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LARGE DEVIATIONS FOR SIMPLE RANDOM WALK ON SUPERCRITICAL PERCOLATION CLUSTERS

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Abstract

We prove quenched large deviation principles governing the position of the random walk on a supercritical site percolation on the integer lattice. A feature of this model is non-ellipticity of transition probabilities. Our analysis is based on the consideration of so-called Lyapunov exponents for the Laplace transform of the first passage time. The rate function is given by the Legendre transform of the Lyapunov exponents.

1. Introduction

In this paper, we study large deviation principles for the simple random walk on a supercritical site percolation in $\mathbf{Z}^d$ with $d \ge 2$. In recent years, this problem has been investigated for the random walk in random environment, see for instance [9, 10, 11]. The above references treat random walks with the elliptic transition probabilities. However, our model treats non-elliptic transition probabilities due to the fluctuation of site percolation clusters, so we get over this problem using the estimates on the graph distance on a supercritical site percolation.

We now describe the setting. Let $d \ge 2$. Set $\Omega := \{0,1\}^{\mathbf{Z}^d}$ and denote a configuration of Ω by $\omega = (\omega(x))_{x \in \mathbf{Z}^d}$. For $p \in (0,1)$ we consider the space Ω equipped with the canonical product σ-field $\mathscr{G}$ and an i.i.d. product probability measure P_p such that $P_p(\omega(x) = 1) = p$ for $x \in \mathbf{Z}^d$. A site $x \in \mathbf{Z}^d$ is called *open* if $\omega(x) = 1$, and *closed* otherwise. A sequence $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ in $\mathbf{Z}^d$ is called *nearest neighbor path* if $|\gamma_m - \gamma_{m+1}| = 1$ holds for all $0 \le m \le n-1$, where we always write $|x| := |x_1| + \cdots + |x_d|$ for $x = (x_1, \dots, x_d) \in \mathbf{R}^d$. Furthermore, a path γ is called *open* if γ_m is open for all $0 \le m \le n$. We now denote $x \overset{\omega}{\leftrightarrow} y$ whenever there is an open path from x to y. The *open cluster* $C(x)$ containing x is then the set of y's such that $x \overset{\omega}{\leftrightarrow} y$. It is well known that there exists $p_c = p_c(d) \in (0,1)$ such that when $p > p_c$, we have $P_p(0 \in \mathscr{C}_\infty) > 0$ and there is

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a unique infinite open cluster $\mathscr{C}_\infty = \mathscr{C}_\infty(\omega)$ (see [7, Theorems 1.10 and 8.1] for instance). Therefore, letting $\Omega_0 := \{0 \in \mathscr{C}_\infty\}$, for $p > p_c$ we define the probability measure $\mathbf{P}$ as

$$\mathbf{P}(A) := \frac{P_p(A \cap \Omega_0)}{P_p(\Omega_0)}, \quad A \in \mathscr{G}.$$

For each configuration $\omega \in \Omega_0$, the simple random walk on $\mathscr{C}_\infty(\omega)$ is the Markov chain $((X_n)_{n=0}^\infty, (P_\omega^x)_{x \in \mathscr{C}_\infty(\omega)})$ on $\mathscr{C}_\infty(\omega)$ defined as follows: $P_\omega^z(X_0 = z) = 1$, and the transition probabilities

$$P_\omega^z(X_{n+1} = x + e \mid X_n = x) = \frac{1}{2d} \mathbf{1}_{\{\omega(e)=1\}} \circ \theta_x, \quad |e| = 1,$$

and

$$P_\omega^z(X_{n+1} = x \mid X_n = x) = \sum_{|e'|=1} \frac{1}{2d} \mathbf{1}_{\{\omega(e')=0\}} \circ \theta_x$$

for all $x, z \in \mathscr{C}_\infty(\omega)$ and $n \in \mathbf{N}_0$, where $(\theta_x)_{x \in \mathbf{Z}^d}$ are the canonical shifts of the space Ω, that is, $\theta_x \omega(y) = \omega(x + y)$ for $x, y \in \mathbf{Z}^d$.

We prove the large deviations for the law of scaled position X_n/n of SRWPC following the same strategy as in [11]. For this, we first study the asymptotics of the cumulant generating function. Let us introduce $H(y) := \inf\{n \geq 0; X_n = y\}$ as the first passage time through y for the path $(X_n)_{n=0}^\infty$. Define for any $\lambda \geq 0$, $\omega \in \Omega_0$ and $x, y \in \mathscr{C}_\infty(\omega)$,

$$\begin{aligned} e_\lambda(x, y, \omega) &:= E_\omega^x[\exp\{-\lambda H(y)\} \mathbf{1}_{\{H(y) < \infty\}}], \\ a_\lambda(x, y, \omega) &:= -\log e_\lambda(x, y, \omega), \\ d_\lambda(x, y, \omega) &:= \max\{a_\lambda(x, y, \omega), a_\lambda(y, x, \omega)\}. \end{aligned} \tag{1.1}$$

THEOREM 1.1 (Lyapunov exponents). *Let $\lambda \geq 0$. Then there exists a nonrandom function $\alpha_\lambda : \mathbf{R}^d \to [0, \infty)$ such that* $\mathbf{P}$-a.s.,

$$\lim_{n \to \infty} \frac{1}{T_x^{(n)}} a_\lambda(0, T_x^{(n)} x) = \alpha_\lambda(x) \tag{1.2}$$

holds for all $x \in \mathbf{Z}^d \backslash \{0\}$, where $T_x^{(n)}$ is the position of n-th intersection of infinite open cluster $\mathscr{C}_\infty$ with the half line $x\mathbf{N}$ (see (2.1) below for details). In addition, $\alpha_\lambda(\cdot)$ has the following properties: for any $q > 0$ and $x, y \in \mathbf{R}^d$,

$$\alpha_\lambda(qx) = q\alpha_\lambda(x), \tag{1.3}$$

$$\alpha_\lambda(x + y) \leq \alpha_\lambda(x) + \alpha_\lambda(y), \tag{1.4}$$

and

$$\lambda|x| \le \alpha_\lambda(x) \le (\lambda + \log(2d))\rho|x|, \tag{1.5}$$

where ρ is a constant appearing in (2.3) *below. Moreover, $\alpha_\lambda(x)$ is concave increasing in λ and convex in x. In particular, it is jointly continuous in λ and x.*

THEOREM 1.2 (Shape theorem). *We have* **P**-a.s. *that*

$$\lim_{\substack{|x|\to\infty \\ x \in \mathscr{C}_\infty}} \frac{a_\lambda(0,x) - \alpha_\lambda(x)}{|x|} = 0.$$

holds for all $\lambda \ge 0$.

The following theorem is our main result.

THEOREM 1.3. *The law of X_n/n obeys the following large deviation principle with the rate function*

$$I(x) := \sup_{\lambda \ge 0}(\alpha_\lambda(x) - \lambda) :$$

- *Upper bound: for any closed subset $A \subset \mathbf{R}^d$, we have* **P**-a.s.,

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega^0(X_n \in nA) \le -\inf_{x \in A} I(x). \tag{1.6}$$

- *Lower bound: for any open subset $B \subset \mathbf{R}^d$, we have* **P**-a.s.,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega^0(X_n \in nB) \ge -\inf_{x \in B} I(x). \tag{1.7}$$

Remark 1.4. The Markov chain $(X_n)_{n=0}^\infty$ represents only one of natural ways to define a simple random walk on the supercritical percolation on $\mathbf{Z}^d$. Other possibilities are the simple random walk on supercritical bond-percolation on $\mathbf{Z}^d$ and the model that, at each unit of time, the walk moves to a site chosen uniformly at random from the accessible neighbors, that is, the walk takes no pauses. We can apply the same strategy taken in this paper to these cases, since the transition probabilities on a unique infinite cluster are uniformly bounded from below and the estimates for the graph distance hold as well.

Organization of the paper. We briefly comment on the proof and the organization of the paper. Our proof basically follows the strategy taken in [11]. The main difficulty compared to it lies in Theorem 1.1. The Lyapunov exponents are easily constructed by the subadditive ergodic theorem, however we have to require the suitable properties for these. In particular, to prove the property (1.4) and (1.5) is more complicated since it directly concerns the configuration of supercritical site percolations. For (1.4), we let $x, y \in \mathscr{C}_\infty$ and consider random walk paths from x to $x+y$. The similar claim is trivial in [11] since random walk paths can choose any sites as its starting and ending points. However, in our model, the site $x+y$ does not necessarily belong to $\mathscr{C}_\infty$ even though both x and y are in $\mathscr{C}_\infty$. Thus, it may be impossible to take a random walk path from x to $x+y$ for every configuration of supercritical site percolations. For (1.5), we use the estimates on the graph distance on supercritical site percolations, since a_λ is bounded by it, see for details (2.7) below.

In Section 2 and 3, we prove Theorems 1.1 and 1.2 with the consideration of the above comments following the strategy taken in [6, Section 3].

In Section 4, we first prove upper bound of Theorem 1.3 in Subsection 4.1. Furthermore, the rough upper and lower bounds and the properties of the rate function are here given by using those of the Lyapunov exponents. We next show the lower bound of Theorem 1.3 in Subsection 4.2. This proof is bit more complicated. In particular, to extend the shape theorem to a general version is needed. We finally study the asymptotic of the rate function as x tending to 0.

Notations. We write E_p, $\mathbf{E}$ and E_ω^z for the expectation with respect to P_p, $\mathbf{P}$ and E_ω^z respectively. Moreover, open $|\cdot|$-balls with center $x \in \mathbf{R}^d$ and radius $r \geq 0$ are denoted by $B(x,r)$ and closed balls by $\bar{B}(x,r)$.

2. Lyapunov exponents

Our goal in this section is to show Theorem 1.1. For the proof, let us first recall some ergodic properties and percolation estimates. We now define the maps $T_x : \Omega_0 \to \mathbf{N} \cup \{\infty\}$ as

$$T_x(\omega) := \inf\{n \geq 1; nx \in \mathscr{C}_\infty(\omega)\}, \quad x \in \mathbf{Z}^d.$$

The Poincaré recurrence theorem then tells us that $\mathbf{P}$-a.s., T_x is finite for all $x \in \mathbf{Z}^d$, see [8, Theorem 2.3.2] for instance. We can therefore define, up to set of measure zero, the maps $\Theta_x : \Omega_0 \to \Omega_0$ by

$$\Theta_x \omega := \theta_x^{T_x(\omega)} \omega, \quad x \in \mathbf{Z}^d.$$

Thanks to [3, Lemma 3.3], Θ_x is invertible, measure preserving and ergodic with respect to $\mathbf{P}$. Setting for $n \in \mathbf{N}_0$,

$$T_x^{(n)}(\omega) := \sum_{k=0}^{n-1} T_x(\Theta_x^k \omega), \tag{2.1}$$

we have by Birkhoff's ergodic theorem and the Kac theorem (see [8, Theorem 2.4.6]) that

$$\lim_{n\to\infty} \frac{T_x^{(n)}}{n} = \mathbf{E}[T_x] = P_p(\Omega_0)^{-1} \tag{2.2}$$

holds $\mathbf{P}$-a.s. and in $L^1(\mathbf{P})$.

Let $D_\omega(x, y)$ be the graph distance on $\mathscr{C}_\infty(\omega)$. Then, [1, Theorem 1.1] implies that if $p > p_c$ then there exist constants $\rho \geq 1$ and $c_1, c_2 > 0$ depending only on p and d such that for all $x \in \mathbf{Z}^d$,

$$P_p(0 \overset{\omega}{\leftrightarrow} x, D_\omega(0, x) > \rho|x|) \leq c_1 e^{-c_2|x|}. \tag{2.3}$$

Moreover, [6, Lemma 2.4] ensures that there exists $c_3 = c_3(p, d)$ such that

$$\mathbf{E}[D_\omega(0, T_x x)] \leq |x| P_p(\Omega_0)^{-1}(\rho + c_3|x|^{-2}) \tag{2.4}$$

holds for all $x \in \mathbf{Z}^d \backslash \{0\}$.

For the proof of Theorem 1.1, let us start with the triangle inequality and integrability properties for a_λ.

Lemma 2.1. *For any $\lambda \geq 0$ and $x, y, z \in \mathscr{C}_\infty$,*

$$a_\lambda(x, y) \leq a_\lambda(x, z) + a_\lambda(z, y).$$

Proof. We set $H_z(y) := \inf\{n \geq H(z); X_n = y\}$. The strong Markov property then shows

$$e_\lambda(x, y) \geq E_\omega^x[\exp\{-\lambda H_z(y)\}\mathbf{1}_{\{H_z(y) < \infty\}}] = e_\lambda(x, z) e_\lambda(z, y),$$

which completes the proof. □

Lemma 2.2. *For all $\lambda \geq 0$, $n \in \mathbf{N}_0$ and $x \in \mathbf{Z}^d \backslash \{0\}$,*

$$\mathbf{E}[a_\lambda(0, T_x x)] \leq |x|(\lambda + \log(2d)) P_p(\Omega_0)^{-1}(\rho + c_3|x|^{-2}), \tag{2.5}$$

and

$$\mathbf{E}[a_\lambda(0, T_x^{(n)} x)] \geq |x| \lambda \mathbf{E}[T_x^{(n)}]. \tag{2.6}$$

Proof. We first show (2.5). If $0 \overset{\omega}{\leftrightarrow} kx$, then let $\gamma_k(\omega) = (\gamma_{k,n}(\omega))_{n=0}^{m_k(\omega)}$ be an open path from 0 to kx with minimal length $m_k(\omega) = D_\omega(0, kx)$, which is chosen by a deterministic algorithm. We then have by the definition of d_λ,

$$d_\lambda(w, w') \leq (\lambda + \log(2d)) D_\omega(w, w'), \quad w, w' \in \mathscr{C}_\infty. \tag{2.7}$$

This implies that

$$\mathbf{E}[a_\lambda(0, T_x x)] \leq (\lambda + \log(2d)) \mathbf{E}[D_\omega(0, T_x x)],$$

and therefore (2.5) follows from (2.4).

For (2.6), note that $H(T_x^{(n)} x) \geq T_x^{(n)}|x|$ P_ω^0-a.s. We thus have

$$\mathbf{E}[a_\lambda(0, T_x^{(n)} x)] \geq \mathbf{E}[-\log E_\omega^0[\exp\{-\lambda T_x^{(n)}|x|\}]] = |x| \lambda \mathbf{E}[T_x^{(n)}],$$

which proves the lemma. □

Now we are in position to prove Theorem 1.1. The following proposition now plays the key rule in the proof.

PROPOSITION 2.3. *There exists a nonrandom function* $W_\lambda : \mathbf{Z}^d \to [0, \infty)$ *such that the following holds* **P**-a.s. *and in* $L^1(\mathbf{P})$: *for all* $x \in \mathbf{Z}^d$,

$$\lim_{n\to\infty} \frac{1}{n} a_\lambda(0, T_x^{(n)} x) = \lim_{n\to\infty} \frac{1}{n} \mathbf{E}[a_\lambda(0, T_x^{(n)} x)] \tag{2.8}$$
$$= \inf_{n\geq 1} \frac{1}{n} \mathbf{E}[a_\lambda(0, T_x^{(n)} x)] = W_\lambda(x).$$

Furthermore, we have for $x \in \mathbf{Z}^d \backslash \{0\}$,

$$|x|\lambda P_p(\Omega_0)^{-1} \leq W_\lambda(x) \leq |x|(\lambda + \log(2d)) P_p(\Omega_0)^{-1} (\rho + c_3 |x|^{-2}). \tag{2.9}$$

Proof. Fix $x \in \mathbf{Z}^d$ and we consider the doubly indexed sequence

$$W_\lambda^{m,n} := a_\lambda(T_x^{(m)} x, T_x^{(n)} x, \omega, \sigma), \quad 0 \leq m < n.$$

This sequence $(W_\lambda^{m,n})_{m<n}$ satisfies the conditions of the subadditive ergodic theorem from the ergodicity of Θ_x and Lemmas 2.1 and 2.2. We can thus find a function W_λ satisfying (2.8).

The upper bound in (2.9) follows immediately from Lemma 2.2. For the lower bound in (2.9), (2.2) and Lemma 2.2 yield

$$W_\lambda(x) \geq |x|\lambda \lim_{n\to\infty} \frac{1}{n} \mathbf{E}[T_x^{(n)}] = |x|\lambda P_p(\Omega_0)^{-1},$$

and the proof is complete. □

The next corollary follows immediately from the above proposition.

COROLLARY 2.4. *Let* $\alpha_\lambda(x) := P_p(\Omega_0) W_\lambda(x)$ *for* $x \in \mathbf{Z}^d$. *Then* **P**-a.s., (1.2) *holds for all* $x \in \mathbf{Z}^d \backslash \{0\}$. *Furthermore*, (1.3) *and* (1.5) *holds for any* $q \in \mathbf{N}$ *and* $x \in \mathbf{Z}^d$.

Proof. It is easy to see (1.2) from (2.2) and Proposition 2.3. Let us next show (1.3). We may assume $x \in \mathbf{Z}^d \backslash \{0\}$ because $W_\lambda(0) = 0$. By the definition of $T_x^{(n)}$ and $T_{qx}^{(n)}$, for all $k \geq 0$ there exists $n_k \in \mathbf{N}_0$ such that $T_x^{(n_k)} x = T_{qx}^{(k)} qx$ holds. In particular, $T_x^{(n_k)} = q T_{qx}^{(k)}$ is verified, and therefore (1.2) implies that

$$\alpha_\lambda(qx) = q \lim_{k\to\infty} \frac{1}{q T_{qx}^{(k)}} a_\lambda(0, T_{qx}^{(k)} qx) = q \lim_{k\to\infty} \frac{1}{T_x^{(n_k)}} a_\lambda(0, T_x^{(n_k)} x) = q\alpha_\lambda(x),$$

which proves (1.3).

We finally show (1.5). The lower bound follows from Proposition 2.3. Using (1.3) and Proposition 2.3, we obtain for $q \in \mathbf{N}$,

$$\alpha_\lambda(x) = \frac{\alpha_\lambda(qx)}{q} \leq |x|(\lambda + \log(2d))(\rho + c_3 |qx|^{-2}).$$

The upper bound is hence shown by letting $q \to \infty$. □

Proof of Theorem 1.1. Given Lemma 2.1 and Corollary 2.4, it suffices to show that (1.4) holds for any $x, y \in \mathbf{Z}^d$, since the proof goes in the same line as that of [11, Proposition 3]. Namely, due to (1.3) we extend $\alpha_\lambda(\cdot)$ to $\mathbf{Q}^d$ by $\alpha_\lambda(x) = \alpha_\lambda(x)/q$ and then to $\mathbf{R}^d$ by continuity. The properties of $\alpha_\lambda(x)$ as a function of λ and x follow from (1.3), (1.4) and the monotonicity of a_λ in λ. See the above references for details.

We first prove the next claim following the proof of [6, Lemma 3.4] (but see Remark 2.5 below): for $x, y, z \in \mathbf{Z}^d \setminus \{0\}$ there exists a sequence $n_k = n_k(|x|, |y|, |z|, |x-z|, |y-z|)$, $k \geq 1$ depending only on $|x|$, $|y|$, $|z|$, $|x-z|$ and $|y-z|$ such that P_p-a.s.,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mathbf{1}_{\{n_i x, n_i z \in \mathscr{C}_\infty\}} = P_p(\Omega_0)^2 = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mathbf{1}_{\{n_i y, n_i z \in \mathscr{C}_\infty\}}. \tag{2.10}$$

We choose $c := 4^{-1} \min\{|x|, |y|, |x-z|, |y-z|\}$ and define the sequence $(m_k)_{k=1}^{\infty}$ as follows: $n_1 := 1$ and by induction for $k \geq 1$, n_{k+1} is defined as

$$n_{k+1} \geq 2n_k \max\left\{ \frac{|x| \vee |z| + c + 1}{|x| \wedge |z| - c}, \frac{|y| \vee |z| + c + 1}{|x| \wedge |y| - c} \right\}.$$

The random variables $(Z_k)_{k=1}^{\infty}$ are then introduced by

$$Z_k := \mathbf{1}_{\{\#C(n_k x), \#C(n_k z) \geq c n_k\}}.$$

It follows that for $k \geq 1$,

$$\begin{aligned} P_p(Z_k \neq \mathbf{1}_{\{n_k x, n_k z \in \mathscr{C}_\infty\}}) &\leq P_p(c n_k \leq \#C(n_k x) < \infty) + P_p(c n_k \leq \#C(n_k z) < \infty) \\ &= 2P_p(c n_k \leq \#C(0) < \infty). \end{aligned}$$

The Borel–Cantelli lemma thereby shows that P_p-a.s., $Z_k = \mathbf{1}_{\{n_k x, n_k z \in \mathscr{C}_\infty\}}$ holds for all sufficiently large k, since $P_p(c n_k \leq \#C(0) < \infty)$ is summable with respect to k, see [7, Section 8.6]. For the proof of the first equality in (2.10), it suffices to show that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} Z_i = P_p(\Omega_0)^2 \tag{2.11}$$

holds P_p-a.s. By the choice of $(n_k)_{k=1}^{\infty}$, $(Z_k)_{k=1}^{\infty}$ is independent random variables under P_p. Using the strong law of large numbers, we obtain P_p-a.s.,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} (Z_i - E_p[Z_i]) = 0. \tag{2.12}$$

In addition,

$$\lim_{k\to\infty} E_p[Z_k] = \lim_{k\to\infty} P_p(\#C(0) \geq cn_k)^2 = P_p(\Omega_0)^2,$$

which proves (2.11) from (2.12) and Cesaro's theorem. According to the same argument as above, the second equality of (2.10) is verified, and therefore we can construct the desired sequence $n_k = n_k(|x|,|y|,|z|,|x-z|,|y-z|)$, $k \geq 1$.

We now turn to prove (1.4). Without loss of generality we can assume $x, y, x+y \in \mathbf{Z}^d \backslash \{0\}$. Let $n_k = n_k(|x|,|y|,|x-z|,|y-z|)$, $k \geq 1$ be the sequence as above and set for $k \geq 1$,

$$\begin{aligned} A_k := \{ & a_\lambda(0, n_k x) \leq (\lambda + \log(2d))\rho n_k |x|, \\ & a_\lambda(0, n_k(x+y)) \leq (\lambda + \log(2d))\rho n_k |x+y|, \\ & a_\lambda(n_k x, n_k(x+y)) \leq (\lambda + \log(2d))\rho n_k |y| \}. \end{aligned}$$

Lemma 2.1 then shows that

$$\begin{aligned} (2.13) \qquad & \frac{1}{k}\sum_{i=1}^{k} \frac{1}{n_i} \mathbf{E}[a_\lambda(0, n_i(x+y)) \mathbf{1}_{A_i} \mathbf{1}_{\{n_i x, n_i(x+y) \in \mathscr{C}_\infty\}}] \\ & \leq \frac{1}{k}\sum_{i=1}^{k} \frac{1}{n_i} \mathbf{E}[a_\lambda(0, n_i x) \mathbf{1}_{A_i} \mathbf{1}_{\{n_i x, n_i(x+y) \in \mathscr{C}_\infty\}}] \\ & \quad + \frac{1}{k}\sum_{i=1}^{k} \frac{1}{n_i} \mathbf{E}[a_\lambda(n_i x, n_i(x+y)) \mathbf{1}_{A_i} \mathbf{1}_{\{n_i x, n_i(x+y) \in \mathscr{C}_\infty\}}]. \end{aligned}$$

On the other hand, we obtain from (2.3) and (2.7) that

$$\begin{aligned} & \mathbf{P}(\mathbf{1}_{A_k} \mathbf{1}_{\{n_k x, n_k(x+y) \in \mathscr{C}_\infty\}} \neq \mathbf{1}_{\{n_k x, n_k(x+y) \in \mathscr{C}_\infty\}}) \\ & \quad \leq \mathbf{P}(n_k x \in \mathscr{C}_\infty, a_\lambda(0, n_k x) > (\lambda + \log(2d))\rho n_k |x|) \\ & \qquad + \mathbf{P}(n_k(x+y) \in \mathscr{C}_\infty, a_\lambda(0, n_k(x+y)) > (\lambda + \log(2d))\rho n_k |x+y|) \\ & \qquad + \mathbf{P}(n_k x, n_k(x+y) \in \mathscr{C}_\infty, a_\lambda(n_k x, n_k(x+y)) > (\lambda + \log(2d))\rho n_k |y|) \\ & \quad \leq 3P_p(\Omega_0)^{-1} c_1 \exp\{-c_2 n_k\}. \end{aligned}$$

The Borel–Cantelli lemma thereby tells us that $\mathbf{P}$-a.s., there is a positive integer $K = K(\omega)$ such that

$$(2.14) \qquad \mathbf{1}_{A_k} \mathbf{1}_{\{n_k x, n_k(x+y) \in \mathscr{C}_\infty\}} = \mathbf{1}_{\{n_k x, n_k(x+y) \in \mathscr{C}_\infty\}}$$

holds for all $k \geq K$. Furthermore, Corollary 2.4 yields that

$$\limsup_{k\to\infty} \left| \frac{1}{n_k} a_\lambda(0, n_k x) - \alpha_\lambda(x) \right| \mathbf{1}_{\{n_k x \in \mathscr{C}_\infty\}} = 0$$

holds $\mathbf{P}$-a.s. By (2.10), we have $\mathbf{P}$-a.s.,

$$
\begin{aligned}
(2.15)\qquad & \limsup_{k\to\infty}\left|\frac{1}{k}\sum_{i=1}^{k}\frac{1}{n_i}a_\lambda(0,n_ix)\mathbf{1}_{A_i}\mathbf{1}_{\{n_ix,\,n_i(x+y)\in\mathscr{C}_\infty\}}-\alpha_\lambda(x)P_p(\Omega_0)^2\right| \\
&\le \limsup_{k\to\infty}\frac{1}{k}\sum_{i=1}^{K-1}\frac{1}{n_i}a_\lambda(0,n_ix)\mathbf{1}_{A_i}\mathbf{1}_{\{n_ix,\,n_i(x+y)\in\mathscr{C}_\infty\}} \\
&\quad+\limsup_{k\to\infty}\frac{1}{k}\sum_{i=K}^{k}\left|\frac{1}{n_i}a_\lambda(0,n_ix)-\alpha_\lambda(x)\right|\mathbf{1}_{\{n_ix\in\mathscr{C}_\infty\}} \\
&\quad+\limsup_{k\to\infty}\alpha_\lambda(x)\left|\frac{1}{k}\sum_{i=K}^{k}\mathbf{1}_{\{n_ix,\,n_i(x+y)\in\mathscr{C}_\infty\}}-P_p(\Omega_0)^2\right| \\
&=0.
\end{aligned}
$$

From the bounded convergence theorem the first term in the right-hand side of (2.13) converges to $\alpha_\lambda(x)P_p(\Omega_0)^2$ as $k\to\infty$. The same argument as above shows that the left-hand side of (2.13) converges to $\alpha_\lambda(x+y)P_p(\Omega_0)^2$ as $k\to\infty$.

It remains to estimate the second term in the right-hand side of (2.13). Note that, by the shift invariance of P_p, the second term in the right-hand side of (2.13) is equal to

$$
\begin{aligned}
&\frac{1}{kP_p(\Omega_0)}\sum_{i=1}^{k}\frac{1}{n_i}E_p[a_\lambda(-n_iy,0)\mathbf{1}_{\theta_{n_i(x+y)}A_i}\mathbf{1}_{\{-n_iy,\,0,\,-n_i(x+y)\in\mathscr{C}_\infty\}}] \\
&\qquad=\frac{1}{k}\sum_{i=1}^{k}\frac{1}{n_i}\mathbf{E}[a_\lambda(-n_iy,0)\mathbf{1}_{\theta_{n_i(x+y)}A_i}\mathbf{1}_{\{-n_iy,\,-n_i(x+y)\in\mathscr{C}_\infty\}}].
\end{aligned}
$$

In addition, the same argument as in the proof of (2.14) shows that **P**-a.s.,

$$
\mathbf{1}_{\theta_{n_k(x+y)}A_k}\mathbf{1}_{\{-n_ky,\,-n_k(x+y)\in\mathscr{C}_\infty\}}=\mathbf{1}_{\{-n_ky,\,-n_k(x+y)\in\mathscr{C}_\infty\}}
$$

holds for all sufficiently large k. On the other hand, using the shift invariance of P_p again, one has for $k\ge 1$,

$$
\begin{aligned}
&\mathbf{E}\left[\frac{1}{k}\sum_{i=1}^{k}\left|\frac{1}{n_i}a_\lambda(-n_iy,0)-\alpha_\lambda(y)\right|\mathbf{1}_{\{-n_iy\in\mathscr{C}_\infty,\,a_\lambda(-n_iy,0)\le(\lambda+\log(2d))\rho n_i|y|\}}\right] \\
&\qquad=\frac{1}{kP_p(\Omega_0)}\sum_{i=1}^{k}E_p\left[\left|\frac{1}{n_i}a_\lambda(0,n_iy)-\alpha_\lambda(y)\right|\mathbf{1}_{\{n_iy,\,0\in\mathscr{C}_\infty,\,a_\lambda(0,n_iy)\le(\lambda+\log(2d))\rho n_i|y|\}}\right].
\end{aligned}
$$

Therefore, Corollary 2.4, the bounded convergence theorem and Cesaro's theorem ensure that the right-hand side of the above expression converges to zero as $k\to\infty$. We can therefore take a sequence $(k_m)_{m=1}^{\infty}$ of $\mathbf{N}$ such that

$$
\lim_{m\to\infty}\frac{1}{k_m}\sum_{i=1}^{k_m}\left|\frac{1}{n_i}a_\lambda(-n_iy,0)-\alpha_\lambda(y)\right|\mathbf{1}_{\{-n_iy\in\mathscr{C}_\infty,\,a_\lambda(-n_iy,0)\le(\lambda+\log(2d))\rho n_i|y|\}}=0
$$

holds **P**-a.s. By (2.10), the same strategy as in (2.15) proves that **P**-a.s.,

$$\lim_{k\to\infty}\left|\frac{1}{k}\sum_{i=1}^{k}\frac{1}{n_i}a_\lambda(-n_i y,0)\mathbf{1}_{\theta_{n_i(x+y)}A_i}\mathbf{1}_{\{-n_i y,\,-n_i(x+y)\in\mathscr{C}_\infty\}}-\alpha_\lambda(y)P_p(\Omega_0)^2\right|=0,$$

and hence, choosing $k=k_m$ in (2.13), we get (1.4) by letting $m\to\infty$. □

Remark 2.5. The argument in [6] contains a small error stating that the events $\{|C(kx)|>\alpha k, |C(ky)|>\alpha k\}$, $k\geq 1$ are independent. The above argument presents one of the way to fix this.

3. Shape theorem

In this section we give the proof of Theorem 1.2. For this end, let us first prove the next lemma which plays the role of the maximal lemma used in the context of random walk in random environment (see [11, Lemma 6]).

LEMMA 3.1. *Let* $c_4:=(\lambda+\log(2d))\rho$. *Then the following holds* **P**-a.s.: *for all* $0<\varepsilon\in\mathbf{Q}$ *there is a positive integer* $N=N(\varepsilon,\omega)$ *such that*

$$\sup\{d_\lambda(x,y);\, y\in\mathscr{C}_\infty, |x-y|\leq\varepsilon|x|\}\leq c_4\varepsilon|x|$$

holds for every $x\in\mathscr{C}_\infty$ *with* $|x|\geq N$, *where* d_λ *is defined as in* (1.1).

Proof. Let $c_5:=3^d(\lambda+\log(2d))$. We first show that there are positive constants c_6 and c_7 depending only on p and d such that for all $r>0$ and $x\in\mathbf{Z}^d$ with $|x|\leq r$,

$$\mathbf{P}(x\in\mathscr{C}_\infty, d_\lambda(0,x)>c_5 r^d)\leq c_6 e^{-c_7 r}. \tag{3.1}$$

Note that due to [6, Lemma 2.2], there exist c_8 and c_9 depending only on p and d such that for any $r>0$ and $x\in\mathbf{Z}^d$ with $|x|\leq r$,

$$P_p(0\overset{\omega}{\leftrightarrow}x, D_\omega(0,x)>(3r)^d)\leq c_8 e^{-c_9 r}.$$

It follows from (2.7) that

$$\begin{aligned}\mathbf{P}(x\in\mathscr{C}_\infty, d_\lambda(0,x)>c_5 r^d)&\leq P_p(\Omega_0)^{-1}P_p(0\overset{\omega}{\leftrightarrow}x, D_\omega(0,x)>(3r)^d)\\&\leq P_p(\Omega_0)^{-1}c_8 e^{-c_9 r},\end{aligned}$$

which proves (3.1) by choosing $c_6:=P_p(\Omega_0)^{-1}c_8$ and $c_7:=c_9$.

Fix $\varepsilon\in(0,\infty)\cap\mathbf{Q}$. For $n\geq 1$ the event A_n is defined as follows: there exist $x,y\in\mathscr{C}_\infty$ with $|x|=n$ and $|x-y|\leq\varepsilon n$ such that $d_\lambda(x,y)>c_4\varepsilon n$ holds. Taking $r_n:=3^{-1}(c_5\varepsilon n)^{1/d}$ for $n\geq 1$, we obtain from (2.3) and (3.1),

$$\begin{aligned}\mathbf{P}(A_n) &\le \sum_{\substack{x\in\mathbf{Z}^d\\|x|=n}}\sum_{\substack{y\in\mathbf{Z}^d\\|x-y|\le r_n}}\mathbf{P}(x,y\in\mathscr{C}_\infty, d_\lambda(x,y)>c_4\varepsilon n)\\&\quad+\sum_{\substack{x\in\mathbf{Z}^d\\|x|=n}}\sum_{\substack{y\in\mathbf{Z}^d\\r_n<|x-y|\le\varepsilon n}}P_p(\Omega_0)^{-1}P_\omega(x\overset{\omega}{\leftrightarrow}y, D_\omega(x,y)>\rho|x-y|)\\&=\mathcal{O}(n^{2d-1}e^{-(c_2\wedge c_7)r_n}).\end{aligned}$$

This means that the sequence $(\mathbf{P}(A_n))_{n=0}^\infty$ is summable, and the proof is complete from the Borel–Cantelli lemma. □

Proof of Theorem 1.2. By the monotonicity and the continuity in λ of a_λ in λ and α_λ respectively, the proof goes in the same line as that of [11, Theorem A]. It suffices to show that for a fixed $\lambda\ge 0$ and all $\varepsilon\in(0,1)\cap\mathbf{Q}$, the following holds $\mathbf{P}$-a.s.: there exists a positive integer $N=N(\varepsilon,\omega)$ such that we have for all $x\in\mathscr{C}_\infty$ with $|x|\ge N$,

$$|a_\lambda(0,x)-\alpha_\lambda(x)|\le\varepsilon|x|.$$

To this end, assume that the above statement fails to hold. Then, there are $\varepsilon_0>0$ and an event of positive probability such that we can choose a sequence $(x_n)_{n=1}^\infty$ of $\mathscr{C}_\infty$ satisfying $|x_n|\to\infty$ and $|a_\lambda(0,x_n)-\alpha_\lambda(x_n)|>\varepsilon_0|x_n|$. Without loss of generality we can assume $x_n/|x_n|\to v\in S^{d-1}$. Let η be an arbitrary positive number, which is chosen small enough later. Pick $v'\in S^{d-1}\cap\mathbf{Q}^d$ with $|v-v'|<\eta$ and M is a positive integer with $Mv'\in\mathbf{Z}^d$. Furthermore, we define for $n\ge 1$,

$$x_n' := \left\lfloor\frac{|x_n|}{M}\right\rfloor Mv'.$$

Thanks to [6, Lemma 5.5], $\mathbf{P}$-a.s., there is a positive integer $k_n=k_n(\eta,\omega)$ such that $(1-\eta)\lfloor|x_n|/M\rfloor\le k_n\le\lfloor|x_n|/M\rfloor$ and $k_nMv'\in\mathscr{C}_\infty$. Therefore,

$$|x_n-x_n'|\le\eta|x_n|+M\le 2\eta|x_n|$$

is valid for all sufficiently large n. In addition, by letting $y_n:=k_nMv'$, the choice of k_n guarantees that

$$|x_n-y_n|\le 2\eta|x_n|+M\eta\left\lfloor\frac{|x_n|}{M}\right\rfloor\le 3\eta|x_n|.$$

Lemma 3.1 thereby implies that $d_\lambda(x_n,y_n)\le 3c_4\eta|x_n|$ holds for all sufficiently large n. It follows form Theorem 1.1 and Proposition 2.1 that

$$\begin{aligned}|a_\lambda(0,x_n)-\alpha_\lambda(x_n)| &\le 3c_6\eta|x_n|+\eta\left\lfloor\frac{|x_n|}{M}\right\rfloor+3(\lambda+\log(2d))\rho\eta|x_n|\\&\le c\eta|x_n|\end{aligned}$$

for some $c = c(p, d) > 0$ and for all sufficiently large n. This is contradiction by letting $\eta \le c^{-1}\varepsilon_0$, and hence the proof is complete. □

Remark 3.2. In [11] Zerner proved more general version of Theorem 1.2, which is called the *uniform shape theorem*. This guarantees that the shape theorem is valid with moving starting points, and is used to relate crossing cost to the Lyapunov exponents in the proof of large deviation lower bound. Unfortunately, the same argument taken in [11] is not applied directly to our model, since we are not able to apply a martingale method used there due to the fluctuation of supercritical site percolations. However, we do not need the stronger theorem as the uniform shape theorem to prove the large deviation principles. We will state the next corollary for more details.

COROLLARY 3.3. *Let* $x \in \mathbf{Q}^d \backslash \{0\}$ *and* $\beta \in (0,1)$. *Then,*

$$\lim_{n\to\infty} \frac{1}{n} a_\lambda(T_{Mv}^{(\lfloor P_p(\Omega_0)\beta n|x|/M\rfloor)} Mv, T_{Mv}^{(\lfloor P_p(\Omega_0)n|x|/M\rfloor)} Mv) = (1-\beta)\alpha_\lambda(z) \tag{3.2}$$

holds $\mathbf{P}$-a.s., *where* $v = x/|x|$ *and* M *is a positive integer such that* $Mv \in \mathbf{Z}^d$.

Proof. The proof goes in the same line as that of [5, Corollary 3.3]. Namely, due to Lemma 2.1, the left-hand side of (3.2) is bounded from below by

$$a_\lambda(0, T_{Mv}^{(\lfloor P_p(\Omega_0)n|x|/M\rfloor)} Mv) - a_\lambda(0, T_{Mv}^{(\lfloor P_p(\Omega_0)\beta n|x|/M\rfloor)} Mv),$$

which converges to $(1-\beta)\alpha_\lambda(x)$ as $n \to \infty$ due to Corollary 2.4. Moreover, the left-hand side of (3.2) is bounded from above by

$$\begin{aligned} &a_\lambda(T_{Mv}^{(\lfloor P_p(\Omega_0)\beta n|x|/(NM)\rfloor N)} Mv, T_{Mv}^{(\lfloor P_p(\Omega_0)n|x|/(NM)\rfloor N)} Mv) \\ &\quad \le \sum_{k=\lfloor P_p(\Omega_0)\beta n|x|/(NM)\rfloor}^{\lfloor P_p(\Omega_0)n|x|/(NM)\rfloor - 1} a_\lambda(T_{Mv}^{(kN)} Mv, T_{Mv}^{((k+1)N)} Mv). \end{aligned}$$

We observe that the right-hand side of the above converges, as $n \to \infty$, to $(1-\beta)\alpha_\lambda(x)$ by Birkhoff's ergodic theorem. See the above reference for details. □

4. Large deviation estimates

Our goal in this section is to prove Theorem 1.3. We prove upper and lower bounds of Theorem 1.3 in Subsection 4.1 and 4.2 respectively.

4.1. Upper bound. In this subsection, we prove the upper bound (1.6) of Theorem 1.3. Let us first mention some properties of the rate function I. We denote the domain of the rate function I by $\mathscr{D}_I$, that is, $\mathscr{D}_I := \{x \in \mathbf{R}^d; I(x) < \infty\}$.

It is easy to see that I is convex on $\mathbf{R}^d$, lower semicontinuous on $\mathscr{D}_I$ and upper semicontinuous on the interior $(\mathscr{D}_I)^{\mathrm{o}}$ of $\mathscr{D}_I$. Furthermore, $I(x) = \infty$ for $|x| > 1$ and

$$0 \le I(x) \le |x|\rho \log(2d) \tag{4.1}$$

for $x \in \mathbf{R}^d$ with $|x| \le \rho^{-1}$.

Proof of the upper bound (1.6) *in Theorem* 1.3. It suffices to consider compact $A \subset \mathbf{R}^d$ since $I(x) = \infty$ for $|x| > 1$. Furthermore, we may assume $0 \notin A$ since $\inf_{x \in A} I(x) = 0$ if $0 \in A$ by (4.1). For every $\delta > 0$ we define the δ-rate function I^δ as

$$I^\delta(x) := (I(x) - \delta) \wedge \frac{1}{\delta}.$$

Set for any $\lambda \ge 0$,

$$A_\lambda(\delta) := \left\{ y \in A; \alpha_\lambda(y) - \lambda > \inf_{x \in A} I^\delta(x) - \delta \right\}.$$

Note that $A = \bigcup_{\lambda \ge 0} A_\lambda(\delta)$. Thanks to compactness of A, there are λ_i, $1 \le i \le m$ such that $A_{\lambda_i}(\delta)$, $1 \le i \le m$ cover A, and therefore we have

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega^0(X_n \in nA) \le \max_{1 \le i \le m} \limsup_{n\to\infty} \frac{1}{n} \log P_\omega^0(X_n \in nA_{\lambda_i}(\delta)).$$

Thus, it suffices to prove that the each components for i is bounded uniformly from above by $\delta - \inf_{x \in A} I^\delta(x)$, which gives (1.6) by letting $\delta \searrow 0$.

Let $\lambda \ge 0$ and we may assume without loss of generality that $nA_\lambda(\delta) \cap \mathscr{C}_\infty \neq \emptyset$ for all $n \ge 1$. Therefore,

$$\frac{1}{n} \log P_\omega^0(X_n \in nA_\lambda(\delta)) \le \lambda + \frac{1}{n} \log \#(nA_\lambda(\delta) \cap \mathbf{Z}^d) - \frac{1}{n} a_\lambda(0, y_{n,\lambda})$$

holds for some maximizing $y_{n,\lambda} \in nA_\lambda(\delta) \cap \mathscr{C}_\infty$. Using Theorem 1.2 and the fact that $\mathrm{dist}(0, A_\lambda(\delta)) > 0$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log P_\omega^0(X_n \in nA_\lambda(\delta)) \\
&\quad \le \lambda - \liminf_{n\to\infty} \frac{1}{n}(a_\lambda(0, y_{n,\lambda}) - \alpha_\lambda(y_{n,\lambda})) - \inf_{y \in A_\lambda(\delta)} \alpha_\lambda(y) \\
&\quad = \lambda - \inf_{y \in A_\lambda(\delta)} \alpha_\lambda(y).
\end{aligned}$$

Due to the definition of $A_\lambda(\delta)$, the most right-hand side of the above expression is bounded from above by $\delta - \inf_{x \in A} I^\delta(x)$. □

4.2. Lower bound. In this subsection, we prove the lower bound (1.7) of Theorem 1.3. To do this, we introduce for $\lambda \geq 0$, $\omega \in \Omega_0$ and $x, y \in \mathscr{C}_\infty(\omega)$ the path measure

$$\hat{P}^{x,y}_{\lambda,\omega}(dX_\cdot) := e_\lambda(x,y)^{-1} \exp\{-\lambda H(y)\}\mathbf{1}_{\{H(y)<\infty\}} P^x_\omega(dX_\cdot),$$

and let us prepare the following lemma, whose proof is, by Corollary 3.3, same as that of [5, Lemma 4.1] and we omit it.

LEMMA 4.1. *Let* $x \in \mathbf{Q}^d \backslash \{0\}$ *and* $\beta \in [0,1)$. *Moreover, let*

$$y_n^{(1)} := T_{Mv}^{(\lfloor P_p(\Omega_0)\beta n|x|/M \rfloor)} Mv,$$

$$y_n^{(2)} := T_{Mv}^{(\lfloor P_p(\Omega_0)n|x|/M \rfloor)} Mv.$$

The following then holds $\mathbf{P}$-a.s.:

$$\lim_{n\to\infty} \hat{P}^{y_n^{(1)},y_n^{(2)}}_{\lambda,\omega}\left(\frac{H(y_n^{(2)})}{(1-\beta)n} \in (\gamma_1,\gamma_2)\right) = 1 \tag{4.2}$$

for all $\lambda > 0$, $\gamma_1, \gamma_2 \in \mathbf{R}$ *satisfying*

$$0 \leq \gamma_1 < \alpha'_{\lambda+}(x) \leq \alpha'_{\lambda-}(x) < \gamma_2. \tag{4.3}$$

Proof of the lower bound (1.7) *in Theorem* 1.3. It is sufficient to prove that $\mathbf{P}$-a.s.,

$$\liminf_{t\to\infty} \frac{1}{n} \log P^0_\omega(X_n \in nB(z,r)) \geq -I(z) \tag{4.4}$$

holds for all $z \in \mathbf{Q}^d\backslash\{0\} \cap \mathscr{D}_I$ and $0 < r \in \mathbf{Q}$. To this end, set $\lambda_* := \sup\{\lambda \geq 0;\ \alpha'_\lambda(z) \geq 1\}$ with the convention $\sup\emptyset = 0$. It is easy to check that $I(z) = \alpha_{\lambda_*}(z) - \lambda_*$ in the case $\lambda_* < \infty$, and $I(z) = \lim_{\lambda\to\infty}(\alpha_\lambda(z) - \lambda)$ otherwise. We first treat the case $\lambda_* < \infty$. Suppose $\lambda_* = 0$, and then $\alpha'_{\lambda-}(z) < 1$ for all $\lambda > 0$. We now consider the site

$$y_n := T_{Mv}^{(\lfloor P_p(\Omega_0)n|z|/M \rfloor)} Mv,$$

where $v = z/|z|$ and M is an positive integer such that $Mv \in \mathbf{Z}^d$. By (2.2), $|y_n - nz| < \varepsilon n$ holds for all $\varepsilon > 0$ and all sufficiently large n. Therefore, choosing $\varepsilon \in (0,r)$, we have $B(y_n,R) \subset nB(z,r)$ for all $R > 0$ for all sufficiently large n. It follows that the left-hand side of (4.4) is bigger than

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n}\{\log P^0_\omega(H(y_n) \leq n) + \log P^{y_n}_\omega(X_{m+H(y_n)} \in B(y_n,R) \text{ for all } m \in [0,n])\} \\
&\quad \geq \liminf_{n\to\infty} \frac{1}{n}\{\log E^0_\omega[\exp\{-\lambda H(y_n)\}\mathbf{1}_{\{H(y_n)\leq n\}}] + \log P^{y_n}_\omega(X_R = y_n)^{n/R}\}
\end{aligned}$$

for $\lambda > 0$. Theorem 1.2 and Lemma 4.1 hence tell us that the above expression is greater than

$$-\alpha_\lambda(z) + \frac{1}{R} \log P_\omega^{y_n}(X_R = y_n).$$

According to [4, Theorem 2.2], $P_\omega^{y_n}(X_R = y_n)$ is bounded below by $c(R \log R)^{d/2}$ for some positive constant $c = c(p, d)$, and therefore we get (4.4) by letting $R \to \infty$ and $\lambda \searrow 0$.

In the cases $\lambda_* \in (0, \infty)$, thanks to Corollary 3.3, we can apply the same strategy of the proof of [11, Theorem B]. Furthermore, in the case $\lambda_* = \infty$, (4.4) follows from the argument as in the proof of [5, Theorem 1.4]. We thus refer the reader to the above references for more details. □

We close this section with a comment on the asymptotics of the rate function as $x \to 0$.

Proposition 4.2. *There exist positive constants C and C' depending only on p and d such that*

$$C|x|^2 \le I(x) \le C'|x|^2 \tag{4.5}$$

holds for all $x \in (\mathscr{D}_I)^o$ sufficiently close to 0. In particular, the zero set of the rate function $\{x \in \mathbf{R}^d; I(x) = 0\}$ is a single point $\{0\}$.

Proof. We first recall that according to [2, Theorem 1, Remark 7 and Proposition 6.1], there exist positive constants c, c', C, C' depending only on p and d such that $\mathbf{P}$-a.s.,

$$c'n^{-2/d} \exp\{-C'|y|^2/n\} \le P_\omega^0(X_{n-1} = y) + P_\omega^0(X_n = y) \le cn^{-2/d} \exp\{-C|y|^2/n\}$$

for all sufficiently large n and for all $y \in \mathscr{C}_\infty$ with $D_\omega(0, y) \le n$. For the proof, it suffices to check by the continuity of I on $(\mathscr{D}_I)^o$ that (4.5) holds for every $x \in \mathbf{Q}^d$. We now choose $\varepsilon > 0$ and $x \in \mathbf{Q}^d$ satisfying

$$\varepsilon + |x| < \rho^{-1}, \tag{4.6}$$

where ρ is the constant appearing in (2.3). Then, set $v = x/|x|$ and let M be an positive integer such that $Mv \in \mathbf{Z}^d$. Moreover, choosing $x_n' := T_{Mv}^{(\lfloor P_p(\Omega_0)n|x|/M \rfloor)} Mv$, by (4.6) and (2.2) we have $x_n' \in n\bar{B}(x, \delta)$ and $D_\omega(0, x_n') \le n$ for sufficiently large n. It follows from Theorem 1.3 that

$$\begin{aligned} -\inf_{y \in \bar{B}(x,\delta)} I(y) &\ge \limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{\substack{y \in n\bar{B}(x,\delta) \\ D_\omega(0,y) \le n}} c'n^{-2/d} \exp\{-C'|y|^2/n\} \right) \\ &\ge \limsup_{n\to\infty} \frac{1}{n} \log(c'n^{-2/d} \exp\{-C'|x_n'|^2/n\}) \\ &\ge C'(|x| - \delta)^2 \end{aligned}$$

for sufficiently small $|x|$ and $\delta > 0$, which proves the upper bound by letting $\delta \searrow 0$. Choosing $B = B(x, \delta)$ in (1.7), we can see the lower bound in the same manner. □

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