Section 6, different exit functions will appear in discussing the experiments of Walborn et al.

THEOREM 5.2. *Let* $\{X_t(\omega); t \in [a,b], \mathbf{Q}^*\}$ *be the stochastic process entangled by double slit. Then the distribution density* $\mu_t^*(x)$ *is given by*

$$\begin{aligned} \mu_t^*(x) &= \beta^2(e^{2R_1(t,x)} + e^{2R_2(t,x)}) \\ &\quad + 2\beta^2 e^{R_1(t,x)+R_2(t,x)} \cos(S_1(t,x) - S_2(t,x)). \end{aligned} \tag{5.3}$$

Proof. By (5.1), (4.3) and (4.6) the distribution density of the entangled motion is

$$\begin{aligned} \mu_t^*(x) &= \phi^*(t,x)\hat{\phi}^*(t,x) = |\psi^*(t,x)|^2 \\ &= \beta^2|\psi_1(t,x) + \psi_2(t,x)\}|^2 \\ &= \beta^2|\psi_1(t,x)|^2 + \beta^2|\psi_2(t,x)|^2 \\ &\quad + 2\beta^2\frac{1}{2}\{\psi_1(t,x)\overline{\psi_2(t,x)} + \overline{\psi_1(t,x)}\psi_2(t,x)\}, \end{aligned}$$

where $\psi_1(t,x) = e^{R_1(t,x)+iS_1(t,x)}$, $\psi_2(t,x) = e^{R_2(t,x)+iS_2(t,x)}$ and

$$|\psi_1(t,x)|^2 = e^{2R_1(t,x)}, \quad |\psi_2(t,x)|^2 = e^{2R_2(t,x)},$$

$$\begin{aligned} \frac{1}{2}\{\psi_1(t,x)\overline{\psi_2(t,x)} + \overline{\psi_1(t,x)}\psi_2(t,x)\} &= \Re(\psi_1(t,x)\overline{\psi_2(t,x)}) \\ &= e^{R_1(t,x)+R_2(t,x)} \cos(S_1(t,x) - S_2(t,x)). \end{aligned}$$

This completes the proof.

We remark that the distribution density $\mu_t^*(x)$ in (5.3) of the entangled stochastic process $\{X_t(\omega); t \in [a,b], \mathbf{Q}^*\}$ has a distribution with stripe-like pattern induced by the cross-term, if $S_1(t,x) - S_2(t,x)$ varies as a function of x.

6. Application of the mathematical theory to the double slit experiments of Walborn et al

As explained in Section 2, a photon (351.1 nm) from an argon ion pump laser is transformed by a special nonlinear crystal called BBO to a couple of photons of 702.2 nm, the polarizations of which are orthogonal of each other. These photons will be employed in the experiments, but they will play different roles. The one named s-photon will be detected by the detector D_s after going through double slit. The other named p-photon will be detected by the detector D_p and play a special role. Namely, for each of four experiments we will pick up an appropriate "sub-population" from the "total population" which consists of all photons from the source. The sub-population consisting of s-photons which should be counted by the detector D_s will be chosen with the help of the p-photon and the coincidence counter in each of four experiments, as will be explained.

We remark that the use of the notion "*population*" and "*sub-population*" was not made in Walborn et al (2002).

Experiment 1.

In this experiment, POL1 in front of detector D_p and QWP1 and QWP2 in front of double slit are not placed in the experimental setup shown in Section 2. This is a standard double spit experiment in which we use all photons from the source.

Definition 6.1. We define the "*total population*" by

$$\{\mathbf{TP}\} = \{\text{all } p\text{-photons from the source}\}.$$

In the experiment 1 the total population $\{\mathbf{TP}\}$ is adopted, and a distribution with a stripe-like pattern was observed.

Applying our mathematical theory, we take the pair of evolution functions $\{\hat{\phi}^*(t,z;\tau_p),\ \phi^*(t,z;\tau_p)\}$ entangled by double slit, where τ_p denotes the polarization of the p-photon, and z is the space variable of the s-photon. We ignore the space variable of the p-photon, since it plays no role. We will ignore the polarization of the s-photon, when it will play no role. (To avoid confusion with the polarization parameters, we use z for the space variable.)

Then the distribution density of an s-photon is

$$\mu_t^*(z) = \hat{\phi}^*(t,z;\tau_p)\phi^*(t,z;\tau_p),$$

by Theorem 5.1. The exit function is

$$\phi_b^*(z) = \phi^*(b,z;\tau_p)$$

which is determined by the entanglement by the double slit and does not depend on the polarization τ_p of the p-photon. In this experiment the p-photon plays no significant role, and the distribution density of an s-photon counted by the detector D_s is given by

$$\mu_b^*(z) = \hat{\phi}^*(b, z; \tau_p)\phi_b^*(z), \tag{6.1}$$

where z is the space variable of the s-photon. Here we can ignore τ_p, since it plays no role.

We have used the total population $\{\mathbf{TP}\}$ without selection in this experiment. Therefore, we can apply Theorem 5.2 and conclude that the cross-term of the evolution functions through slit 1 and slit 2 induce a stripe-like pattern in the distribution by the entanglement.

This experiment is a photon version of Young's experiment. It should be emphasized, however, that we treat no "*wave*", but a "*particle* (photon)".

Experiment 2.

In the second experiment, QWP1 and QWP2 are placed in front of double slit, but POL1 is not placed in front of detector D_p.

If an s-photon is x-polarized, then QWP1 in front of slit 1 transforms it to a left circularly polarized photon, and QWP2 in front of slit 2 transforms the s-photon to a right circularly polarized photon.

And if an s-photon is y-polarized, then QWP1 in front of slit 1 transforms it to a right circularly polarized photon, and QWP2 in front of slit 2 transforms the s-photon to a left circularly polarized photon.

Working setup. To clarify things, in front of detector D_p we place a linear polarizer oriented so that only y-polarized p-photons go through. (This was not done in the experiment of Walborn et al.) Then s-photons which are x-polarized in front of QWP1 and QWP2 will be counted by the detector D_s.

Definition 6.2. We define "*sub-population* 2" by

$$\{\mathbf{SP}\}_2 = \{p\text{-photons which are } y\text{-polarized}\}.$$

We adopt the sub-population $\{\mathbf{SP}\}_2$ in the experiment 2. In this experiment the stripe-like pattern was not observed. (See Remark 1 below.)

Suppose an s-photon is caught by the detector D_s. If the polarization of the corresponding p-photon is not y-polarized, then it will not go through the linear polarizer, hence will not be detected by the detector D_p. Therefore, we won't count such an s-photon, even though the detector D_s caught it.

Therefore, the s-photon which is counted was x-polarized in front of QWP1 and QWP2, since the polarization of the s-photon is orthogonal to that of the corresponding p-photon in $\{\mathbf{SP}\}_2$. Hence the s-photon was transformed by QWP1 and QWP2 to a left circularly polarized photon and a right circularly polarized photon, respectively.

Walborn et al (2002) wrote that since a photon through slit 1 and a photon through slit 2 have orthogonal polarizations, there is no probability of inducing a stripe-like distribution. (However, we remark that "the left circular polarization and the right circular polarization of photons are orthogonal" is not obvious. No proof for this statement was given in Walborn et al (2002).)

Moreover, we can see which slit an s-photon went through, if we measure the polarization of the s-photon at the detector D_s. Suppose the s-photon is x-polarized. If it is left circularly polarized at the detector D_s, then the s-photon came through slit 1. If it is right circularly polarized at the detector D_s, then the s-photon came through slit 2.

We now apply our mathematical model. For the experiment 2 we must have

$$1_{\{\mathbf{SP}\}_2}(\tau_p)\phi_b^*(z)$$

as the exit function, where $\phi_b^*(z)$ is the exit function of an s-photon at the detector D_s, which depends on the polarization of s-photon, and $1_{\{\mathbf{SP}\}_2}(\tau_p)$ is the indicator function of the sub-population $\{\mathbf{SP}\}_2$, that is, if τ_p is y-polarized, then $1_{\{\mathbf{SP}\}_2}(\tau_p) = 1$, and $1_{\{\mathbf{SP}\}_2}(\tau_p) = 0$, otherwise. Namely, an s-photon detected by the detector D_s is counted if and only if the corresponding p-photon is in the sub-population $\{\mathbf{SP}\}_2$. We remark that the exit function and the entrance function must be normalized as in (3.1) by multiplying a constant.

Therefore, the distribution density of an s-photon counted at the detector D_s is

$$\mu_b^*(z) = \hat{\phi}^*(b, z; \tau_p) 1_{\{\mathbf{SP}\}_2}(\tau_p)\phi_b^*(z). \tag{6.2}$$

Since the exit function is $1_{\{\mathbf{SP}\}_2}(\tau_p)\phi_b^*(z)$, the s-photon which is counted at the detector D_s was x-polarized in front of QWP1 and QWP2. Then QWP1 and QWP2 transform the s-photon into a left circularly polarized photon and a right circularly polarized photon, respectively.

Then the complex evolution functions $\psi_{+1}(t, r, \eta)$ and $\psi_{-1}(t, r, \eta)$ of "stochastic process 1" and "stochastic process 2", which describe the s-photon going through slit 1 and slit 2, have factors $e^{i\eta}$ and $e^{-i\eta}$, respectively, where (r, η) denotes two dimensional polar coordinate. The factors $e^{i\eta}$ and $e^{-i\eta}$ induce the left circular motion and the right circular motion, respectively, see Section 8. In particular, we look at the dependence on the variable η, since the polarization of photons plays an essential role in this experiment.

Let

$$\psi(t, r, \eta) = \beta(\psi_{+1}(t, r, \eta) + \psi_{-1}(t, r, \eta))$$

be the entangled complex evolution function, where β is a normalizing constant. Then it is easy to see that the distribution density $\psi(t, r, \eta)\overline{\psi}(t, r, \eta)$ has a factor $(1 - \cos 2\eta)$, which vanishes at $\eta = 0$ and $\eta = \pi$. Therefore, the distribution density is separated into two, namely $0 < \eta < \pi$ and $\pi < \eta < 2\pi$. Hence paths of an s-photon through slit 1 and slit 2 are separated, and no stripe-like pattern appears.

Remark 1. In the experiment of Walborn et al "a linear polarizer oriented so that only y-polarized p-photons go through" is actually not placed in front of detector D_p. Therefore, their source is not the sub-population $\{\mathbf{SP}\}_2$ but the total population $\{\mathbf{TP}\}$ which is mixed, hence the observed distribution shown in Section 2 as a figure (on the right-hand side) was somewhat blurred, but their experiment is essentially the same as explained above.

Experiment 3.
In the third experiment, QWP1 and QWP2 are placed in front of double-slit, namely, the experimental setup for s-photons is the same as in the experiment 2.

In the experiment 3, however, a linear polarizer POL1 is placed in front of detector D_p in addition, and the POL1 is oriented (suitably for QWP1 or QWP2) so that it will pass p-photons which are linearly polarized of a combination of x and y.

Because of the linear polarizer POL1 placed in front of detector D_p, the polarization of the p-photon that will be counted is a combination of x and y.

Definition 6.3. We define "*sub-population* 3" by

$$\{\mathbf{SP}\}_3 = \{p\text{-photons whose polarizations are a combination of } x \text{ and } y\}.$$

In the experiment 3 the sub-population $\{\mathbf{SP}\}_3$ is adopted. In this experiment a stripe-like pattern was observed.

Suppose an s-photon is caught by the detector D_s. If the polarization of the corresponding p-photon is not a combination of x and y, then it will not go through POL1, hence will not be detected by the detector D_p. Therefore, we won't count such an s-photon, even though the detector D_s caught it.

We note that the polarization of an s-photon which will be counted is also a combination of x and y in front of QWP1 and QWP2, since it is orthogonal to that of the corresponding p-photon in $\{\mathbf{SP}\}_3$.

We now apply our mathematical model. For the experiment 3 we must have

$$1_{\{\mathbf{SP}\}_3}(\tau_p)\phi_b^*(z)$$

as the exit function. Then the distribution density of an s-photon at the detector D_s is

$$\mu_b^*(z) = \hat{\phi}^*(b, z; \tau_p) 1_{\{\mathbf{SP}\}_3}(\tau_p)\phi_b^*(z). \tag{6.3}$$

As noted above, the polarization of the s-photon which will be counted is a combination of x and y. Hence QWP1 and QWP2 placed in front of double slit won't transform the s-photon to a circularly polarized photon. Accordingly, the "stochastic process 1" and "stochastic process 2" have polarizations τ_{s1} and τ_{s2}

which are not orthogonal. Therefore, the cross term of the evolution functions through slit 1 and slit 2 induce stripe-like pattern by the entanglement.

Remark 2. The experiment 2 and experiment 3 are not standard double-slit experiments. They are further detailed experiments. In other words, they are a sort of *paired* experiments. As a matter of fact, the experimental setup of the experiment 2 and experiment 3 for s-photons are exactly the same. Moreover, we do not touch s-photons in the experiment 2 neither in the experiment 3.

Nevertheless, the observed distributions are completely different.

This is caused as a result that, by placing POL1, the sub-population is changed from

$$\{\mathbf{SP}\}_2 = \{p\text{-photons which are } y\text{-polarized}\},$$

to the other sub-population

$$\{\mathbf{SP}\}_3 = \{p\text{-photons whose polarizations are a combination of } x \text{ and } y\},$$

that is, different statistics are used in the experiment 2 and experiment 3. This point about different statistics was, however, not mentioned in Walborn et al (2002).

In our mathematical model, equation (6.2) which has the exit function $1_{\{\mathbf{SP}\}_2}(\tau_p)\phi_b^*(z)$ is changed to equation (6.3) which has the other exit function $1_{\{\mathbf{SP}\}_3}(\tau_p)\phi_b^*(z)$, namely, we use different entangled stochastic processes for the experiment 2 and experiment 3, respectively.

Remark 3. In Walborn et al (2002), QWP1 and QWP2 placed in front of double slit are called "marker", and POL1 placed in front of detector D_p "eraser". However, QWP1 and QWP2 alone can't mark "which way". "To tell which way" we need to know the polarization of the s-photon in front of QWP1 and QWP2, that is, x-polarized or y-polarized, which we can fix by changing the orientation of the linear polarizer POL1. As a matter of fact, POL1 in front of detector D_p plays both roles of "marker" and "eraser" by suitably adjusting its orientation. Hence it is probably better to call QWP1 and QWP2 with $\{\mathbf{SP}\}_2$ "marker", and with $\{\mathbf{SP}\}_3$ "eraser" in experiments 2 and 3.

Experiment 4.

The experimental setup is the same as in the third experiment, but the polarizer POL1 and detector D_p are placed farther away from BBO crystal so that the path of the p-photon is lengthened.

In this experiment if the detector D_s catches an s-photon first and the detector D_p then catches a p-photon, we resister "count 1". Such counts are repeated as in other experiments.

Since only the order of detection of s-photons and p-photons is exchanged, the sub-population of this experiment is exactly the same as in the experiment 3.

Definition 6.4. We define "*sub-population* 4" by

$$\{\mathbf{SP}\}_4 = \{p\text{-photons whose polarizations are a combination of } x \text{ and } y\}.$$

In the experiment 4 the sub-population $\{\mathbf{SP}\}_4$ is adopted. In this experiment a stripe-like pattern was observed.

Suppose an s-photon is caught by the detector D_s. If the polarization of the corresponding p-photon is not a combination of x and y, then it will not go through POL1, hence will not be detected by the detector D_p. Therefore, we won't count such an s-photon, even though the detector D_s caught it. To count or not to count is decided by the sub-population $\{\mathbf{SP}\}_4$. Namely, exchanging the order of detection of s-photons and p-photons plays no role.

We now apply our mathematical model. Since $\{\mathbf{SP}\}_4 = \{\mathbf{SP}\}_3$, the distribution density of an s-photon which we count at the detector D_s is

$$\mu_b^*(z) = \hat{\phi}^*(b, z; \tau_p) 1_{\{\mathbf{SP}\}_4}(\tau_p)\phi_b^*(z) = \hat{\phi}^*(b, z; \tau_p) 1_{\{\mathbf{SP}\}_3}(\tau_p)\phi_b^*(z). \tag{6.4}$$

We get naturally a stripe-like distribution pattern as in the experiment 3.

Remark 4. There exists a sort of explanation on the experiments of Walborn et al by using "uncertainty principle" (Heisenberg (1927)) and "non-locality" (Bell (1964)), (or "Bohr's complementarity"). But this is an erroneous explanation.

In fact, we have clarified in the present paper that "uncertainty principle" and "non-locality" play no role in the double-slit problem, and that "Bohr's complementarity" has no place in our discussion.

We remark, moreover, that Heisenberg's uncertainty principle and Bell's non-locality claim are both incorrect (cf. Nagasawa (1997), (2009), (2012)).

For the double-slit experiment in another experimental setup different from that of Walborn et al, we refer to Scully-Drühl (1982) and Kim, Scully et al. (2000).

Appendix

7. Equation of motion and equation of paths

Main theorems of the dynamic theory of random motion of a particle that have been applied in the preceding sections will be explained, details for which we refer to Nagasawa (1993), (2000), (2003), (2007), (2012*).

In the dynamic theory of random motion, the *equation of motion* is given by

$$\begin{aligned} &\frac{\partial \phi}{\partial t} + \frac{1}{2}\sigma^2 \triangle \phi + c(t, x)\phi = 0, \\ -&\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2}\sigma^2 \triangle \hat{\phi} + c(t, x)\hat{\phi} = 0, \end{aligned} \tag{7.1}$$

where $a \leq t \leq b$, σ^2 is a constant and $c(t,x)$ is a scaler potential. (Cf. e.g. Nagasawa (1993), (2000), (2002), (2007), (2012*).)

Let $p(s,x,t,y)$ be the fundamental solution of (7.1). The function $p(s,x,t,y)$ is a *transit function* in Postulate made in Section 3.

Then under the normality condition of a triplet $\{p(s,x,t,y), \hat{\phi}_a(x), \phi_b(y)\}$ there exists a stochastic process $\{X_t(\omega); t \in [a,b], \mathbf{Q}\}$ by Theorem 3.1.

THEOREM 7.1. (i) *Let $p(s,x,t,y)$ be the transit function determined by the equation of motion* (7.1), *and take an evolution function $\phi(s,x) = e^{R(t,x)+S(t,x)}$. Then*

$$q(s,x,t,y) = \frac{1}{\phi(s,x)} p(s,x,t,y)\phi(t,y)$$

is the transition probability density of the random motion $\{X_t(\omega); t \in [a,b], \mathbf{Q}\}$.

(ii) *The function $q(s,x,t,y)$ is the fundamental solution of an evolution equation*

$$\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2 \Delta u + \boldsymbol{a}(s,x) \cdot \nabla u = 0, \tag{7.2}$$

where

$$\boldsymbol{a}(t,x) = \sigma^2 \frac{\nabla \phi(t,x)}{\phi(t,x)} = \sigma^2(\nabla R(t,x) + \nabla S(t,x)) \tag{7.3}$$

is the evolution drift determined by the evolution function $\phi(s,x) = e^{R(t,x)+S(t,x)}$.

Equation (7.2) is called the *equation of kinematics* of random motion.

Proof. It is clear that $q(s,x,t,y)$ is a transition probability density. On the right-hand side of (3.2), multiply $\phi(t_i,x_i)$ and divide by $\phi(t_i,x_i)$ at each dx_i. Then the right-hand side of the formula (3.2) can be written as

$$\begin{aligned}
&= \int dx_0 \hat{\phi}_a(x_0)\phi(a,x_0) \frac{1}{\phi(a,x_0)} p(a,x_0,t_1,x_1)\phi(t_1,x_1)\, dx_1 \\
&\quad \times \frac{1}{\phi(t_1,x_1)} p(t_1,x_1,t_2,x_2)\phi(t_2,x_2)\, dx_2 \\
&\quad \cdots \frac{1}{\phi(t_n,x_n)} p(t_n,x_n,b,y)\phi_b(y)\, dy f(x_1,\ldots,x_n) \\
&= \int dx_0 \hat{\phi}_a(x_0)\phi(a,x_0) q(a,x_0,t_1,x_1)\, dx_1 q(t_1,x_1,t_2,x_2)\, dx_2 \\
&\quad \cdots dx_{n-1} q(t_{n-1},x_{n-1},t_n,x_n)\, dx_n q(t_n,x_n,b,y)\, dy f(x_1,\ldots,x_n),
\end{aligned}$$

denoting $\mu_a(x) = \hat{\phi}_a(x_0)\phi(a,x_0)$, we have therefore

$$\mathbf{Q}[f(X_{t_1},\dots,X_{t_n})] = \int dx_0\mu_a(x_0)q(a,x_0,t_1,x_1)\,dx_1q(t_1,x_1,t_2,x_2)\,dx_2$$
$$\cdots dx_{n-1}q(t_{n-1},x_{n-1},t_n,x_n)\,dx_nf(x_1,\dots,x_n),$$

which proves the assertion (i).

LEMMA 7.1 (Nagasawa (1989)). *Let $p(s,x,t,y)$ be the fundamental solution of* (7.1) *and $\phi(s,x)$ be an evolution function. Then*

$$u(s,x) = \frac{p(s,x,t,y)}{\phi(s,x)}$$

satisfies

$$Lu + \sigma^2\frac{1}{\phi}\nabla\phi\cdot\nabla u = \frac{1}{\phi}(Lp + c(s,x)p) - \frac{u}{\phi}(L\phi + c(s,x)\phi), \tag{7.4}$$

where

$$L = \frac{\partial}{\partial s} + \frac{1}{2}\sigma^2\triangle.$$

Since the right hand side of (7.4) vanishes, the assertion (ii) follows from Lemma 7.1. This completes the proof of Theorem 7.1.

THEOREM 7.2. (i) *Let $\{X_t(\omega); t\in[a,b],\mathbf{Q}\}$ be the stochastic process determined by a triplet $\{p(s,x,t,y),\hat{\phi}_a(x),\phi_b(y)\}$, where $p(s,x,t,y)$ is the transit function determined by the equation of motion* (7.1). *Then the paths* (*trajectories*) *of the random motion of a particle are given by a stochastic differential equation*

$$X_t = X_a + \sigma B_{t-a} + \int_a^t \sigma^2\nabla\log\phi(s,X_s)\,ds, \tag{7.5}$$

where $\sigma^2\nabla\log\phi = \sigma^2\dfrac{\nabla\phi}{\phi}$ is the evolution drift determined by the evolution function $\phi(t,x)$, B_t is a d-dimensional Brownian motion, and X_a is a random variable independent of the Brownian motion B_t.

(ii) *The distribution density $\mu_t(x)$ of the stochastic process X_t is given by*

$$\mu_t(x) = \phi(t,x)\hat{\phi}(t,x) = e^{2R(t,x)}, \quad t\in[a,b].$$

Equation (7.5) is called the *equation of paths* of random motion.

THEOREM 7.3. *The following two assertions are equivalent*

(i) *evolution functions*

$$\phi(t,x) = e^{R(t,x)+S(t,x)}, \quad \hat{\phi}(t,x) = e^{R(t,x)-S(t,x)}$$

satisfy the equation of motion (7.1);

(ii) *a complex evolution function*

$$\psi(t,x) = e^{R(t,x)+iS(t,x)}$$

satisfies the complex evolution equation

$$\frac{\partial \psi}{\partial t} - i\left(\frac{1}{2}\sigma^2 \triangle - V(t,x)\right)\psi = 0; \tag{7.6}$$

under a condition

$$V(t,x) + c(t,x) + \tilde{V}(t,x) = 0,$$

where

$$\tilde{V}(t,x) = 2\frac{\partial S}{\partial t} + \sigma^2(\nabla S)^2.$$

Remark 5. Schrödinger interpreted equation (7.6) as a complex-valued wave equation (the "Schrödinger equation") in his wave mechanics, cf. Schrödinger (1926). His "wave-interpretation" was, however, a mistake. The equation (7.6) is *not a wave equation but an evolution equation in our dynamic theory of random motion*. It is the equation which generates semi-groups. In (7.6) the generator is complex-valued. For the theory of semi-groups of operators, see Yosida (1948). We note in this context that Feynman (1948) regarded the Schrödinger equation as a complex-valued evolution equation and developed a complex stochastic theory, which has no relation to our dynamic theory of random motion.

THEOREM 7.4. *Let* $\{R(t,x), S(t,x)\}$ *be given and set*

$$\phi(t,x) = e^{R(t,x)+S(t,x)}, \quad \hat{\phi}(t,x) = e^{R(t,x)-S(t,x)}.$$

Then there exist a transit function $p(s,x,t,y)$, *and a stochastic process*

$$\{X_t(\omega); t \in [a,b], \mathbf{Q}\}$$

such that the probability measure $\mathbf{Q}$ *satisfies* (3.2) *and*

$$\hat{\phi}(t,x) = \int dz \hat{\phi}_a(z) p(a,z,t,x),$$

$$\phi(t,x) = \int p(t,x,b,y)\phi_b(y)\, dy,$$

where $\hat{\phi}_a(z) = \hat{\phi}(a,z)$ *and* $\phi_b(y) = \phi(b,y)$.

Proof. Let $\{X_t(\omega); t \in [a,b], \mathbf{Q}\}$ be a solution of a stochastic differential equation

$$X_t = X_a + \sigma B_{t-a} + \int_a^t \sigma^2 \nabla \log \phi(s, X_s)\, ds,$$

where the distribution density of X_a is $\hat{\phi}_a(x)\phi(a,x)$. Let $q(s,x,t,y)$ be the transition probability density of $\{X_t(\omega),\mathbf{Q}\}$, and set

$$p(s,x,t,y) = \phi(t,x)q(s,x,t,y)\frac{1}{\phi(t,y)}$$

Then $p(s,x,t,y)$ is a transit function and it is easy to see that $\{X_t(\omega),\mathbf{Q}\}$ satisfies equation (3.2) with the transit function $p(s,x,t,y)$ defined above. This completes the proof.

8. Theory of a photon

One can't apply Maxwell's theory to the motion of "*a single photon*". This is clear. Nevertheless, so far as my knowledge, there is no theory of motion of a single photon.

"*A theory of motion of a photon*" based on the dynamic theory of random motion will be given.

In the dynamic theory of random motion, the *equation of motion*

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\sigma^2\triangle\phi + c(x)\phi = 0,$$

$$-\frac{\partial\hat{\phi}}{\partial t} + \frac{1}{2}\sigma^2\triangle\hat{\phi} + c(x)\hat{\phi} = 0,$$

determines the motion of a particle, where $a \le t \le b$, and $c(x)$ is a scalar function.

The coefficient σ^2 in the above equations is determined by the mass m of a particle, namely, $\sigma^2 = \dfrac{1}{m}\dfrac{h}{2\pi}$, where h is the Planck constant.

Since a photon has no mass, we postulate that the coefficient σ^2 is determined by the energy of a photon.

We note that if the frequency of an electro-magnetic field is ν, a photon in it has no frequency but the energy $h\nu$. Therefore, we set $\sigma^2 = h\nu$, which is the intensity of random motion.

We consider a photon moving along the z-axis with the speed c, and assume that it makes random motion with the Hooke potential on the xy-plane orthogonal to z. The random motion on the xy-plane carries photon's energy.

The lowest energy: For linearly polarized motion of a photon in the xy-plane we consider first of all the equation of motion

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial x^2} - \frac{1}{2}\kappa^2x^2\phi = 0,$$

$$-\frac{\partial\hat{\phi}}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2\hat{\phi}}{\partial x^2} - \frac{1}{2}\kappa^2x^2\hat{\phi} = 0,$$

and the stationary motion. Then we have the evolution functions

$$\phi(t,x) = \beta e^{(\sigma\kappa t/2)-(\kappa x^2/2\sigma)}, \quad \hat{\phi}(t,x) = \beta e^{-(\sigma\kappa t/2)-(\kappa x^2/2\sigma)},$$

of the lowest energy. By (7.3) the evolution drift $a(x)$ is given by

$$a(x) = \sigma^2 \frac{1}{\phi(t,x)} \frac{\partial\phi(t,x)}{\partial x} = -\sigma\kappa x.$$

Hence the equation of kinematics is

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - \sigma\kappa x \frac{\partial u}{\partial x} = 0, \tag{8.1}$$

by (7.2). Therefore, the equation of paths is

$$x_t = x_a + \sigma B_{t-a} - \sigma\kappa \int_a^t ds x_s, \tag{8.2}$$

by (7.5). This is a random harmonic oscillation.

Then we have

THEOREM 8.1. (i) *Let x_t be given by* (8.2). *Then in the xy-plane there is a random motion of the lowest energy $X_t = (x_t, \alpha x_t)$, where α is a real constant. This is a random harmonic oscillation on a linear line $y = \alpha x$ in the xy-plane. The motion orthogonal to this is $X_t = (-\alpha x_t, x_t)$, which is a random harmonic oscillation on a linear line $-\alpha y = x$.*

(ii) *As a special case $\alpha = 0$, the x-polarized random harmonic oscillation is given by $X_t = (x_t, 0)$. The motion orthogonal to this is $X_t = (0, x_t)$, which is the y-polarized random harmonic oscillation.*

The first exited energy: More generally the *equation of motion of a photon* is

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\sigma^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + c\frac{\partial\phi}{\partial z} - \frac{1}{2}\kappa^2(x^2+y^2)\phi = 0,$$
$$-\frac{\partial\hat{\phi}}{\partial t} + \frac{1}{2}\sigma^2\left(\frac{\partial^2\hat{\phi}}{\partial x^2} + \frac{\partial^2\hat{\phi}}{\partial y^2}\right) + c\frac{\partial\phi}{\partial z} - \frac{1}{2}\kappa^2(x^2+y^2)\hat{\phi} = 0.$$

By separating variables, the equation of motion along the z-axis is

$$\frac{\partial\phi}{\partial t} + c\frac{\partial\phi}{\partial z} = 0,$$

$$-\frac{\partial\hat{\phi}}{\partial t} + c\frac{\partial\phi}{\partial z} = 0.$$

Therefore, it is a uniform motion with the speed c.

The equation of motion in the xy-plane is

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-\frac{1}{2}\kappa^2(x^2+y^2)\phi=0,$$
$$-\frac{\partial \hat{\phi}}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2}\right)-\frac{1}{2}\kappa^2(x^2+y^2)\hat{\phi}=0.$$

We then consider one more mode of motion of a photon of the first exited energy.

We now use the polar coodinates (r,η) in two dimensions. Then the equation of motion given above is

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial \eta^2}\right)-\frac{1}{2}\kappa^2r^2\phi=0,$$
$$-\frac{\partial \hat{\phi}}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{\phi}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\hat{\phi}}{\partial \eta^2}\right)-\frac{1}{2}\kappa^2r^2\hat{\phi}=0.$$

We consider the stationary motion. For the quantum numbers $m=\pm1$, $n=0$ we have the eigenvalue $\lambda=2\sigma\kappa$, and a complex evolution function

$$\psi_{\pm1}(t,r,\eta)=\beta r e^{-(\kappa/2\sigma)r^2+i(-2\sigma\kappa t\pm\eta)},$$

(cf. e.g. Pauling-Wilson (1935)). We introduce notations

$$R=\log r-\frac{\kappa}{2\sigma}r^2,\quad S_{\pm1}=-2\sigma\kappa t\pm\eta,$$

and write $\psi_{\pm1}$ in the exponential form

$$\psi_{\pm1}(t,r,\eta)=\beta e^{R+iS_{\pm1}}.$$

We remark, however, that from the complex evolution function we can't get random motion.

To get random motion we need evolution functions

$$\phi_{\pm1}(t,r,\eta)=\beta e^{R+S_{\pm1}},\quad \hat{\phi}_{\pm1}(t,r,\eta)=\beta e^{R-S_{\pm1}},$$

which are equivalent to the complex evolution function $\psi_{\pm1}(t,r,\eta)$. We note that they are determined by the same pair $\{R,S_{\pm1}\}$.

We consider the case $m=+1$, namely

$$\psi_{+1}(t,r,\eta)=\beta r e^{-(\kappa/2\sigma)r^2+i(-2\sigma\kappa t+\eta)}=\beta e^{R+iS_{+1}},$$

and set

$$\phi_{+1}(t,r,\eta)=\beta e^{R+S_{+1}}=\beta r e^{-(\kappa/2\sigma)r^2+(-2\sigma\kappa t+\eta)},$$
$$\hat{\phi}_{+1}(t,r,\eta)=\beta e^{R-S_{+1}}=\beta r e^{-(\kappa/2\sigma)r^2-(-2\sigma\kappa t+\eta)}.$$

Then it can be shown that they satisfy the equation of motion

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial \eta^2}\right)-\left(\frac{1}{2}\kappa^2 r^2+\tilde{V}(r)\right)\phi=0,$$

$$-\frac{\partial \hat{\phi}}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{\phi}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\hat{\phi}}{\partial \eta^2}\right)-\left(\frac{1}{2}\kappa^2 r^2+\tilde{V}(r)\right)\hat{\phi}=0,$$

with an additional potential

$$\tilde{V}(r)=\sigma^2\frac{1}{r^2}-4\sigma\kappa,$$

and the distribution density is given by

$$\mu(t,r,\eta)=\phi_{+1}(t,r,\eta)\hat{\phi}_{+1}(t,r,\eta)=\beta^2 r^2 e^{-(\kappa/\sigma)r^2}.$$

Moreover, by (7.3), the evolution drift determined by the evolution function $\phi_{+1}=\beta e^{R+S_{+1}}$ is

$$a^r(r,\eta)=\sigma^2\frac{1}{\phi_{+1}}\frac{\partial \phi_{+1}}{\partial r}=\sigma^2\frac{\partial R}{\partial r}=\sigma^2\frac{1}{r}-\sigma\kappa,$$

$$a^\eta(r,\eta)=\sigma^2\frac{1}{\phi_{+1}}\frac{1}{r}\frac{\partial \phi_{+1}}{\partial \eta}=\sigma^2\frac{1}{r}\frac{\partial S_{+1}}{\partial \eta}=\sigma^2\frac{1}{r}.$$

Hence the equation of kinematics is

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \eta^2}\right)+\left(\sigma^2\frac{1}{r}-\sigma\kappa r\right)\frac{\partial u}{\partial r}+\sigma^2\frac{1}{r}\frac{1}{r}\frac{\partial u}{\partial \eta}=0,$$

by (7.2).

Therefore, the equation of paths in the polar coordinates (r,η) is

$$r_t=r_a+\sigma B^1_{t-a}+\int_a^t ds\left(\frac{3}{2}\sigma^2\frac{1}{r_s}-\sigma\kappa r_s\right),$$

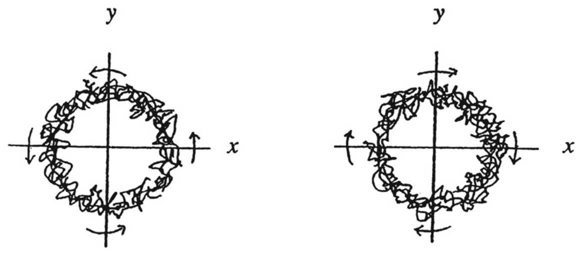
$$\eta_t=\eta_a+\int_a^t \sigma\frac{1}{r_s}\,dB^2_{s-a}+\int_a^t \sigma^2\frac{1}{r_s^2}\,ds,$$

by (7.5), where B^1_t and B^2_t are independent one-dimensional Brownian motions.

Looking at the radial motion r_t, we see that drift $\frac{3}{2}\sigma^2\frac{1}{r}-\sigma\kappa r$ has a zero $\bar{r}=\sqrt{\frac{3}{2}\frac{\sigma}{\kappa}}$, and drift is positive, if $r<\bar{r}$, and negative if $r>\bar{r}$. Therefore, $\bar{r}$ is an attractive point.

The angular motion η_t has anti-clockwise drift σ^2/r_s^2.

Therefore, $X_t=(r_t,\eta_t)$ makes anti-clockwise random circular motion as illustrated at the left-hand side of the figure below.



$m = 1$ $m = -1$

We consider the case $m = -1$, namely

$$\psi_{-1}(t, r, \eta) = \beta r e^{-(\kappa/2\sigma)r^2 + i(-2\sigma\kappa t - \eta)} = \beta e^{R + iS_{-1}},$$

and set

$$\phi_{-1}(t, r, \eta) = \beta e^{R + S_{-1}} = \beta r e^{-(\kappa/2\sigma)r^2 + (-2\sigma\kappa t - \eta)},$$
$$\hat{\phi}_{-1}(t, r, \eta) = \beta e^{R - S_{-1}} = \beta r e^{-(\kappa/2\sigma)r^2 - (-2\sigma\kappa t - \eta)}.$$

The evolution drift determined by the evolution function $\phi_{-1} = \beta e^{R + S_{-1}}$ is

$$a^r(r, \eta) = \sigma^2 \frac{1}{\phi_{-1}} \frac{\partial \phi_{-1}}{\partial r} = \sigma^2 \frac{\partial R}{\partial r} = \sigma^2 \frac{1}{r} - \sigma\kappa,$$

$$a^\eta(r, \eta) = \sigma^2 \frac{1}{\phi_{-1}} \frac{1}{r} \frac{\partial \phi_{-1}}{\partial \eta} = \sigma^2 \frac{1}{r} \frac{\partial S_{-1}}{\partial \eta} = -\sigma^2 \frac{1}{r}.$$

Therefore, the equation of paths in the polar coordinates (r, η) is

$$r_t = r_a + \sigma B^1_{t-a} + \int_a^t ds \left(\frac{3}{2} \sigma^2 \frac{1}{r_s} - \sigma\kappa r_s \right),$$

$$\eta_t = \eta_a + \int_a^t \sigma \frac{1}{r_s} \, dB^2_{s-a} - \int_a^t \sigma^2 \frac{1}{r_s^2} \, ds,$$

where B^1_t and B^2_t are independent one-dimensional Brownian motions. The angular motion η_t has clockwise drift $-\sigma^2/r_s^2$ in this case.

Therefore, $X_t = (r_t, \eta_t)$ makes clockwise random circular motion as illustrated at the right-hand side of the figure above.

Thus we have shown

Theorem 8.2. *There are three modes of random motion in the xy-plane. Let x_t be given by* (8.2).

The first one is $X_t = (x_t, 0)$ and $X_t = (0, x_t)$. These are the x-polarized and y-polarized random harmonic oscillation, respectively.

The second one is $X_t = (x_t, \alpha x_t)$, where α is a real constant. This is a random harmonic oscillation on a linear line $y = \alpha x$ in the xy-plane. The motion orthogonal to this is $X_t = (-\alpha x_t, x_t)$. This is a random harmonic oscillation on a linear *line* $-\alpha y = x$. (*These random motion are called sometimes "a combination of* x *and* y *polarized motion"*.)

The third one is $X_t = (r_t, \eta_t)$ *in the polar coordinates, which makes anti-clockwise (and clockwise) random circular motion.*

We regard these three modes of random motion on the xy-plane as the mode of the polarization of a photon. For instance, if the motion is on the x-axis, a photon is "x-polarized". If the motion is on a linear line in the xy-plane, a photon is "linearly polarized" by a combination of x and y. If a photon makes left circular random motion, the photon is "left circularly polarized". If a photon makes right circular random motion, the photon is "right circularly polarized".

We remark that the photon model given in this section is applicable to the double-slit experiment of Walborn et al discussed in Section 6.

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References

[1] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics **1** (1964), 195–202.

[2] R. P. Feynman, Space-time approach to non-relativistic quantum mechanics, Reviews of Modern Phys. **22** (1948), 367–387.

[3] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, ZS. für Phys. **43** (1927), 172–198.

[4] Y-H. Kim, R. Yu, S. P. Kulik, Y. Shih and M. O. Scully, Delayed "choice" quantum eraser, Phy. Rev. letters **84** (2000), 1–5.

[5] A. Kolmogoroff, Über die Analytischen Methoden in Wahrscheinlichkeitsrechnung, Math. Ann. **104** (1931), 415–458.

[6] M. Nagasawa, Transformations of diffusion and Schrödinger processes, Probab. Th. Rel. Fields **82** (1989), 109–136.

[7] M. Nagasawa, Schrödinger equations and diffusion theory, Monographs in mathematics **86**, Birkhäuser Verlag, Basel, Boston, Berlin, 1993.

[8] M. Nagasawa, On the locality of hidden variable theories in quantum physics, Chaos, Solitons and Fractals **8** (1997), 1773–1792.

[9] M. Nagasawa, Stochastic processes in quantum physics, Monographs in mathematics **94**, Birkhäuser Verlag, Basel, Boston, Berlin, 2000.

[10] M. Nagasawa, On quantum particles, Chaos, Solitons & Fractals **13** (2002), 1393–1405.

[11] M. Nagasawa, Dynamic theory of stochastic movement of systems, Stochastic economic dynamics (B. J. Jensen and T. Palokangas, eds.), Copenhagen Business School Press, 2007, 133–164.

[12] M. Nagasawa, A note on the expectation and deviation of physical quantities, Chaos, Solitons & Fractals **39** (2009), 2311–2315.

[13] M. NAGASAWA, On Heisenberg's inequality and Bell's inequality, Kodai Math. J. **35** (2012), 33–51.
[14] M. NAGASAWA, A new theory of quantum physics by theory of Markov processes, Sanseido Shoten, Tokyo, 2012 (In Japanese).
[15] K. R. PARTHASARATHY, Probability measures on metric spaces, Academic press, New York and London, 1967.
[16] L. PAULING AND E. B. WILSON, Introduction to quantum mechanics with applications to chemistry, McGraw-Hill Book Co. Inc., New York, 1935.
[17] E. SCHRÖDINGER, Quantisierung als Eigenwertproblem (4. Mitteilung), Ann. der Physik **81** (1926), 109–139.
[18] E. SCHRÖDINGER, Über die Umkehrung der Naturgesetze. Sitzungsberichte der preussischen Akad, der Wissenschaften Physikalisch-Mathematische Klasse, 1931, 144–153.
[19] M. O. SCULLY AND K. DRÜHL, Quantum eraser: A proposed photon correlation experiment concerning observation and "delayed choice" in quantum mechanics, Phy. Rev. A. **25** (1982), 2208–2213.
[20] S. P. WALBORN, M. O. T. CUNHA, M. O. PADUA AND C. H. MONKEN, Double-slit quantum eraser, Phy. Rev. A. **65** (2002), 033818-1–033818-6.
[21] K. YOSIDA, On the differentiability and representation of one-parameter semi-group of linear operators, J. Math. Soc. Japan **1** (1948), 15–21.

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