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ASYMPTOTIC BEHAVIOUR OF GOOD SYSTEMS OF PARAMETERS OF SEQUENTIALLY GENERALIZED COHEN-MACAULAY MODULES

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Abstract

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. A finitely generated R-module M is called sequentially generalized Cohen-Macaulay module if there is a filtration $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ of submodules of M such that $0 = \dim M_0 < \dim M_1 < \cdots < \dim M_t$ and each M_i/M_{i-1} is a generalized Cohen-Macaulay module. In this paper we study the asymptotic behavior of good systems of parameters, introduced in [N. T. Cuong, D. T. Cuong, On sequentially Cohen-Macaulay modules, Kodai Math. J. 30 (2007), 409–428], of sequentially generalized Cohen-Macaulay modules.

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and let M be a finitely generated R-module of dimension $d > 0$. Let $\underline{x} = x_1, \dots, x_d$ be a system of parameters (s.o.p. for short) of M. It is well known that M is a Cohen-Macaulay module if $\ell(M/\underline{x}M) = e(\underline{x}; M)$ for every s.o.p. $\underline{x}$, where $e(\underline{x}; M)$ is the Serre multiplicity of M relative to $\underline{x}$. A generalization of Cohen-Macaulay module is the notion of *generalized Cohen-Macaulay module* introduced in [5] by N. T. Cuong, P. Schenzel and N. V. Trung: M is a generalized Cohen-Macaulay module if the differences $I_M(\underline{x}) = \ell(M/\underline{x}M) - e(\underline{x}; M)$ for every s.o.p. $\underline{x}$ are bounded above. In this case the number $I(M) = \sup_{\underline{x}:\text{s.o.p.}} I_M(\underline{x})$ is called *the Buchsbaum invariant of* M, and it holds $I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(M))$, where $H^i_{\mathfrak{m}}(M)$ is the i-th local cohomology module with respect to $\mathfrak{m}$. Moreover, there exists a large enough integer n ($n \gg 0$ for short) such that $I_M(\underline{x}) = I(M)$ for every s.o.p. $\underline{x} \subseteq \mathfrak{m}^n$ ([11, Proposition 2.10]). A s.o.p. $\underline{x}$ satisfying $I_M(\underline{x}) = I(M)$ is called *standard*. An R-module M is called *Buchsbaum* if every parameter ideal is standard. We recall that the

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index of reducibility of a parameter ideal $(\underline{x})$ on M is defined as $N_R((\underline{x}), M) = \dim_{R/\mathfrak{m}} \operatorname{Soc}(M/(\underline{x})M)$, where $\operatorname{Soc}(N) \cong 0 :_N \mathfrak{m} \cong \operatorname{Hom}(R/\mathfrak{m}, N)$ for an arbitrary R-module N. It is well-known that if M is a Cohen-Macaulay module, then $N_R((\underline{x}), M)$ is independent of the choice of $\underline{x}$. If M is a Buchsbaum module, S. Goto and H. Sakurai have proved that $N_R((\underline{x}), M)$ is independent of the choice of $\underline{x} \subseteq \mathfrak{m}^n$ with $n \gg 0$ (cf. [7]). N. T. Cuong and H. L. Truong in [6, Theorem 1.1] extended Goto and Sakurai's result for generalized Cohen-Macaulay modules. Recently, N. T. Cuong and the author reproved this result, based on a splitting theorem for local cohomology ([4, Theorem 1.1, Corollary 4.2]).

Another generalization of Cohen-Macaulay module is the notion of *sequentially Cohen-Macaulay module* introduced for graded rings by R. P. Stanley in [10] and for modules over local rings by P. Schenzel in [9], and by N. T. Cuong and L. T. Nhan in [3]. In [3], the authors also introduced the notion of *sequentially generalized Cohen-Macaulay modules*: M is called a sequentially (generalized) Cohen-Macaulay module if there is a filtration $\mathscr{F} : M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ of submodules of M such that $0 = \dim M_0 < \dim M_1 < \cdots < \dim M_t$ and each M_i/M_{i-1} is (generalized) Cohen-Macaulay. Such a filtration $\mathscr{F}$ is called a (generalized) Cohen-Macaulay filtration. In order to study sequentially (generalized) Cohen-Macaulay modules we can consider a filtration $\mathscr{F} : M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ of submodules of M that satisfies *the dimension condition* that $0 = \dim M_0 < \dim M_1 < \cdots < \dim M_t = d$. For such a filtration $\mathscr{F}$ with $d_i = \dim M_i$, the notion of *good system of parameters with respect to* $\mathscr{F}$ introduced in [1] is useful for the study of sequential (generalized) Cohen-Macaulayness of M. A s.o.p. $\underline{x} = x_1, \ldots, x_d$ is called *good* with respect to $\mathscr{F}$ if $M_i \cap (x_{d_i+1}, \ldots, x_d)M = 0$ for all $0 \le i \le t-1$. Then, $x_1, \ldots, x_{d_i}$ is a s.o.p. of M_i for every $0 \le i \le t-1$. In [1], N. T. Cuong and D. T. Cuong considered the difference

$$I_{\mathscr{F},M}(\underline{x}) = \ell(M/(\underline{x})M) - \sum_{i=0}^{t} e(x_1, \ldots, x_{d_i}; M_i).$$

They proved that $I_{\mathscr{F},M}(\underline{x})$ is non-negative for every s.o.p. $\underline{x}$ that is good with respect to $\mathscr{F}$, and that M is a sequentially Cohen-Macaulay module if and only if $I_{\mathscr{F},M}(\underline{x}^{\underline{n}}) = 0$ for some (and therefore all) s.o.p. $\underline{x}$ that is good with respect to $\mathscr{F}$, where $\underline{x}^{\underline{n}} = x_1^{n_1}, \ldots, x_d^{n_d}$ for any d-tuple of positive integers $\underline{n} = (n_1, \ldots, n_d)$ ([1, Theorem 4.2]). Furthermore, N. T. Cuong and D. T. Cuong proved in [2] that M is a sequentially generalized Cohen-Macaulay if and only if there are a filtration $\mathscr{F}$ and a good s.o.p. $\underline{x}$ such that $I_{\mathscr{F},M}(\underline{x}^{\underline{n}})$ is constant for all $n_1, \ldots, n_d \gg 0$ ([2, Theorems 3.8, 5.2]).

The purpose of this paper is to show that the method used in [4] can be applied to study the asymptotic behavior of good s.o.p. of sequentially generalized Cohen-Macaulay modules. For a sequentially generalized Cohen-Macaulay module M with a generalized Cohen-Macaulay filtration $\mathscr{F}$ we show that $I_{\mathscr{F},M}(\underline{x})$ and $N_R((\underline{x}), M)$ are independent of the choice of good s.o.p. $\underline{x}$ of M with respect to $\mathscr{F}$ contained in $\mathfrak{m}^n$ with $n \gg 0$. The main results will be proved in Section 3. In the next Section we recall briefly some facts about filtrations satisfying the dimension condition, good systems of parameters, and (sequentially) generalized Cohen-Macaulay modules.

2. Preliminaries

Let $\underline{x} = x_1, \ldots, x_d$ be a s.o.p. of M. We consider the difference $I_M(\underline{x}) = \ell(M/\underline{x}M) - e(\underline{x}; M)$, where $e(\underline{x}; M)$ denotes the Serre multiplicity of M relative to $\underline{x}$. Set $I(M) = \sup_{\underline{x}} I_M(\underline{x})$ where the supremum is taken over all systems of parameters of M.

Definition 2.1. An R-module M is called a *generalized Cohen-Macaulay module* if $I(M) < \infty$.

Some basic facts of generalized Cohen-Macaulay modules can be found in [5] and [11].

Remark 2.2. Let M be a generalized Cohen-Macaulay module. Then:

(i) There exists an integer n such that $I_M(\underline{x}) = I(M)$ for every s.o.p. $\underline{x} \subseteq \mathfrak{m}^n$.

(ii) The i-th local cohomology $H^i_{\mathfrak{m}}(M)$ has finite length for all $i < d$, and

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(M)).$$

(iii) (cf. [6, Theorem 1.1], [4, Corollary 4.2]) The *index of reducibility* of a parameter ideal $(\underline{x})$ on M is defined by $N_R((\underline{x}), M) = \dim_{R/\mathfrak{m}} \operatorname{Soc}(M/(\underline{x})M)$, where $\operatorname{Soc}(N) \cong 0 :_N \mathfrak{m} \cong \operatorname{Hom}(R/\mathfrak{m}, N)$ for an arbitrary R-module N. Then,

$$N_R((\underline{x}), M) = \sum_{i=0}^{d} \binom{d}{i} \dim_{R/\mathfrak{m}} \operatorname{Soc}(H^i_{\mathfrak{m}}(M))$$

for every s.o.p. $\underline{x} \subseteq \mathfrak{m}^n$ with $n \gg 0$.

Next we recall briefly some basic facts about filtrations satisfying the dimension condition and good s.o.p. as defined in [1].

Definition 2.3.

(i) We say that a finite filtration

$$\mathscr{F} : M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$$

of submodules of M satisfies the *dimension condition* if $\dim M_0 < \dim M_1 < \cdots < \dim M_t$, and then $\mathscr{F}$ is said to have the length t. For convenience, we always consider that $\dim M_1 > 0$.

(ii) A filtration of submodules $\mathscr{D} : D_0 \subseteq D_1 \subseteq \cdots \subseteq D_t = M$ of M is called the *dimension filtration* of M if the following two conditions are satisfied:
   (a) D_{i-1} is the largest submodule of D_i with $\dim D_{i-1} < \dim D_i$ for $i = t, t-1, \ldots, 1$.
   (b) $D_0 = H^0_{\mathfrak{m}}(M)$ is the 0-th local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$.

Definition 2.4. Let $\mathscr{F} : M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ be a filtration satisfying the dimension condition. Put $d_i = \dim M_i$. A s.o.p. $\underline{x} = x_1, \ldots, x_d$ of M is called a *good system of parameters with respect to* $\mathscr{F}$ if $M_i \cap (x_{d_i+1}, \ldots, x_d)M = 0$ for $i = 0, 1, \ldots, t-1$. A good s.o.p. with respect to the dimension filtration is simply called a good s.o.p. of M.

Remark 2.5 (see, [1]).
(i) The dimension filtration always exists and is unique. We will always denote the dimension filtration of M by $\mathscr{D} : D_0 \subseteq D_1 \subseteq \cdots \subseteq D_t = M$.
(ii) A good s.o.p. of M is a good s.o.p. with respect to every filtration satisfying the dimension condition.

Let $\mathscr{F} : M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ be a filtration satisfying the dimension condition with $d_i = \dim M_i$, and let $\underline{x} = x_1, \ldots, x_d$ be a good s.o.p. of M with respect to $\mathscr{F}$. It is clear that $x_1, \ldots, x_{d_i}$ is a s.o.p. of M_i for all $i \leq t$. Therefore, we can define

$$I_{\mathscr{F},M}(\underline{x}) = \ell(M/(\underline{x})M) - \sum_{i=0}^{t} e(x_1, \ldots, x_{d_i}; M_i),$$

where $e(x_1, \ldots, x_{d_i}; M_i)$ is the Serre multiplicity and $e(x_1, \ldots, x_{d_0}; M_0) = \ell(M_0)$ if $\dim M_0 = 0$. It should be noted here that $I_{\mathscr{F},M}(\underline{x})$ is non-negative for every good s.o.p. $\underline{x}$ with respect to $\mathscr{F}$ ([1, Lemma 2.6]). Finally, we recall the notions of generalized Cohen-Macaulay filtrations and sequentially generalized Cohen-Macaulay modules, and their relation to $I_{\mathscr{F},M}(\underline{x})$.

Definition 2.6. Let $\mathscr{F} : M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ be a filtration of submodules of M. Then, $\mathscr{F}$ is called a *generalized Cohen-Macaulay filtration* if it satisfies the dimension condition, $\dim M_0 = 0$ and $M_1/M_0, \ldots, M_t/M_{t-1}$ are generalized Cohen-Macaulay modules. Moreover, M is called a *sequentially generalized Cohen-Macaulay module* if it has a generalized Cohen-Macaulay filtration.

Of course, every generalized Cohen-Macaulay module of dimension d is a sequentially generalized Cohen-Macaulay module with the generalized Cohen-Macaulay filtration $\mathscr{F} : 0 \subseteq M$. The following results can be found in [2].

Remark 2.7. Let M be a sequentially generalized Cohen-Macaulay module with a generalized Cohen-Macaulay filtration $\mathscr{F} : M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$. Then:

(i) The dimension filtration $\mathscr{D}$ of M is a generalized Cohen-Macaulay filtration of length t. Moreover $\ell(D_i/M_i) < \infty$ for all $i \leq t$.

(ii) M_0 and $H^j_{\mathfrak{m}}(M/M_i)$ have finite length for all $i \leq t-1$ and all $j \leq \dim M_{i+1} - 1$.

(iii) Put $I_{\mathscr{F}}(M) = \sup_{\underline{x}} I_{\mathscr{F},M}(\underline{x})$, where the supremum is taken over the set of good systems of parameters of M with respect to $\mathscr{F}$. Then,

$$I_{\mathscr{F}}(M) = \ell(H^0_{\mathfrak{m}}(M/M_0)) + \sum_{i=0}^{t-1} \sum_{j=1}^{d_{i+1}-1} \left(\binom{d_{i+1}-1}{j} - \binom{d_i - 1}{j} \right) \ell(H^j_{\mathfrak{m}}(M/M_i)).$$

Moreover, if $\underline{x}$ is a good s.o.p. of M with respect to $\mathscr{F}$, then $I_{\mathscr{F},M}(\underline{x}^{\underline{n}}) = I_{\mathscr{F}}(M)$ for all $n_1, \ldots, n_d \gg 0$.

3. The main results

First, we recall some results from [4] that play the key role in this paper. Suppose we are given an integer t, an ideal $\mathfrak{a}$ of R and a submodule U of M. Set $\overline{M} = M/U$. We say that an element $x \in \mathfrak{a}$ satisfies the condition (#) if $0 :_M x = U$ and the short exact sequence

$$0 \to \overline{M} \xrightarrow{x} M \to M/xM \to 0$$

induces short exact sequences

$$0 \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M/xM) \to H^{i+1}_{\mathfrak{a}}(\overline{M}) \to 0$$

for all $i < t-1$. In this is the case, we consider the above exact sequence as an extension of $H^i_{\mathfrak{a}}(M)$ by $H^{i+1}_{\mathfrak{a}}(\overline{M})$, therefore as an element of $\operatorname{Ext}^1_R(H^{i+1}_{\mathfrak{a}}(\overline{M}), H^i_{\mathfrak{a}}(M))$ (see [8, Chapter 3]). We denote this element by E^i_x. Especially, if $H^t_{\mathfrak{a}}(\overline{M}) \cong H^t_{\mathfrak{a}}(M)$, then we have a short exact sequence

$$0 \to H^{t-1}_{\mathfrak{a}}(M) \to H^{t-1}_{\mathfrak{a}}(M/xM) \to 0 :_{H^t_{\mathfrak{a}}(\overline{M})} x \to 0.$$

Suppose that the short exact sequence above induces a short exact sequence

$$0 \to 0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a} \to 0 :_{H^{t-1}_{\mathfrak{a}}(M/xM)} \mathfrak{a} \to 0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a} \to 0.$$

Then, we can consider this exact sequence as an element of $\operatorname{Ext}^1_R(0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}, 0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a})$, denoted by F^{t-1}_x. It should be noted here that an extension of R-module A by an R-module C is split if it is the zero-element of $\operatorname{Ext}^1_R(C, A)$.

Lemma 3.1 ([4] Theorem 2.2). *Let M, U, $\overline{M}$, $\mathfrak{a}$ and t be as above, and let $x, y \in \mathfrak{a}$. Then, the following statements are true.*

(i) *Suppose that x, y satisfy the condition (#), and that $0 :_M (x+y) = U$. Then, $x+y$ also satisfies the condition (#), and $E^i_{x+y} = E^i_x + E^i_y$ for all*

$i < t-1$. *Furthermore, if* $H^t_{\mathfrak{a}}(\overline{M}) \cong H^t_{\mathfrak{a}}(M)$ *and* F^{t-1}_x *and* F^{t-1}_y *are determined, then* F^{t-1}_{x+y} *is also determined, and we have* $F^{t-1}_{x+y} = F^{t-1}_x + F^{t-1}_y$.

(ii) *Suppose that* x *satisfies the condition* (#) *and that* $0 :_M xy = U$. *Then,* xy *also satisfies the condition* (#), *and* $E^i_{xy} = yE^i_x$ *for all* $i < t-1$. *Moreover, if* $H^t_{\mathfrak{a}}(\overline{M}) \cong H^t_{\mathfrak{a}}(M)$ *and* F^{t-1}_x *is determined, then* F^{t-1}_{xy} *is also determined and* $F^{t-1}_{xy} = yF^{t-1}_x$. *Especially, if* $yH^i_{\mathfrak{a}}(M) = 0$ *for all* $i < t$, *then* $F^{t-1}_{xy} = E^i_{xy} = 0$ *for all* $i < t-1$.

Lemma 3.2 ([4] Lemma 3.1). *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring,* $\mathfrak{a}$, $\mathfrak{b}$ *ideals and* $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ *prime ideals such that* $\mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}_j$ *for all* $j \le n$. *Let* $x \in \mathfrak{a}\mathfrak{b}$ *with* $x \notin \mathfrak{p}_j$ *for all* $j \le n$. *Then, there are elements* $a_1, \dots, a_r \in \mathfrak{a}$, $b_1, \dots, b_r \in \mathfrak{b}$ *such that* $x = a_1b_1 + \cdots + a_rb_r$, *and that* $a_ib_i \notin \mathfrak{p}_j$ *and* $a_1b_1 + \cdots + a_ib_i \notin \mathfrak{p}_j$ *for all* $i \le r$ *and all* $j \le n$.

For the rest of this paper, let M be a sequentially generalized Cohen-Macaulay module of dimension $d > 0$ with a generalized Cohen-Macaulay filtration $\mathscr{F} : M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$. Let $\mathscr{D} : H^0_{\mathfrak{m}}(M) = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_t = M$ be the dimension filtration of M. Set $\mathfrak{c}_i = \operatorname{Ann} M_i$ for all $i = 0, \dots, t$, and let n_0 be a positive integer such that $\mathfrak{m}^{n_0} H^j_{\mathfrak{m}}(M/M_i) = 0$ for all $i \le t-1$ and all $j \le d_{i+1} - 1$.

Lemma 3.3. *Let* $x \in \mathfrak{m}^{n_0}\mathfrak{c}_{t-1}$ *and* $y \in \mathfrak{m}^{n_0}$ *be parameters of* M. *For every submodule* N *of* M *such that* $N \subseteq D_{t-1}$ *and for all* $i < d-1$ *we have an exact sequence*

$$0 \to H^i_{\mathfrak{m}}(M/N) \to H^i_{\mathfrak{m}}(M/(xyM + N)) \to H^{i+1}_{\mathfrak{m}}(M/D_{t-1}) \to 0.$$

Proof. Notice that $D_{t-1}/M_{t-1} = H^0_{\mathfrak{m}}(M/M_{t-1})$. Hence,

$$0 :_M x \supseteq 0 :_M \mathfrak{m}^{n_0}\mathfrak{c}_{t-1} = (0 :_M \mathfrak{c}_{t-1}) :_M \mathfrak{m}^{n_0} \supseteq M_{t-1} :_M \mathfrak{m}^{n_0} = D_{t-1}.$$

So, $D_{t-1} \subseteq N :_M x \subseteq D_{t-1} :_M x = D_{t-1}$. Thus, $N :_M x = D_{t-1}$, and hence $N :_M xy = D_{t-1}$. Put $\overline{M} = M/D_{t-1}$. The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{M} & \xrightarrow{xy} & M/N & \xrightarrow{p_1} & M/(xyM+N) & \longrightarrow & 0 \\
 & & \Big\downarrow y & & \Big\downarrow id & & \Big\downarrow & & \\
0 & \longrightarrow & \overline{M} & \xrightarrow{x} & M/N & \xrightarrow{p_2} & M/(xM+N) & \longrightarrow & 0,
\end{array}$$

where p_1, p_2 are the natural projections, commutes. By applying the functor $H^i_{\mathfrak{m}}(\bullet)$ to the above diagram we get a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^i_{\mathfrak{m}}(\overline{M}) & \xrightarrow{\psi_i} & H^i_{\mathfrak{m}}(M/N) & \longrightarrow & \cdots \\
 & & \Big\downarrow y & & \Big\downarrow id & & \\
\cdots & \longrightarrow & H^i_{\mathfrak{m}}(\overline{M}) & \xrightarrow{\varphi_i} & H^i_{\mathfrak{m}}(M/N) & \longrightarrow & \cdots,
\end{array}$$

where ψ_i, φ_i are derived from the maps $\overline{M} \xrightarrow{xy} M/N$, $\overline{M} \xrightarrow{x} M/N$, respectively. It is easily seen that $H^0_{\mathfrak{m}}(\overline{M}) = 0$ and $H^i_{\mathfrak{m}}(\overline{M}) \cong H^i_{\mathfrak{m}}(M/M_{t-1})$ for all $i > 0$. Thus, $yH^i_{\mathfrak{m}}(\overline{M}) = 0$ for all $0 < i < d$ since $y \in \mathfrak{m}^{n_0}$. Hence, $\psi_i = 0$ for all $i < d$, and we have a short exact sequence

$$0 \to H^i_{\mathfrak{m}}(M/N) \to H^i_{\mathfrak{m}}(M/(xyM + N)) \to H^{i+1}_{\mathfrak{m}}(\overline{M}) \to 0$$

for every $i < d - 1$. □

Suppose that for a parameter $x \in \mathfrak{m}^{2n_0}\mathfrak{c}_{t-1}$ and for $i \leq t-1$ and $j < d-1$, the sequence

$$0 \to H^j_{\mathfrak{m}}(M/M_i) \to H^j_{\mathfrak{m}}(M/(xM + M_i)) \to H^{j+1}_{\mathfrak{m}}(M/D_{t-1}) \to 0$$

is exact. We denote this short exact sequence by $E^{i,j}_x \in \mathrm{Ext}^1_R(H^{j+1}_{\mathfrak{m}}(M/D_{t-1}), H^j_{\mathfrak{m}}(M/M_i))$.

Proposition 3.4. (i) *Suppose that $x \in \mathfrak{m}^{2n_0}\mathfrak{c}_{t-1}$ is a parameter of M. Then, $E^{i,j}_x$ is determined for all $i \leq t-1$ and all $j < d-1$.*
(ii) *Suppose that $x \in \mathfrak{m}^{3n_0}\mathfrak{c}_{t-1}$ is a parameter of M. Then, $E^{i,j}_x = 0$ for all $i \leq t-1$ and all $j < d-1$.*

Proof. (i) It follows from Lemma 3.3 that whenever $a \in \mathfrak{m}^{n_0}\mathfrak{c}_{t-1}$ and $b \in \mathfrak{m}^{n_0}$ are parameters of M, then $E^{i,j}_{ab}$ is determined for all $i \leq t-1$ and all $j < d-1$. By Lemma 3.2 for each parameter x in $\mathfrak{m}^{2n_0}\mathfrak{c}_{t-1}$ there are parameters $a_1, \ldots, a_r \in \mathfrak{m}^{n_0}\mathfrak{c}_{t-1}$ and $b_1, \ldots, b_r \in \mathfrak{m}^{n_0}$ such that $x = a_1b_1 + \cdots + a_rb_r$ and that $a_1b_1 + \cdots + a_kb_k$ is parameter for every $k \leq r$. Hence, $E^{i,j}_x = E^{i,j}_{a_1b_1} + \cdots + E^{i,j}_{a_rb_r}$ is determined for all $i \leq t-1$ and all $j < d-1$ by Lemma 3.1 (i).

(ii) By Lemma 3.2 there are parameters $a_1, \ldots, a_r \in \mathfrak{m}^{2n_0}\mathfrak{c}_{t-1}$ and $b_1, \ldots, b_r \in \mathfrak{m}^{n_0}$ such that $x = a_1b_1 + \cdots + a_rb_r$ and that $a_1b_1 + \cdots + a_kb_k$ are parameters for all $k \leq r$. It follows from Lemma 3.1 that

$$E^{i,j}_x = E^{i,j}_{a_1b_1+\cdots+a_rb_r} = b_1E^{i,j}_{a_1} + \cdots + b_rE^{i,j}_{a_r} = 0$$

for all $i \leq t-1$ and all $j < d-1$. □

Let $\underline{x} = x_1, \ldots, x_d$ be a good s.o.p. of M with respect to $\mathscr{F}$. By [2, Lemma 3.6] M/x_dM is also a sequentially generalized Cohen-Macaulay module with the generalized Cohen-Macaulay filtration

$$\mathscr{F}_d : M_0 \cong (M_0 + x_dM)/x_dM \subset \cdots \subset M_s \cong (M_s + x_dM)/x_dM \subset M/x_dM,$$

where $s = t-1$ if $d_{t-1} < d-1$, and $s = t-2$ if $d_{t-1} = d-1$. Moreover, $x_1, \ldots, x_{d-1}$ is a good s.o.p. of M/x_dM with respect to $\mathscr{F}_d$.

Lemma 3.5. *Let $d > 1$ and $\underline{x'} = x_1, \ldots, x_{d-1}$. Then, $I_{\mathscr{F}_d, M/x_dM}(\underline{x'}) = I_{\mathscr{F}, M}(\underline{x})$.*

Proof. It suffices to show that $e(\underline{x'}; M/x_dM) = e(\underline{x}; M)$ if $d_{t-1} < d-1$, and $e(\underline{x'}; M/x_dM) = e(\underline{x}; M) + e(\underline{x'}; M_{t-1})$ if $d_{t-1} = d-1$. By definition of Serre multiplicity we have

$$e(\underline{x'}; M/x_dM) = e(\underline{x}; M) + e(\underline{x'}; 0:_M x_d).$$

As D_{t-1} is the largest submodule of M with dimension less than d, it holds $0:_M x_d \subseteq D_{t-1}$. Obviously, $M_{t-1} \subseteq 0:_M x_d$. By Remark 2.7 (i) we have $\dim M_{t-1} = \dim 0:_M x_d$ and $\ell((0:_M x_d)/M_{t-1}) < \infty$. Thus, $e(\underline{x'}; 0:_M x_d) = e(\underline{x'}; M_{t-1})$. Therefore, $e(\underline{x'}; M/x_dM) = e(\underline{x}; M)$ if $d_{t-1} < d-1$, and $e(\underline{x'}; M/x_dM) = e(\underline{x}; M) + e(\underline{x'}; M_{t-1})$ if $d_{t-1} = d-1$. □

The following is a generalization of [1, Theorem 4.3].

THEOREM 3.6. *The following assertions are true*

(i) *For every good s.o.p.* $\underline{x} = x_1, \ldots, x_d$ *of* M *with respect to* $\mathscr{F}$ *such that* $x_j \in \mathfrak{m}^{3n_0}\mathfrak{c}_i$ *for all* $0 \le i \le t-1$ *and all* $d_i < j \le d_{i+1}$ *then* $I_{\mathscr{F},M}(\underline{x}) = I_{\mathscr{F}}(M)$ *and*

$$I_{\mathscr{F},M}(\underline{x}) = \ell(H^0_{\mathfrak{m}}(M/M_0)) + \sum_{i=0}^{t-1}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)\ell(H^j_{\mathfrak{m}}(M/M_i)).$$

(ii) $I_{\mathscr{F},\mathscr{M}}(\underline{x}) = I_{\mathscr{F}}(M)$ *for every good s.o.p.* $\underline{x} = x_1, \ldots, x_d$ *of* M *with respect to* $\mathscr{F}$ *contained in* $\mathfrak{m}^n$ *for* $n \gg 0$.

Proof. (i) We prove the assertion by induction on d. The case $d = 1$ is trivial since M is a generalized Cohen-Macaulay module. Assume that $d > 1$ and that the assertion is proved for all smaller values of d. Notice that M/x_dM is also a sequentially generalized Cohen-Macaulay module with the generalized Cohen-Macaulay filtration $\mathscr{F}_d$

$$\mathscr{F}_d : M_0 \cong (M_0 + x_dM)/x_dM \subset \cdots \subset M_s \cong (M_s + x_dM)/x_dM \subset M/x_dM,$$

where $s = t-1$ if $d_{t-1} < d-1$, and $s = t-2$ if $d_{t-1} = d-1$. Since $x_d \in \mathfrak{m}^{3n_0}\mathfrak{c}_{t-1}$ we have

$$H^j_{\mathfrak{m}}(M/(M_i + x_dM)) \cong H^j_{\mathfrak{m}}(M/M_i) \oplus H^{j+1}_{\mathfrak{m}}(M/D_{t-1})$$

for all $i \le t-1$ and all $j < d-1$ by Proposition 3.4. Hence, $\mathfrak{m}^{n_0}H^j_{\mathfrak{m}}(M/(M_i + x_dM)) = 0$ for all $i \le s$ and all $j \le d_{i+1}-1$ and

$$\begin{aligned}\ell(H^j_{\mathfrak{m}}(M/(M_i + x_dM))) &= \ell(H^j_{\mathfrak{m}}(M/M_i)) + \ell(H^{j+1}_{\mathfrak{m}}(M/D_{t-1})) \\ &= \ell(H^j_{\mathfrak{m}}(M/M_i)) + \ell(H^{j+1}_{\mathfrak{m}}(M/M_{t-1}))\end{aligned}$$

for all $i \le t-1$ and all $j \le d_{i+1}-1$, since D_{t-1}/M_{t-1} has finite length. We will denote $\ell(H^i_{\mathfrak{m}}(\bullet))$ by $h^i(\bullet)$. By Lemma 3.5 and the inductive hypothesis we have

$$
\begin{aligned}
I_{\mathscr{F},M}(\underline{x}) &= I_{\mathscr{F}_d,\,M/x_dM}(\underline{x'}) \\
&= h^0(M/(M_0 + x_dM)) \\
&\quad + \sum_{i=0}^{s-1}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^j(M/(M_i + x_dM)) \\
&\quad + \sum_{j=1}^{d-2}\left(\binom{d-2}{j} - \binom{d_s-1}{j}\right)h^j(M/(M_s + x_dM)).
\end{aligned}
$$

We now consider two cases.

Case 1. $d_{t-1} < d-1$, then $s = t-1$. We have

$$
\begin{aligned}
I_{\mathscr{F},M}(\underline{x}) &= h^0(M/M_0) + h^1(M/M_{t-1}) \\
&\quad + \sum_{i=0}^{t-2}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)(h^j(M/M_i) + h^{j+1}(M/M_{t-1})) \\
&\quad + \sum_{j=1}^{d-2}\left(\binom{d-2}{j} - \binom{d_{t-1}-1}{j}\right)(h^j(M/M_{t-1}) + h^{j+1}(M/M_{t-1})) \\
&= h^0(M/M_0) + \sum_{i=0}^{t-2}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^j(M/M_i) \\
&\quad + h^1(M/M_{t-1}) + \sum_{i=0}^{t-2}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^{j+1}(M/M_{t-1}) \\
&\quad + \sum_{j=1}^{d-1}\left(\binom{d-1}{j} - \binom{d_{t-1}}{j}\right)h^j(M/M_{t-1}) \\
&= h^0(M/M_0) + \sum_{i=0}^{t-2}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^j(M/M_i) \\
&\quad + h^1(M/M_{t-1}) + \sum_{j=2}^{d_{t-1}}\binom{d_{t-1}-1}{j-1}h^j(M/M_{t-1}) \\
&\quad + \sum_{j=1}^{d-1}\left(\binom{d-1}{j} - \binom{d_{t-1}}{j}\right)h^j(M/M_{t-1}) \\
&= h^0(M/M_0) + \sum_{i=0}^{t-1}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^j(M/M_i).
\end{aligned}
$$

CASE 2. $d_{t-1} = d-1$, then $s = t-2$. We have

$$
\begin{aligned}
I_{\mathscr{F},M}(\underline{x}) &= h^0(M/M_0) + h^1(M/M_{t-1}) \\
&\quad + \sum_{i=0}^{t-3}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)(h^j(M/M_i) + h^{j+1}(M/M_{t-1})) \\
&\quad + \sum_{j=1}^{d-2}\left(\binom{d-2}{j} - \binom{d_{t-2}-1}{j}\right)(h^j(M/M_{t-2}) + h^{j+1}(M/M_{t-1})) \\
&= h^0(M/M_0) + \sum_{i=0}^{t-2}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^j(M/M_i) \\
&\quad + h^1(M/M_{t-1}) + \sum_{i=0}^{t-3}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^{j+1}(M/M_{t-1}) \\
&\quad + \sum_{j=1}^{d-2}\left(\binom{d-2}{j} - \binom{d_{t-2}-1}{j}\right)h^{j+1}(M/M_{t-1}) \\
&= h^0(M/M_0) + \sum_{i=0}^{t-2}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^j(M/M_i) \\
&\quad + h^1(M/M_{t-1}) + \sum_{j=2}^{d_{t-2}}\binom{d_{t-2}-1}{j-1}h^j(M/M_{t-1}) \\
&\quad + \sum_{j=2}^{d-1}\left(\binom{d-2}{j-1} - \binom{d_{t-2}-1}{j-1}\right)h^j(M/M_{t-1}) \\
&= h^0(M/M_0) + \sum_{i=0}^{t-2}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^j(M/M_i) \\
&\quad + \sum_{j=1}^{d-1}\binom{d-2}{j-1}h^j(M/M_{t-1}) \\
&= h^0(M/M_0) + \sum_{i=0}^{t-1}\sum_{j=1}^{d_{i+1}-1}\left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)h^j(M/M_i).
\end{aligned}
$$

Notice that $I_{\mathscr{F},\mathscr{M}}(\underline{y}^{\underline{n}})$ is a non-decreasing function in $\underline{n} = (n_1, \dots, n_d) \in \mathbf{N}^d$ for every good s.o.p. $\underline{y} = y_1, \dots, y_d$ with respect to $\mathscr{F}$ (cf. [1, Proposition 2.9]). Thus we have $I_{\mathscr{F},\mathscr{M}}(\underline{x}) = I_{\mathscr{F}}(M)$.

(ii) By the Artin-Rees Lemma there exists an positive integer k such that

$$\mathfrak{m}^n \cap \mathfrak{c}_i = \mathfrak{m}^{n-k}(\mathfrak{m}^k \cap \mathfrak{c}_i) \subseteq \mathfrak{m}^{n-k}\mathfrak{c}_i$$

for all $n \geq k$ and all $i = 0, \dots, t-1$. Therefore, it follows from (i) that $I_{\mathcal{F},\mathcal{M}}(\underline{x}) = I_{\mathcal{F}}(M)$ for every good s.o.p. $\underline{x} = x_1, \dots, x_d$ of M with respect to $\mathcal{F}$ contained in $\mathfrak{m}^{3n_0+k}$. □

If $x \in \mathfrak{m}^{2n_0}\mathfrak{c}_{t-1}$ is a parameter, then since $E_x^{i,j}$ is determined for all $i \leq t-1$ and all $j < d-1$, hences the short exact sequences

$$0 \to \overline{M} \xrightarrow{x\cdot} M/M_i \to M/(M_i + xM) \to 0$$

for $i \leq t-1$, where $\overline{M} = M/D_{t-1}$, induce short exact sequences

$$0 \to H_{\mathfrak{m}}^{d-1}(M/M_i) \to H_{\mathfrak{m}}^{d-1}(M/(M_i + xM)) \to 0 :_{H_{\mathfrak{m}}^d(M)} x \to 0$$

for $i \leq t-1$ by Proposition 3.4(i). Suppose that these sequences indece exact sequences

$$0 \to 0 :_{H_{\mathfrak{m}}^{d-1}(M/M_i)} \mathfrak{m} \to 0 :_{H_{\mathfrak{m}}^{d-1}(M/(M_i+xM))} \mathfrak{m} \to 0 :_{H_{\mathfrak{m}}^d(M)} \mathfrak{m} \to 0$$

for $i \leq t-1$. We shall denote these exact sequences by $F_x^{i,d-1}$ for $i \leq t-1$.

PROPOSITION 3.7. (i) *Suppose that* $x \in \mathfrak{m}^{2n_0}\mathfrak{c}_{t-1}$ *and* $y \in \mathfrak{m}$ *are parameters of* M. *Then,* $F_{xy}^{i,d-1}$ *is determined for all* $i \leq t-1$.
(ii) *Suppose that* $x \in \mathfrak{m}^{2n_0+1}\mathfrak{c}_{t-1}$ *is a parameter of* M. *Then* $F_x^{i,d-1}$ *is determined for all* $i \leq t-1$.

Proof. (i) For $i \leq t-1$ we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{M} & \xrightarrow{x} & M/M_i & \xrightarrow{p_1} & M/(M_i + xM) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle y} & & \downarrow & & \\ 0 & \longrightarrow & \overline{M} & \xrightarrow{xy} & M/M_i & \xrightarrow{p_2} & M/(M_i + xyM) & \longrightarrow & 0, \end{array}$$

where $\overline{M} = M/D_{t-1}$ and p_1 and p_2 are the natural projections. This diagram induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{\mathfrak{m}}^{d-1}(M/M_i) & \longrightarrow & H_{\mathfrak{m}}^{d-1}(M/(M_i + xM)) & \longrightarrow & 0 :_{H_{\mathfrak{m}}^d(M)} x & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle y} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_{\mathfrak{m}}^{d-1}(M/M_i) & \longrightarrow & H_{\mathfrak{m}}^{d-1}(M/(M_i + xyM)) & \longrightarrow & 0 :_{H_{\mathfrak{m}}^d(M)} xy & \longrightarrow & 0. \end{array}$$

By applying the functor $\mathrm{Ext}_R^i(R/\mathfrak{m}, \bullet)$ to this diagram we obtain a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 :_{H_{\mathfrak{m}}^d(M)} \mathfrak{m} & \xrightarrow{\alpha} & \mathrm{Ext}_R^1(R/\mathfrak{m}, H_{\mathfrak{m}}^{d-1}(M/M_i)) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle y} & & \\ \cdots & \longrightarrow & 0 :_{H_{\mathfrak{m}}^d(M)} \mathfrak{m} & \xrightarrow{\beta} & \mathrm{Ext}_R^1(R/\mathfrak{m}, H_{\mathfrak{m}}^{d-1}(M/M_i)) & \longrightarrow & \cdots, \end{array}$$

where α, β are connecting homomorphisms. Thus, $\beta = y \circ \alpha = 0$ since $y \in \mathfrak{m}$. Hence, $F_{xy}^{i,d-1}$ is determined for all $i \leq t-1$.

(ii) follows form (i) by using the same method as in the proof of Proposition 3.4 (i). □

COROLLARY 3.8. *Let $x \in \mathfrak{m}^{3n_0+1}\mathfrak{c}_{t-1}$ be a parameter of M, and let $\overline{M} = M/D_{t-1}$. Then,*

$$\begin{aligned}&\dim_{R/\mathfrak{m}} \operatorname{Soc}(H_{\mathfrak{m}}^{j}(M/(xM + M_i)))\\ &\qquad = \dim_{R/\mathfrak{m}} \operatorname{Soc}(H_{\mathfrak{m}}^{j}(M/M_i)) + \dim_{R/\mathfrak{m}} \operatorname{Soc}(H_{\mathfrak{m}}^{j+1}(\overline{M}))\end{aligned}$$

for all $i \leq t-1$ and all $j \leq d-1$.

Proof. This follows from Proposition 3.4 (ii) and Proposition 3.7 (ii). □

By using Corollary 3.8 and the same method as in the proof of Theorem 3.6 we get the following result.

THEOREM 3.9. *Then the following assertions are true*

(i) *For every good s.o.p. $\underline{x} = x_1, \ldots, x_d$ of M with respect to $\mathcal{F}$ such that $x_j \in \mathfrak{m}^{3n_0+1}\mathfrak{c}_i$ for all $0 \leq i \leq t-1$ and all $d_i < j \leq d_{i+1}$, the index of reducibility of $(\underline{x})$ on M is independent of the choice of $\underline{x}$, and it holds*

$$\begin{aligned}N_R((\underline{x}), M) &= \dim_{R/\mathfrak{m}} \operatorname{Soc}(H_{\mathfrak{m}}^{0}(M))\\ &\quad + \sum_{i=0}^{t-1}\sum_{j=1}^{d_{i+1}}\left(\binom{d_{i+1}}{j} - \binom{d_i}{j}\right) \dim_{R/\mathfrak{m}} \operatorname{Soc}(H_{\mathfrak{m}}^{j}(M/M_i)).\end{aligned}$$

(ii) *Let $\underline{x} = x_1, \ldots, x_d$ be a good s.o.p. of M with respect to $\mathcal{F}$ contained in $\mathfrak{m}^n$ for $n \gg 0$. Then, the index of reducibility of the parameter ideal $(\underline{x})$ on M is independent of the choice of $\underline{x}$.*

Notice that if M is a sequentially Cohen-Macaulay module, then the Cohen-Macaulay filtration is unique and is just the dimension filtration $\mathcal{D}$ of M (cf. [1]). In this case, $H_{\mathfrak{m}}^{j}(M/D_i) = 0$ for all $j < d_{i+1}$ and $H_{\mathfrak{m}}^{d_{i+1}}(M/D_i) \cong H_{\mathfrak{m}}^{d_{i+1}}(M)$. By applying Theorem 3.9 for sequentially Cohen-Macaulay module we obtain the main result of [12].

COROLLARY 3.10. *Let M be a sequentially Cohen-Macaulay module of dimension d. Then there is a positive integer n such that for every good s.o.p. $\underline{x} = x_1, \ldots, x_d$ of M contained in $\mathfrak{m}^n$ the index of reducibility $N_R((\underline{x}); M)$ is independent of the choice of $\underline{x}$ and*

$$N_R((\underline{x}); M) = \sum_{i=0}^{d} \dim_{R/\mathfrak{m}} \operatorname{Soc}(H_{\mathfrak{m}}^{i}(M)).$$

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