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BUNDLE DECOMPOSITION AND INFINITESIMAL CR AUTOMORPHISM APPROACHES TO CR AUTOMORPHISM GROUP OF GENERALIZED ELLIPSOIDS

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Abstract

Two types of elementary and direct proofs of the classification theorem for CR automorphisms of generalized ellipsoids by [MM10] are given. The first proof is to use an invariant decomposition of holomorphic tangent bundle, and the second is to use infinitesimal CR automorphisms. The advantages of our proofs are: (1) The conditions on the number n_j of the variables and the exponents m_j in the defining equation are weakened, (2) the proofs are more direct than [MM10], (3) our proofs may be applicable to wider class of hypersurfaces.

1. Introduction

In this note, we give two types of elementary proofs of the classification theorem in [MM10] posted in archive "arXiv:1004.1922v1[math.CV]". Let M be a strictly pseudoconvex part of the boundary of a generalized ellipsoid defined by the equation

$$\operatorname{Im} z_{n+1} = |z_1|^{2m_1} + \cdots + |z_{s-1}|^{2m_{s-1}} + |z_s|^2. \tag{1}$$

Here, we use multi-index notation for $z = (z_1, \ldots, z_s) \in \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_s} = \mathbf{C}^n$ and $|z_j|^{2m_j} = (|z_j^1|^2 + \cdots + |z_j^{n_j}|^2)^{m_j}$. For more precise notation, refer to section 2. R. Monti and D. Morbidelli proved the following classification theorem in [MM10].

Theorem 1.1. *Let N and $\tilde{N}$ be connected open subsets of M and $f : N \to \tilde{N}$ a CR diffeomorphism. Then for a suitable choice of ψ below, $r > 0$, $a = (a_s, a_{n+1}) \in \mathbf{C}^{n_s} \times \mathbf{C}$, and $a_{n+1} = t^0 + i|a_s|^2$, we have $f = \psi \circ \delta_r \circ J \circ \phi_a$, where*

$$I(z_1, \ldots, z_{s-1}, z_s, z_{n+1}) = (z_1/(z_{n+1})^{1/m_1}, \ldots, z_{s-1}/(z_{n+1})^{1/m_{s-1}}, z_s/z_{n+1}, -1/z_{n+1}), \tag{2}$$

$$\delta_r(z_1, \ldots, z_{s-1}, z_s, z_{n+1}) = (r^{1/m_1} z_1, \ldots, r^{1/m_{s-1}} z_{s-1}, r z_s, r^2 z_{n+1}), \tag{3}$$

$$\psi(z, z_{n+1}) = (B_1 z_{\sigma(1)}, \ldots, B_{s-1} z_{\sigma(s-1)}, B_s z_s + b_s, b_{n+1} + z_{n+1} + 2i(B_s z_s \cdot \bar{b}_s)), \tag{4}$$

$$\phi_a(z_1, \ldots, z_{s-1}, z_s, z_{n+1}) = (z_1, \ldots, z_{s-1}, z_s + a_s, z_{n+1} + a_{n+1} + 2i z_s \cdot \bar{a}_s). \tag{5}$$

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σ is a permutation of indices $\{1, \dots, s-1\}$, B_j's are unitary matrices, $b_s \in \mathbf{C}^{n_s}$, $b_{n+1} = t^1 + i|b_s|^2 \in \mathbf{C}$. $m_j, n_j \geq 2$ for $j = 1, \dots, s-1$ and $n_s \geq 0$. J is a mapping either I as above or the identity mapping.

D. Morbidelli [M09] proved a classification theorem for conformal homeomorphisms with respect to a certain distance. As an application of his technique, R. Monti and D. Morbidelli [MM10] proved Theorem 1.1 by constructing transformation rules of Chern curvatures, Ricci tensors and some invariants. Since we treat generalized ellipsoids, the transformation rules of them, which are usually difficult to compute, are calculated.

As an application of a Riemannian case to a CR case, their proof is successful and very interesting, but from the view point of the CR geometry, we can give an elementary and more direct proof without using any curvature, tensors and invariants.

The technique of the first proof is to use the decomposition of the holomorphic tangent bundle $T^{1,0}M$ of the boundary M of a generalized ellipsoid into its subbundles, which are invariant under CR mappings as introduced in [MM10]. Expand each component f_j of the CR mapping f and write down the conditions that the bundle decomposition is invariant under the push forward of f. It determines a Taylor expansion of f. The advantages of this proof are the following: [MM10] assumes that $m_j \geq 2$ for any j. In case of $m_j = 1$, since the mapping ψ in Theorem 1.1 can not be reconstructed from its CR factor $u_\psi = 1$ as in Theorem 4.3 in [MM10], the condition $m_j \geq 2$ can not be removed in their proof. Such case is studied in [M09]. On the contrary, our proof works also for any positive integers m_j after small modification of a bundle decomposition as noted in the section 4. Note that if all $n_j = 1$, the hypersurface M is a boundary of pseudoellipsoid and its CR automorphism group is well known. Therefore interesting case is that some n_j equals to one, and our argument also works for such a case after modification. We assume $m_j, n_j \geq 2$ throughout this paper except for section 4.2 (and section 7.2). About the proof of Theorem 3.5 in [MM10] (Lemma 2.1 in the present paper), they wrote in the head of section 5, "*The proof is rather involved, but we were not able to find a more direct one*." But our proof is more direct and simple one.

Our proof must work for the hypersurfaces defined by a variables splitting defining equation:

$$\operatorname{Im} z_{n+1} = f_1(z_1, \bar{z}_1) + \cdots + f_n(z_n, \bar{z}_n) \tag{6}$$

for $z_j \in \mathbf{C}^{n_j}$, $z_{n+1} \in \mathbf{C}$ and f_j's are real analytic functions.

The technique of the second proof is to use an infinitesimal CR automorphism. A real vector field X is an infinitesimal CR automorphism if the one-parameter group of transformation $\exp tX$ is a CR diffeomorphism of M. Let $\operatorname{hol}(M, p_0)$ be a set of infinitesimal CR automorphisms. [BER98] gave conditions on which $\operatorname{hol}(M, p_0)$ is a Lie algebra of $\operatorname{Aut}(M, p_0)$, the automorphism group of M fixing p_0. Remark that the boundary of generalized ellipsoid M defined by the equation (1) satisfies the condition. Refer §7 for more precise statement of their theorem. Therefore, we know the classification of CR automorphisms if we classify the infinitesimal CR automorphisms. Note that computing infinitesimal CR automorphisms is much easier than doing transformation rules of curvatures and tensors. From the proof of the classification theorem in [MM10], we do not know the reason why mappings I, δ_r, ψ and ϕ_a appear. But using infinitesimal CR automorphisms method, these mappings appear naturally as one-parameter group of transformations.

The organization of this note is the following. In the part 1, we use a bundle decomposition method. In section 2, we decompose a holomorphic tangent bundle and give some properties of it. In section 3, some relations among coefficients of the mapping are obtained. In section 4, we specify the coefficients in order to compare our result with [MM10]. We also give some remark about modification of a holomorphic tangent bundle decomposition there. In the part 2, we use infinitesimal CR automorphisms to obtain CR automorphisms. In section 5, we classify infinitesimal CR automorphisms, and in section 6, we obtain the local one-parameter group of diffeomorphism of $\exp tX$ for $X \in \mathrm{hol}(M, p_0)$, which leads to a classification of CR automorphisms of M. In the last section, we compare the one-parameter group of transformations with the mappings in Theorem 1.1 and give remark on the integers m_j and n_j.

Part 1. Bundle decomposition method

2. Decomposition of holomorphic tangent bundle and its invariancy

In this section, we define some notation following the paper [MM10]. Throughout this paper, we use a multi-index notation, for example, for $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ and $a = (a_1, \dots, a_n) \in \mathbf{N}^n$, $(z)^a = z_1^{a_1} \cdots z_n^{a_n}$ and $|a| = a_1 + \cdots + a_n$. Since we shall use upper index z_j^α as the αth component of $z_j = (z_j^1, \dots, z_j^{n_j}) \in \mathbf{C}^{n_j}$, we use $(z)^a$ for the a-th power of z instead of z^a. Let M_0 be a boundary of a generalized ellipsoid defined by $M_0 = \{(z, z_{n+1}) \in \mathbf{C}^{n+1} : \operatorname{Im} z_{n+1} = p(z, \bar{z})\}$, where $p(z, \bar{z}) = |z_1|^{2m_1} + \cdots + |z_{s-1}|^{2m_{s-1}} + |z_s|^2$ for $z = (z_1, \dots, z_s) \in \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_s} = \mathbf{C}^n$. We assume that m_i, n_j are integers such that $m_j, n_j \geq 2$ for $1 \leq j \leq s-1$ and $n_s \geq 0$. For a set of integers $I_j = \{1, 2, \dots, n_j\}$, we put $|z_j|^2 = \sum_{\alpha \in I_j} |z_j^\alpha|^2$. Let M be the strictly pseudoconvex part of M_0, namely,

$$M = \left\{ (z, z_{n+1}) \in M_0 : \prod_{j=1}^{s-1} |z_j| \neq 0 \right\}. \tag{7}$$

Note that $\Omega = \{(z, z_{n+1}) \in \mathbf{C}^{n+1} : \operatorname{Im} z_{n+1} > p(z, \bar{z})\}$ is the unbounded set which is biholomorphically equivalent to the bounded domain

$$E = \left\{ (z, z_{n+1}) \in \mathbf{C}^{n+1} : \sum_{j=1}^{s} |z_j|^{2m_j} + |z_{n+1}|^2 < 1 \right\}. \tag{8}$$

On M, we define vector fields Z_j^α, E_j, W_j^α for $j = 1, \dots, s$ such as

$$Z_j^\alpha = \frac{\partial}{\partial z_j^\alpha} + i m_j \bar{z}_j^\alpha |z_j|^{2(m_j-1)} \frac{\partial}{\partial t} \quad \text{for } t = \operatorname{Re} z_{n+1} \text{ and } \alpha \in I_j, \tag{9}$$

$$E_j = \frac{1}{m_j} \sum_{\alpha \in I_j} z_j^\alpha Z_j^\alpha, \tag{10}$$

$$W_j^\alpha = \frac{\partial}{\partial z_j^\alpha} - \sum_{\beta \in I_j} \frac{\bar{z}_j^\alpha z_j^\beta}{|z_j|^2} \frac{\partial}{\partial z_j^\beta} \quad \text{for } \alpha \in I_j \tag{11}$$

Let $\mathscr{E}$ be the subbundle of a holomorphic tangent bundle $T^{1,0}M$ spanned by the vector fields $E_1, \dots, E_s$, and $\mathscr{W}_j$ the subbundle of $T^{1,0}M$ spanned by the vector fields W_j^α with $\alpha \in I_j$. It follows from $z_j^1 W_j^1 + \cdots + z_j^{n_j} W_j^{n_j} = 0$ that

$$\mathscr{W}_j = \operatorname{span}\{W_j^1, \dots, W_j^{n_j}\} = \operatorname{span}\{W_j^2, \dots, W_j^{n_j}\}. \tag{12}$$

Then by [MM10], an orthogonal decomposition

$$T^{1,0}M = \mathscr{W}_1 \oplus \cdots \oplus \mathscr{W}_{s-1} \oplus \mathscr{W}_s \oplus \mathscr{E} \tag{13}$$

holds. Observe that $\mathscr{W}_s \oplus \mathscr{E} = \operatorname{span}\{E_1, \dots, E_{s-1}, Z_s^\alpha : \alpha \in I_s\}$. Let $N, \tilde{N} \subset M$ be connected open sets and $f : N \to \tilde{N}$ a CR automorphism. We put "tilde" on the target objects. Following [MM10], there are two possible cases:

$$f_*(\mathscr{W}_s \oplus \mathscr{E})_p = (\tilde{\mathscr{W}}_s \oplus \tilde{\mathscr{E}})_{f(p)}, \tag{14}$$

$$f_*(\mathscr{W}_s \oplus \mathscr{E})_p = (\tilde{\mathscr{W}}_j)_{f(p)} \quad \text{for some } j = 1, \dots, s-1. \tag{15}$$

The first case (14) means that the decomposition (13) is invariant under the push forward by a CR automorphism. [MM10] proved that the second case does not occur by using the transformation rules for Chern tensors, scalar curvatures and so on. We shall give a simple and elementary proof of it. In what follows, we shall omit indices p and $f(p)$ in (14) and (15), which will not make any confusion.

Lemma 2.1. *Let $N, \tilde{N} \subset M$ be open subsets and $f : N \to \tilde{N}$ a CR automorphism. Then $f_*(\mathscr{W}_s \oplus \mathscr{E}) = (\tilde{\mathscr{W}}_s \oplus \tilde{\mathscr{E}})$ holds. Therefore, after permutation of indices $j = 1, \dots, s-1$, we have $f_*(\mathscr{W}_j) = \tilde{\mathscr{W}}_j$.*

Proof. Put $f = (f_1, \dots, f_s)$ and $f_j = (f_j^1, \dots, f_j^{n_j})$. Assume that there exists j_0 such that $f_*(\mathscr{W}_s \oplus \mathscr{E}) = \tilde{\mathscr{W}}_{j_0}$ and fix it. Since, as noted, $\mathscr{W}_s \oplus \mathscr{E} = \operatorname{span}\{E_1, \dots, E_{s-1}, Z_s^\alpha : \alpha \in I_s\}$, the push forwards of E_l and Z_s^β by f are expressed by a combinations of $\tilde{W}_{j_0}^\alpha$, $\alpha = 2, \dots, n_j$ in view of (12) as follows:

$$f_* E_l = \sum_{\alpha=2}^{n_{j_0}} P_{l,\alpha} \tilde{W}_{j_0}^\alpha, \quad f_* Z_s^\beta = \sum_{\alpha=2}^{n_{j_0}} R_{s,\alpha}^\beta \tilde{W}_{j_0}^\alpha \tag{16}$$

for some functions $P_{l,\alpha}$ and $R^{\beta}_{s,\alpha}$ on $\tilde{N}$. Applying the coordinate function $\tilde{z}^{\gamma}_m$ with $m = 1, \dots s$, $m \neq j_0$ and $\gamma \in I_m$ to the both sides of (16), we get $E_l f^{\gamma}_m = Z^{\beta}_s f^{\gamma}_m = 0$. Expand f^{γ}_m as

$$f^{\gamma}_m(z, t + ip) = \sum_{|a|, b \geq 0} A^{\gamma}_{m:a,b}(z)^a (t + ip)^b. \tag{17}$$

Then we have

$$Z^{\beta}_s f^{\gamma}_m = \sum_{\substack{b \geq 0 \\ a^{\beta}_s \geq 1}} A^{\gamma}_{m:a,b} a^{\beta}_s (z^1_1)^{a^1_1} \cdots (z^{\beta}_s)^{a^{\beta}_s - 1} \cdots (z^{n_s}_s)^{a^{n_s}_s} (t + ip)^b \tag{18}$$
$$+ i\bar{z}^{\beta}_s \sum_{\substack{b \geq 0 \\ a^{\beta}_s \geq 1}} A^{\gamma}_{m:a,b} b (z)^a (t + ip)^b = 0.$$

Here we have used a multi-indices notation for $a = (a_1, \dots a_s)$ and $a_j = (a^1_j, \dots a^{n_j}_j)$. Therefore the expansion of f^{γ}_m is reduced to

$$f^{\gamma}_m = \sum_{a = (a_1, \dots, a_{s-1})} A^{\gamma}_{m:a} (z_1)^{a_1} \cdots (z_{s-1})^{a_{s-1}}. \tag{19}$$

Next we apply E_l to this expansion to get

$$E_l f^{\gamma}_m = \sum_{\alpha \in I_l} \sum_{a^{\alpha}_l \geq 1} A^{\gamma}_{m:a} a^{\alpha}_l (z_1)^{a_1} \cdots (z_{s-1})^{a_{s-1}} = 0. \tag{20}$$

This implies that $f^{\gamma}_m = A^{\gamma}_{m:0}$, which contradicts to f being a diffeomorphism. This completes the proof of Lemma 2.1. □

Now we may assume that

$$f_*(\mathscr{W}_j) = \tilde{\mathscr{W}}_j, \quad f_*(\mathscr{W}_s \oplus \mathscr{E}) = \tilde{\mathscr{W}}_s \oplus \tilde{\mathscr{E}} \tag{21}$$

after reordering $\mathscr{W}_1, \dots, \mathscr{W}_{s-1}$.

3. Expansion of CR automorphisms

In this section, we shall give expansions of the components of $f = (f_1, \dots, f_s, f_{n+1})$ making use of (21).

Recall that $f_k = (f^1_k, \dots, f^{n_k}_k)$. First we prove that f^{γ}_k is a function with variables z_k, z_s and $z_{n+1} = t + ip(z, \bar{z})$ for $k = 1, \dots, s$, $\gamma = 1, \dots, n_k$, and f_{n+1} is a function with z_s and $z_{n+1} = t + ip(z, \bar{z})$.

We expand f^{γ}_k as

$$f^{\gamma}_k(z, t + ip) = \sum_{|a|, b \geq 0} A^{\gamma}_{k:a,b}(z)^a (t + ip)^b. \tag{22}$$

For any fixed $j = 1, \dots, s-1$ and $\alpha \in I_j$, put

$$f_* W_j^\alpha = \sum_{\beta \in I_j} P_\beta^\alpha \tilde{W}_j^\beta \tag{23}$$

and apply it to the coordinate function $\tilde{z}_k^\gamma$ for $k = 1, \dots, s$ with $k \neq j$ and $\gamma = 1, \dots, n_k$, to get $W_j^\alpha f_k^\gamma = 0$ and it is calculated as follows.

$$\begin{aligned} W_j^\alpha f_k^\gamma = \left(1 - \frac{|z_j^\alpha|^2}{|z_j|^2}\right) & \left\{ \sum_{a_j^\alpha \geq 1} A_{k:a,b}^\gamma a_j^\alpha (z_1^1)^{a_1^1} \cdots (z_j^\alpha)^{a_j^\alpha - 1} \cdots (z_s^{n_s})^{a_s^{n_s}} (t + ip)^b \right. \\ & \left. + \sum_{b \geq 1} A_{k:a,b}^\gamma b(z)^a (t+ip)^{b-1} i \frac{\partial p}{\partial z_j^\alpha} \right\} \\ - \frac{\bar{z}_j^\alpha}{|z_j|^2} \sum_{\substack{\beta \in I_j \\ \beta \neq \alpha}} & \left\{ \sum_{a_j^\beta \geq 1} A_{k:a,b}^\gamma a_j^\beta (z)^a (t+ip)^b \right. \\ & \left. + \sum_{b \geq 1} A_{k:a,b}^\gamma b(z)^a (t+ip)^{b-1} i \frac{\partial p}{\partial z_j^\beta} \right\} = 0. \end{aligned} \tag{24}$$

First, multiply $|z_j|^2$ to (24), and pick up the terms which contain the first order of $\bar{z}$ and do not contain $\bar{z}_j^\alpha$ to get

$$\sum_{a_j^\alpha \geq 1} A_{k:a,b}^\gamma a_j^\alpha (z_1^1)^{a_1^1} \cdots (z_j^\alpha)^{a_j^\alpha - 1} \cdots (z_s^{n_s})^{a_s^{n_s}} t^b = 0 \tag{25}$$

for $j = 1, \dots, s-1$, $j \neq k$. This means that $A_{k:a,b}^\gamma = 0$ if at least one of $a_j^1, \dots a_j^{n_j}$ is positive for $j \neq k$. Therefore we conclude that f_k^γ is expanded in z_k, z_s and $t + ip$ as

$$\begin{aligned} f_k^\gamma(z, t+ip) = & \sum_{|a_s|, b \geq 0} A_{k:a_s,b}^\gamma (z_s)^{a_s} (t+ip)^b \\ & + \sum_{\substack{|a_k| \geq 1 \\ |a_s|, b \geq 0}} A_{k:a_k,a_s,b}^\gamma (z_k)^{a_k} (z_s)^{a_s} (t+ip)^b \end{aligned} \tag{26}$$

for $k = 1, \dots, s-1$ and

$$f_s^\gamma(z, t+ip) = \sum_{|a_s|, b \geq 0} A_{s:a_s,b}^\gamma (z_s)^{a_s} (t+ip)^b. \tag{27}$$

Similarly, we apply the coordinate function $\tilde{z}_{n+1}$ to (23), and we conclude that f_{n+1} is a function with variables z_s and z_{n+1}.

The rest of this section, we shall show that $A^{\gamma}_{k:a_s,b} = 0$ and that $A^{\gamma}_{k:a_k,a_s,b}$ appears only for $|a_k| = 1$ in (26) and obtain relations among coefficients.

Claim 1. *$A^{\gamma}_{k:a_s,b} = 0$ for $k = 1, \dots, s-1$ in* (26).

By mean of Lemma 2.1, the push forward

$$f_* W^{\alpha}_j = \sum_{l=1}^{s-1} \sum_{\beta \in \{2,\dots,n_l\}} P^{\alpha,l}_{j,\beta} \tilde{W}^{\beta}_l + \sum_{l=1}^{s-1} Q^{\alpha,l}_j \tilde{E}_l + \sum_{\beta \in I_s} R^{\alpha}_{j,\beta} \tilde{Z}^{\beta}_s \tag{28}$$

should imply $Q^{\alpha,l}_j = R^{\alpha}_{j,\beta} = 0$. Apply the coordinate function $\tilde{z}^{\gamma}_j$ to (28) for $j = 1, \dots, s-1$ to get

$$W^{\alpha}_j f^{\gamma}_j = -\frac{\bar{f}^2_j f^{\gamma}_j}{|f_j|^2} P^{\alpha,j}_{j,2} - \cdots + \left(1 - \frac{\bar{f}^{\gamma}_j f^{\gamma}_j}{|f_j|^2}\right) P^{\alpha,j}_{j,\gamma} - \cdots - \frac{\bar{f}^{n_j}_j f^{\gamma}_j}{|f_j|^2} P^{\alpha,j}_{j,n_j} + \frac{Q^{\alpha,j}_j}{m_j} f^{\gamma}_j. \tag{29}$$

Since α and γ move from 1 to n_j, this can be written as the matrices form:

$$\begin{pmatrix} W^1_j f^1_j & \cdots & W^1_j f^{n_j}_j \\ \vdots & \ddots & \vdots \\ W^{n_j}_j f^1_j & \cdots & W^{n_j}_j f^{n_j}_j \end{pmatrix} \tag{30}$$
$$= \begin{pmatrix} P^{1,j}_{j,2} & \cdots & P^{1,j}_{j,n_j} & Q^{1,j}_j \\ \vdots & \ddots & \vdots & \vdots \\ P^{n_j,j}_{j,2} & \cdots & P^{n_j,j}_{j,n_j} & Q^{n_j,j}_j \end{pmatrix}$$
$$\times \begin{pmatrix} -\frac{\bar{f}^2_j f^1_j}{|f_j|^2} & 1 - \frac{|f^2_j|^2}{|f_j|^2} & -\frac{\bar{f}^2_j f^3_j}{|f_j|^2} & \cdots & -\frac{\bar{f}^2_j f^{n_j}_j}{|f_j|^2} \\ -\frac{\bar{f}^3_j f^1_j}{|f_j|^2} & -\frac{\bar{f}^3_j f^2_j}{|f_j|^2} & 1 - \frac{|f^3_j|^2}{|f_j|^2} & \cdots & -\frac{\bar{f}^3_j f^{n_j}_j}{|f_j|^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\bar{f}^{n_j}_j f^1_j}{|f_j|^2} & -\frac{\bar{f}^{n_j}_j f^2_j}{|f_j|^2} & -\frac{\bar{f}^{n_j}_j f^3_j}{|f_j|^2} & \cdots & 1 - \frac{|f^{n_j}_j|^2}{|f_j|^2} \\ \frac{f^1_j}{m_j} & \frac{f^2_j}{m_j} & \frac{f^3_j}{m_j} & \cdots & \frac{f^{n_j}_j}{m_j} \end{pmatrix}.$$

Since the inverse of the second matrix in the right hand side is calculated as

$$(31)\qquad -\frac{m_j}{f_j^1}\begin{pmatrix} \frac{f_j^2}{m_j} & \frac{f_j^3}{m_j} & \frac{f_j^4}{m_j} & \cdots & \frac{f_j^{n_j}}{m_j} & -\frac{|f_j^1|^2}{|f_j|^2} \\ -\frac{f_j^1}{m_j} & 0 & 0 & \cdots & 0 & -\frac{f_j^1\bar{f}_j^2}{|f_j|^2} \\ 0 & -\frac{f_j^1}{m_j} & 0 & \cdots & 0 & -\frac{f_j^1\bar{f}_j^3}{|f_j|^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{f_j^1}{m_j} & -\frac{f_j^1\bar{f}_j^{n_j}}{|f_j|^2} \end{pmatrix},$$

the coefficients $Q_l^{\alpha,l}$ can be solved from (30) and they should be zero. It leads to a system of equations:

$$(32)\qquad \begin{cases} W_j^1 f_j^1\cdot\bar{f}_j^1+\cdots+W_j^1 f_j^{n_j}\cdot\bar{f}_j^{n_j}=0 \\ \quad\vdots \\ W_j^{n_j} f_j^1\cdot\bar{f}_j^1+\cdots+W_j^{n_j} f_j^{n_j}\cdot\bar{f}_j^{n_j}=0. \end{cases}$$

Substitute the expansion (26) and vector field (11) into this system and multiply $|z_j|^2$ to the both sides. Then the terms in the αth equation in (32) with the variables $z_j^\beta\bar{z}_j^\alpha(z_s)^{a_s}(\bar{z}_s)^{a_s}t^b$ with $\alpha\neq\beta$ satisfy

$$(33)\qquad \sum_{v=1}^{n_j}\left\{\sum_{\substack{|a_s|\geq 0\\ b\geq 0}}\bar{A}^v_{j:a_s,b}(\bar{z})_s^{a_s}t^b\sum_{\substack{|a_s|\geq 0\\ b\geq 0}}A^v_{j:a_j^\beta=1,a_s,b}(z_s)^{a_s}t^b\right\}=0.$$

Here, $A^v_{j:a_j^\beta=1,a_s,b}$ means that $A^v_{j:a_j,a_s,b}$ with $a_j=(0,\dots,1,\dots,0)$ (the βth component is 1 and the others are all 0). This equation holds for $j=1,\dots,s-1$ and $\beta\in I_j$. Therefore this is written in the matrix equation as follows:

$$(34)\qquad \begin{pmatrix} \sum_{\substack{|a_s|\geq 0\\ b\geq 0}}A^1_{j:a_j^1=1,a_s,b}(z_s)^{a_s}t^b & \cdots & \sum_{\substack{|a_s|\geq 0\\ b\geq 0}}A^{n_j}_{j:a_j^1=1,a_s,b}(z_s)^{a_s}t^b \\ \vdots & \ddots & \vdots \\ \sum_{\substack{|a_s|\geq 0\\ b\geq 0}}A^1_{j:a_j^{n_j}=1,a_s,b}(z_s)^{a_s}t^b & \cdots & \sum_{\substack{|a_s|\geq 0\\ b\geq 0}}A^{n_j}_{j:a_j^{n_j}=1,a_s,b}(z_s)^{a_s}t^b \end{pmatrix}$$

$$\times\begin{pmatrix} \sum_{\substack{|a_s|\geq 0\\ b\geq 0}}\bar{A}^1_{j:a_s,b}(\bar{z})_s^{a_s}t^b \\ \vdots \\ \sum_{\substack{|a_s|\geq 0\\ b\geq 0}}\bar{A}^{n_j}_{j:a_s,b}(\bar{z})_s^{a_s}t^b \end{pmatrix}=0.$$

Since the mapping f is a diffeomorphism on N, the determinant of the first matrix in (34) is nonzero there. Therefore it implies

$$\sum_{b\geq 0}\bar{A}^{1}_{j:a_s,b}(\bar{z})^{a_s}_{s}t^b=\cdots=\sum_{b\geq 0}\bar{A}^{n_j}_{j:a_s,b}(\bar{z})^{a_s}_{s}t^b=0, \tag{35}$$

which proves Claim 1. □

Claim 2. *$A^{\gamma}_{k:a_k,a_s,b}=0$ for $|a_k|\geq 2$ in* (26).

We use an induction on $|a_k|$. First, pick up the terms with the first order of z_j and the third order of $\bar{z}_j$ for $j=1,\dots,s-1$ from the αth equation in (32), then we get

$$\sum_{\gamma\in I_j}\left[\sum_{|a_j|=2}\bar{A}^{\gamma}_{j:a_j,a_s,b}(\bar{z}_j)^{a_j}(\bar{z}_s)^{a_s}t^b \times\left\{|z_j|^2\sum A^{\gamma}_{j:a^{\alpha}_j=1,a_s,b}(z_s)^{a_s}t^b-\bar{z}^{\alpha}_j\sum_{\substack{b\geq 0\\ \beta\in I_j}}A^{\gamma}_{j:a^{\beta}_j=1,a_s,b}z^{\beta}_j(z_s)^{a_s}t^b\right\}\right]=0. \tag{36}$$

for any fixed $\alpha=1,\dots,n_j$. Since the terms with $\bar{z}^{\alpha}_j$ satisfy the matrix equation:

$$\begin{pmatrix}\sum A^{1}_{j:a^1_j=1,a_s,b}(z_s)^{a_s}t^b & \cdots & \sum A^{n_j}_{j:a^1_j=1,a_s,b}(z_s)^{a_s}t^b\\ \vdots & \ddots & \vdots\\ \sum A^{1}_{j:a^{n_j}_j=1,a_s,b}(z_s)^{a_s}t^b & \cdots & \sum A^{n_j}_{j:a^{n_j}_j=1,a_s,b}(z_s)^{a_s}t^b\end{pmatrix} \times\begin{pmatrix}\sum_{|a_j|=2}\bar{A}^{1}_{j:a_j,a_s,b}(\bar{z}_j)^{a_j}(z_s)^{a_s}t^b\\ \vdots\\ \sum_{|a_j|=2}\bar{A}^{n_j}_{j:a_j,a_s,b}(\bar{z}_j)^{a_j}(z_s)^{a_s}t^b\end{pmatrix}=0, \tag{37}$$

we obtain $A^{\gamma}_{k:a_k,a_s,b}=0$ for $|a_k|=2$. Secondly, we assume $A^{\gamma}_{k:a_k,a_s,b}=0$ for $|a_k|=d-1$. Now pick up the $(d+2)$nd order terms of z_j and $\bar{z}_j$ for $j=1,\dots s-1$. The same argument leads to the matrix equation:

$$\begin{pmatrix}\sum A^{1}_{j:a^1_j=1,a_s,b}(z_s)^{a_s}t^b & \cdots & \sum A^{n_j}_{j:a^1_j=1,a_s,b}(z_s)^{a_s}t^b\\ \vdots & \ddots & \vdots\\ \sum A^{1}_{j:a^{n_j}_j=1,a_s,b}(z_s)^{a_s}t^b & \cdots & \sum A^{n_j}_{j:a^{n_j}_j=1,a_s,b}(z_s)^{a_s}t^b\end{pmatrix} \tag{38}$$

$$\times \begin{pmatrix} \sum_{|a_j|=d} \bar{A}^1_{j:a_j,a_s,b} (\bar{z}_j)^{a_j} (\bar{z}_s)^{a_s} t^b \\ \vdots \\ \sum_{|a_j|=d} \bar{A}^{n_j}_{j:a_j,a_s,b} (\bar{z}_j)^{a_j} (\bar{z}_s)^{a_s} t^b \end{pmatrix} = 0.$$

Therefore we obtain $A^{\gamma}_{k:a_k,a_s,b} = 0$ for $|a_k| = d$, which proves Claim 2. □

As a result, the expansion (26) of f^{γ}_k is reduced to

$$f^{\gamma}_k(z, t+ip) = \sum_{\substack{|a_s|\geq 0 \\ b\geq 0}} A^{\gamma}_{k:1,a_s,b} z^1_k (z_s)^{a_s} (t+ip)^b + \cdots + \sum_{\substack{|a_s|\geq 0 \\ b\geq 0}} A^{\gamma}_{k:n_k,a_s,b} z^{n_k}_k (z_s)^{a_s} (t+ip)^b \tag{39}$$

for $k = 1, \ldots, s-1$.

Next we shall obtain the relations among coefficients $A^{\gamma}_{k:1,a_s,b}, \ldots, A^{\gamma}_{k:n_k,a_s,b}$.

Claim 3. *The relation*

$$A^{\gamma}_{k,\lambda,a_s,b} = \frac{A^{\gamma}_{k:\lambda,0,0}}{A^1_{k:\lambda,0,0}} A^1_{k:\lambda,a_s,b} \tag{40}$$

holds for $\lambda = 1, \ldots, n_k$.

We apply the coordinate function $\tilde{z}^{\gamma}_j$ $(j = 1, \ldots, s-1)$ to

$$f_* E_j = \sum_{l=1}^{s-1} \sum_{\beta\in\{2,\ldots,n_l\}} P^l_{j,\beta} \tilde{W}^{\beta}_l + \sum_{l=1}^{s-1} Q^l_j \tilde{E}_l + \sum_{\alpha\in I_s} R_{j,\alpha} \tilde{Z}^{\alpha}_s \tag{41}$$

to get a matrix equation

$$\begin{pmatrix} E_j f^1_j \\ \vdots \\ E_j f^{n_j}_j \end{pmatrix} = \begin{pmatrix} -\frac{\bar{f}^2_j f^1_j}{|f_j|^2} & -\frac{\bar{f}^3_j f^1_j}{|f_j|^2} & \cdots & -\frac{\bar{f}^{n_j}_j f^1_j}{|f_j|^2} & \frac{f^1_j}{m_j} \\ 1 - \frac{|f^2_j|^2}{|f_j|^2} & -\frac{\bar{f}^3_j f^2_j}{|f_j|^2} & \cdots & -\frac{\bar{f}^{n_j}_j f^2_j}{|f_j|^2} & \frac{f^2_j}{m_j} \\ -\frac{\bar{f}^2_j f^3_j}{|f_j|^2} & 1 - \frac{|f^3_j|^2}{|f_j|^2} & \cdots & -\frac{\bar{f}^{n_j}_j f^3_j}{|f_j|^2} & \frac{f^3_j}{m_j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{\bar{f}^2_j f^{n_j}_j}{|f_j|^2} & -\frac{\bar{f}^3_j f^{n_j}_j}{|f_j|^2} & \cdots & 1 - \frac{|f^{n_j}_j|^2}{|f_j|^2} & \frac{f^{n_j}_j}{m_j} \end{pmatrix} \begin{pmatrix} P^j_{j,2} \\ \vdots \\ P^j_{j,n_j} \\ Q^j_j \end{pmatrix}. \tag{42}$$

Note that the inverse matrix in the right hand side is the transposition of (31). It follows from $P^l_{j,\beta} = 0$ that

$$E_j f_j^1 \cdot f_j^\beta - E_j f_j^\beta \cdot f_j^1 = 0 \tag{43}$$

for $\beta = 1, \dots, n_j$ and $j = 1, \dots, s-1$. This means that

$$f_j^\beta \frac{\partial f_j^1}{\partial t} - f_j^1 \frac{\partial f_j^\beta}{\partial t} = 0. \tag{44}$$

Similarly we write down $f_* Z_s^\alpha$ by the combination of $\tilde{W}_l^\beta$, $\tilde{E}_l$, and $\tilde{Z}_s^\beta$, and apply the coordinate function $\tilde{z}_j^\gamma$ to get

$$f_j^\beta \frac{\partial f_j^1}{\partial z_s^\alpha} - f_j^1 \frac{\partial f_j^\beta}{\partial z_s^\alpha} = 0. \tag{45}$$

The equations (44) and (45) imply that f_j^β / f_j^1 is independent of z_s^α and t. Pick up the terms with $(z_j^\lambda)^2$ from

$$f_j^1(z, t+ip)|_{z_s=t=0} f_j^\beta(z, t+ip) = f_j^\beta(z, t+ip)|_{z_s=t=0} f_j^1(z, t+ip), \tag{46}$$

we obtain Claim 3. □

The coefficient $A^\gamma_{j:\lambda, a_s, b}$ is decomposed into "λ-part" and "non λ-part".

Claim 4. *The coefficient* $A^\gamma_{j:\lambda, a_s, b} = A^\gamma_{j:\lambda,0,0} A^1_{j:\lambda, a_s, b} / A^1_{j:\lambda,0,0}$ *is decomposed as*

$$A^\gamma_{j:\lambda, a_s, b} = \frac{A^\gamma_{j:\lambda,0,0}}{A^1_{j:1,0,0}} e^{i\theta_\lambda} \sqrt{\frac{|A^1_{j:1,0,0}|^2 + \cdots + |A^{n_j}_{j:1,0,0}|^2}{|A^1_{j:\lambda,0,0}|^2 + \cdots + |A^{n_j}_{j:\lambda,0,0}|^2}} A^1_{j:1, a_s, b}. \tag{47}$$

For simplicity, we divide into the "λ-part" and "non λ-part".

$$\mathscr{A}^\gamma_{j:\lambda} = \frac{A^\gamma_{j:\lambda,0,0}}{A^1_{j:1,0,0}} e^{i\theta_\lambda} \sqrt{\frac{|A^1_{j:1,0,0}|^2 + \cdots + |A^{n_j}_{j:1,0,0}|^2}{|A^1_{j:\lambda,0,0}|^2 + \cdots + |A^{n_j}_{j:\lambda,0,0}|^2}} \tag{48}$$

$$\mathscr{A}_{j:a_s, b} = A^1_{j:1, a_s, b}. \tag{49}$$

We use the first equation in (32). Since $W_j^1 f_j^\beta$ is calculated as

$$W_j^1 f_j^\beta = \sum_{|a_s|, b \geq 0} \frac{A^\beta_{j:1,0,0}}{A^1_{j:1,0,0}} A^1_{j:1, a_s, b} (z_s)^{a_s} (t+ip)^b - \frac{\bar{z}_j^1}{|z_j|^2} f_j^\beta, \tag{50}$$

the first equation in the system (32) is written as

$$\begin{aligned} |z_j|^2 (A^1_{j:1,0,0} \bar{f}_j^1 + \cdots + A^{n_j}_{j:1,0,0} \bar{f}_j^{n_j}) \sum_{|a_s|, b \geq 0} A^1_{j:1, a_s, b} (z_s)^{a_s} (t+ip)^b \\ - \bar{z}_j^1 A^1_{j:1,0,0} (|f_j^1|^2 + \cdots + |f_j^{n_j}|^2) = 0. \end{aligned} \tag{51}$$

Pick up the coefficient of $|z_j^\lambda|^2 \bar{z}_j^1$ with $\lambda \neq \gamma$ to get

$$\left| \sum_{|a_s|, b \geq 0} A^1_{j:\lambda, a_s, b} (z_s)^{a_s} t^b \right|^2 \tag{52}$$

$$= \left| \sum_{|a_s|, b \geq 0} A^1_{j:1, a_s, b} (z_s)^{a_s} t^b \right|^2 \frac{|A^1_{j:\lambda,0,0}|^2}{|A^1_{j:1,0,0}|^2} \frac{|A^1_{j:1,0,0}|^2 + \cdots + |A^{n_j}_{j:1,0,0}|^2}{|A^1_{j:\lambda,0,0}|^2 + \cdots + |A^{n_j}_{j:\lambda,0,0}|^2}.$$

This leads to the relation:

$$A^1_{j:\lambda, a_s, b} = e^{i\theta_\lambda} A^1_{j:1, a_s, b} \frac{A^1_{j:\lambda,0,0}}{A^1_{j:1,0,0}} \sqrt{\frac{|A^1_{j:1,0,0}|^2 + \cdots + |A^{n_j}_{j:1,0,0}|^2}{|A^1_{j:\lambda,0,0}|^2 + \cdots + |A^{n_j}_{j:\lambda,0,0}|^2}}, \tag{53}$$

and therefore to the conclusion of Claim 4. □

Now we have proved that the expansion (39) of f_j^γ is reduced to

$$f_j^\gamma = \sum_{\lambda=1}^{n_j} \mathscr{A}^\gamma_{j:\lambda} z_j^\lambda \sum_{|a_s|, b \geq 0} \mathscr{A}_{j:a_s, b} (z_s)^{a_s} (t + ip)^b \tag{54}$$

for $j = 1, \ldots, s-1$.

Next, we show that the matrix $(\mathscr{A}^\gamma_{j:\lambda})_{\gamma,\lambda}$ is a constant multiple of a unitary matrix.

Claim 5.

$$\begin{pmatrix} \mathscr{A}^1_{j:1} & \cdots & \mathscr{A}^{n_j}_{j:1} \\ \vdots & \ddots & \vdots \\ \mathscr{A}^1_{j:n_j} & \cdots & \mathscr{A}^{n_j}_{j:n_j} \end{pmatrix} \in aU(n_j). \tag{55}$$

Here $a \in \mathbf{R}$ and $U(n_j)$ is a unitary matrix of size n_j.

Substitute (54) into the αth equation in (32) and we get

$$\sum_{\lambda=1}^{n_j} \bar{\mathscr{A}}^1_{j:\lambda} \bar{z}_j^\lambda \{ |z_j|^2 \mathscr{A}^1_{j:\alpha} - \bar{z}_j^\alpha z_j^\beta \mathscr{A}^1_{j:\beta} \} + \cdots + \sum_{\lambda=1}^{n_j} \bar{\mathscr{A}}^{n_j}_{j:\lambda} \bar{z}_j^\lambda \{ |z_j|^2 \mathscr{A}^{n_j}_{j:\alpha} - \bar{z}_j^\alpha z_j^\beta \mathscr{A}^{n_j}_{j:\beta} \} = 0. \tag{56}$$

The coefficients of $|z_j^\lambda|^2 z_j^\alpha$ satisfy the equation:

$$|\mathscr{A}^1_{j:1}|^2 + \cdots + |\mathscr{A}^{n_j}_{j:1}|^2 = |\mathscr{A}^1_{j:2}|^2 + \cdots + |\mathscr{A}^{n_j}_{j:2}|^2 = \cdots = |\mathscr{A}^1_{j:n_j}|^2 + \cdots + |\mathscr{A}^{n_j}_{j:n_j}|^2. \tag{57}$$

We denote this constant by a. The coefficients with the variable $\bar{z}_j^\alpha \bar{z}_j^\gamma z_j^\beta$ in (56) with α, β and γ being different mutually, satisfy

$$\bar{\mathscr{A}}^1_{j:\gamma} \mathscr{A}^1_{j:\beta} + \cdots + \bar{\mathscr{A}}^{n_j}_{j:\gamma} \mathscr{A}^{n_j}_{j:\beta} = 0. \tag{58}$$

These two relations imply the conclusion of Claim 5. □

By substituting (48) into (57), we conclude that

$$|A^1_{j:1,0,0}|^2 + \cdots + |A^{n_j}_{j:1,0,0}|^2 = |A^1_{j:\lambda,0,0}|^2 + \cdots + |A^{n_j}_{j:\lambda,0,0}|^2 \tag{59}$$

for $\lambda = 2, \dots, n_j$. Therefore we get $\mathscr{A}^\beta_{j:\lambda} = A^\beta_{j:\lambda,0,0} e^{i\theta_\lambda} / A^1_{j:1,0,0}$ from (48). Therefore f^γ_j can be written as

$$f^\gamma_j = \sum_{\lambda=1}^{n_j} A^\gamma_{j:\lambda,0,0} e^{i\theta_\lambda} z_j^\lambda \sum_{a_s, b} \frac{A^1_{j:1,a_s,b}}{A^1_{j:1,0,0}} (z_s)^{a_s} (t + ip)^b, \tag{60}$$

where $((1/a)A^\gamma_{j:\lambda,0,0})_{\gamma,\lambda} \in U(n_j)$. Applying the coordinate functions $\tilde{z}^\gamma_k$, $\tilde{z}^\gamma_s$ and $\tilde{z}_{n+1}$ to the push forward

$$f_* E_j = \sum_{l=1}^{s-1} Q^l_j \tilde{E}_l + \sum_{\alpha \in I_s} R_{j,\alpha} \tilde{Z}^\alpha_s \tag{61}$$

for $j \neq k$, we get

$$E_j f^\gamma_k = \frac{Q^k_j}{m_k} f^\gamma_k, \tag{62}$$

$$E_j f^\gamma_s = R_{j,\gamma}, \tag{63}$$

$$E_j f_{n+1} = 2\left(i \sum_{l=1}^{s-1} Q^l_j |f_l|^{2m_l} + i \sum_{\alpha \in I_s} R_{j,\alpha} \bar{f}^\alpha_s \right). \tag{64}$$

The $j = k$ case, Q^j_j is calculated from (42). Substitute the expansion of f into (62), then we obtain

$$Q^k_j = \delta^k_j + 2m_k i |z_j|^{2m_j} \frac{\sum A^1_{k:1,a_s,b} (z_s)^{a_s} (t + ip)^{b-1} b}{\sum A^1_{k:1,a_s,b} (z_s)^{a_s} (t + ip)^b}. \tag{65}$$

On the other hand, $E_j f^\alpha_s$ and $E_j f_{n+1}$ are calculated explicitly as

$$E_j f^\alpha_s = 2i|z_j|^{2m_j} \frac{\partial f^\alpha_s}{\partial t}, \quad E_j f_{n+1} = 2i|z_j|^{2m_j} \frac{\partial f_{n+1}}{\partial t}. \tag{66}$$

Combining these with (62), (63), (64) and (65), and the fact that $((1/a)A^\gamma_{j:\lambda,0,0})_{\gamma,\lambda} \in U(n_j)$, and setting $z_1 = \cdots = z_{s-1} = 0$, we conclude that

$$\begin{aligned} &\frac{\partial f_{n+1}}{\partial t}\bigg|_{z_1=\cdots=z_{s-1}=0} - 2i \sum_{\alpha \in I_s} \left(\bar{f}^\alpha_s \frac{\partial f^\alpha_s}{\partial t} \right)\bigg|_{z_1=\cdots=z_{s-1}=0} \\ &\quad = \left| \sum \frac{A^1_{j:1,a_s,b}}{A^1_{j:1,0,0}} (z_s)^{a_s} (t + i|z_s|^2)^b \sqrt{a} \right|^{2m_j}, \end{aligned} \tag{67}$$

which implies that the right hand side does not depend on j and therefore we can write as

$$\left(\sum \frac{A^1_{j:1,a_s,b}}{A^1_{j:1,0,0}}(z_s)^{a_s}(t+ip)^b\sqrt{a}\right)^{m_j} = \sum \hat{A}_{a_s,b}(z_s)^{a_s}(t+ip)^b. \tag{68}$$

This does not vanish locally as the next Claim.

Claim 6. *$\hat{A}_{0,0} \neq 0$, therefore we can write as*

$$\sum \frac{A^1_{j:1,a_s,b}}{A^1_{j:1,0,0}}(z_s)^{a_s}(t+ip)^b\sqrt{a} = \frac{1}{(\sum D_{a_s,b}(z_s)^{a_s}(t+ip)^b)^{1/m_j}}. \tag{69}$$

Assume that $D_{0,1} \neq 0$. Remind that the decomposition in Theorem 1.1 has two cases: $J = I$ or identity. The case $D_{0,1} \neq 0$ corresponds to the case of $J = I$. We shall treat the case $D_{0,1} = 0$ at the end of this section. It follows from (68) that

$$|f_j|^{2m_j} = |z_j|^{2m_j}\left|\sum \hat{A}_{a_s,b}(z_s)^{a_s}(t+ip)^b\right|^2 \tag{70}$$

for $j = 1, \ldots, s-1$. Expand f_s^α and f_{n+1} as

$$f_s^\alpha = \sum S_{a_s,b}(z_s)^{a_s}(t+ip)^b \sum \hat{A}_{a_s,b}(z_s)^{a_s}(t+ip)^b, \tag{71}$$

$$f_{n+1} = \sum N_{a_s,b}(z_s)^{a_s}(t+ip)^b \sum \hat{A}_{a_s,b}(z_s)^{a_s}(t+ip)^b. \tag{72}$$

Substitute these expansions into

$$\frac{1}{2i}(f_{n+1} - \bar{f}_{n+1}) = \sum_{j=1}^{s-1} |f_j|^{2m_j} + |f_s|^2 \tag{73}$$

and pick up the terms without z_s and $\bar{z}_s$, which satisfy the following:

$$\begin{aligned}\sum \hat{A}_{0,b}(t+ip_s)^b \sum N_{0,b}(t+ip_s)^b - \sum \bar{\hat{A}}_{0,b}(t-ip_s)^b \sum \overline{N}_{0,b}(t-ip_s)^b \\ = 2i\left|\sum \hat{A}_{0,b}(t+ip_s)^b\right|^2 p_s + 2i\left|\sum \hat{A}_{0,b}(t+ip_s)^b\right|^2\left|\sum S_{0,b}(t+ip_s)^b\right|^2.\end{aligned} \tag{74}$$

Here we denote by $p_s = \sum_{l=1}^{s-1} |z_l|^{2m_l}$. In this equation, the terms with t and with p_s are the following:

$$\begin{aligned}&\sum \hat{A}_{0,b}t^b \sum N_{0,b}t^b - \sum \bar{\hat{A}}_{0,b}t^b \sum \overline{N}_{0,b}t^b \\ &\quad = 2i\left|\sum \hat{A}_{0,b}t^b\right|^2\left|\sum S_{0,b}t^b\right|^2,\end{aligned} \tag{75}$$

$$\begin{aligned}&\sum \hat{A}_{0,b}(ip_s)^b \sum N_{0,b}(ip_s)^b - \sum \bar{\hat{A}}_{0,b}(-ip_s)^b \sum \overline{N}_{0,b}(-ip_s)^b \\ &\quad = 2i\left|\sum \hat{A}_{0,b}(ip_s)^b\right|^2 p_s + 2i\left|\sum \hat{A}_{0,b}(ip_s)^b\right|^2\left|\sum S_{0,b}(ip_s)^b\right|^2.\end{aligned} \tag{76}$$

Assume that $\hat{A}_{0,0} = 0$. Pick up the terms with t^1 from (75) and terms with p_s from (76), then we get $\hat{A}_{0,1} N_{0,0} = 0$. So, we have three cases, Case (1) $\hat{A}_{0,0} = \hat{A}_{0,1} = 0$ and $N_{0,0} \neq 0$, Case (2) $\hat{A}_{0,0} = N_{0,0} = 0$ and $\hat{A}_{0,1} \neq 0$, Case (3) $\hat{A}_{0,0} = \hat{A}_{0,1} = N_{0,0} = 0$. First we consider Case (1). We pick up the terms with t^b from (75) and $t^{b-1} p_s$ from (74) for $b = 2, \ldots$ repeatedly, then we get $\hat{A}_{0,b} = 0$. Substitute this into (70) and expansions of f_s^α and f_{n+1}. Pick up the terms which contain only t and z_s from (73). Then we conclude that $f_{n+1} \equiv 0$, which is a contradiction. Next we consider the Case (2). Pick up the terms with t^2 in (75), with $(p_s)^2$ in (76), with tp_s in (74), with $t^2 p_s$ in (74) and with $(p_s)^3$ in (76), we obtain $\hat{A}_{0,1} = 0$, which is a contradiction. Therefore only Case (3) remains. In Case (3), by picking up the terms with t^3 in (75) and with $(p_s)^3$ in (76), we have three cases: Case (1) $\hat{A}_{0,0} = \hat{A}_{0,1} = \hat{A}_{0,2} = N_{0,0} = 0$ and $N_{0,1} \neq 0$, Case (2) $\hat{A}_{0,0} = \hat{A}_{0,1} = N_{0,0} = N_{0,1} = 0$ and $\hat{A}_{0,2} \neq 0$, Case (3) $\hat{A}_{0,0} = \hat{A}_{0,1} = \hat{A}_{0,2} = N_{0,0} = N_{0,1} = 0$. Making use of the same arguments as above, we prove that only Case (3) remains. Then we repeat this proccess and by induction, we obtain that only the case $\hat{A}_{0,0} = \hat{A}_{0,1} = \hat{A}_{0,2} = \cdots = 0$ and $N_{0,0} = N_{0,1} = \cdots = 0$ happens, which implies that $f_{n+1} \equiv 0$. Therefore we conclude Claim 6. □

Now we have shown that components of f are expanded as

$$(77) \qquad f_j^\alpha = \sum_{\lambda=1}^{n_j} \frac{A_{j:\lambda}^\alpha}{\sqrt{a}} z_j^\lambda e^{i\theta_\lambda} \frac{1}{\left(\sum_{|a_s|, b \geq 0} D_{a_s,b} (z_s)^{a_s} (t+ip)^b\right)^{1/m_j}},$$

$$(78) \qquad f_s^\beta = \sum_{|a_s|, b \geq 0} S_{a_s,b} (z_s)^{a_s} (t+ip)^b \frac{1}{\sum_{|a_s|, b \geq 0} D_{a_s,b} (z_s)^{a_s} (t+ip)^b},$$

$$(79) \qquad f_{n+1} = \sum_{|a_s|, b \geq 0} N_{a_s,b} (z_s)^{a_s} (t+ip)^b \frac{1}{\sum_{|a_s|, b \geq 0} D_{a_s,b} (z_s)^{a_s} (t+ip)^b},$$

and, for each j, the matrix $(A_{j:\lambda}^\alpha e^{i\theta_\lambda}/\sqrt{a})_{\lambda,\alpha}$ is a unitary matrix.

CLAIM 7. *$D_{a_s,b}$, $S_{a_s,b}$ and $N_{a_s,b}$ vanish for $|a_s| + b \geq 2$.*

Applying vector fields $\bar{Z}_s^\alpha$, E_j, $\bar{Z}_s^\alpha \bar{Z}_s^\beta$, $E_j \bar{Z}_s^\alpha$ and $E_j \bar{E}_i$ to (73), the following relations hold:

$$(80) \quad \sum \bar{D}_{a_s,b} (\bar{z}_s)^{a_s} (t-ip)^b \sum N_{a_s,b} \left(\frac{\partial}{\partial z_s^\alpha}\right) (z_s)^{a_s} (t+ip)^b$$
$$+ 2i\bar{z}_s^\alpha - \sum \bar{N}_{a_s,b} (\bar{z}_s)^{a_s} (t-ip)^b \sum D_{a_s,b} \left(\frac{\partial}{\partial z_s^\alpha}\right) (z_s)^{a_s} (t+ip)^b$$
$$- 2i \sum_{\gamma \in I_s} \left(\sum \bar{S}_{a_s,b}^\gamma (\bar{z}_s)^{a_s} (t-ip)^b \sum S_{a_s,b}^\gamma \left(\frac{\partial}{\partial z_s^\alpha}\right) (z_s)^{a_s} (t+ip)^b \right) = 0,$$

$$(81) \quad \sum \bar{N}_{a_s,b} (\bar{z}_s)^{a_s} (t-ip)^b \sum D_{a_s,b} (z_s)^{a_s} b (t+ip)^{b-1} + 1$$

$$-\sum \bar{D}_{a_s,b}(\bar{z}_s)^{a_s}(t-ip)^b \sum N_{a_s,b}(z_s)^{a_s} b(t+ip)^{b-1}$$
$$+2i\sum_{\gamma\in I_s}\Big(\sum \bar{S}^{\gamma}_{a_s,b}(\bar{z}_s)^{a_s}(t-ip)^b \sum S^{\gamma}_{a_s,b}(z_s)^{a_s} b(t+ip)^{b-1}\Big)=0,$$

(82) $$\left(\sum \bar{N}_{a_s,b}\left(\frac{\partial}{\partial \bar{z}_s^{\beta}}\right)(\bar{z}_s)^{a_s}(t-ip)^b - 2iz_s^{\beta}\sum \bar{N}_{a_s,b}(\bar{z})_s^{a_s} b(t-ip)^{b-1}\right)$$
$$\times \sum D_{a_s,b}\left(\frac{\partial}{\partial z_s^{\alpha}}\right)(z_s)^{a_s}(t+ip)^b$$
$$-\left(\sum \bar{D}_{a_s,b}\left(\frac{\partial}{\partial \bar{z}_s^{\beta}}\right)(\bar{z}_s)^{a_s}(t-ip)^b - 2iz_s^{\beta}\sum \bar{D}_{a_s,b}(\bar{z}_s)^{a_s} b(t-ip)^{b-1}\right)$$
$$\times \sum N_{a_s,b}\left(\frac{\partial}{\partial z_s^{\alpha}}\right)(z_s)^{a_s}(t+ip)^b$$
$$+2i\sum_{\gamma\in I_s}\left(\sum \bar{S}^{\gamma}_{a_s,b}\left(\frac{\partial}{\partial \bar{z}_s^{\beta}}\right)(\bar{z}_s)^{a_s}(t-ip)^b - 2iz_s^{\beta}\sum \bar{S}^{\gamma}_{a_s,b}(\bar{z}_s)^{a_s} b(t-ip)^{b-1}\right)$$
$$\times \sum S^{\gamma}_{a_s,b}\left(\frac{\partial}{\partial z_s^{\alpha}}\right)(z_s)^{a_s}(t+ip)^b - 2i\delta^{\alpha}_{\beta}=0,$$

(83) $$\sum \bar{N}_{a_s,b}\left(\frac{\partial}{\partial \bar{z}_s^{\alpha}}\right)(\bar{z}_s)^{a_s}(t-ip)^b \sum D_{a_s,b}(z_s)^{a_s} b(t+ip)^{b-1}$$
$$-\sum \bar{D}_{a_s,b}\left(\frac{\partial}{\partial \bar{z}_s^{\alpha}}\right)(\bar{z}_s)^{a_s}(t-ip)^b \sum N_{a_s,b}(z_s)^{a_s} b(t+ip)^{b-1}$$
$$+2i\sum_{\gamma\in I_s}\left(\sum \bar{S}^{\gamma}_{a_s,b}\left(\frac{\partial}{\partial \bar{z}_s^{\alpha}}\right)(\bar{z}_s)^{a_s}(t-ip)^b \sum S^{\gamma}_{a_s,b}(z_s)^{a_s} b(t+ip)^{b-1}\right)=0,$$

(84) $$\sum \bar{N}_{a_s,b}(\bar{z}_s)^{a_s} b(t-ip)^{b-1}\sum D_{a_s,b}(z_s)^{a_s} b(t+ip)^{b-1}$$
$$-\sum \bar{D}_{a_s,b}(\bar{z}_s)^{a_s} b(t-ip)^{b-1}\sum N_{a_s,b}(z_s)^{a_s} b(t+ip)^{b-1}$$
$$+2i\sum_{\gamma\in I_s}\Big(\sum \bar{S}^{\gamma}_{a_s,b}(\bar{z}_s)^{a_s} b(t-ip)^{b-1}\sum S^{\gamma}_{a_s,b}(z_s)^{a_s} b(t+ip)^{b-1}\Big)=0.$$

The constant terms in (82), (83) and (84) satisfy

(85) $$\bar{N}_{a_s^{\beta}=1,0}D_{a_s^{\alpha}=1,0} - \bar{D}_{a_s^{\beta}=1,0}N_{a_s^{\alpha}=1,0} + 2i\sum_{\gamma\in I_s}\bar{S}^{\gamma}_{a_s^{\beta}=1,0}S^{\gamma}_{a_s^{\alpha}=1,0} - 2i\delta^{\beta}_{\alpha}=0,$$

(86) $$N_{a_s^{\alpha}=1,0}=\frac{1}{\bar{D}_{0,1}}\left(D_{a_s^{\alpha}=1,0}\bar{N}_{0,1}+2i\sum_{\gamma\in I_s}S^{\gamma}_{a_s^{\alpha}=1,0}\bar{S}^{\gamma}_{0,1}\right),$$

(87) $$\bar{N}_{0,1}D_{0,1}-\bar{D}_{0,1}N_{0,1}+2i\sum_{\gamma\in I_s}\bar{S}^{\gamma}_{0,1}S^{\gamma}_{0,1}=0.$$

Define $C_{s,\alpha}^{\gamma}$ to be $S_{a_s^{\alpha}=1,0}^{\gamma} = C_{s,\alpha}^{\gamma} + D_{a_s^{\alpha}=1,0} S_{0,1}^{\gamma} / D_{0,1}$. Then the above three equations imply that the matrix $(C_{s,\alpha}^{\gamma})_{\gamma,\alpha}$ is a unitary matrix. Next we pick up the coefficients of t, p_1 and $\bar{z}_s^{\alpha}$ from (80), (83) and (82), those of t and p_1 from (84) and those of t from (81). The constant term of (84) together with these coefficients lead the matrix equations:

$$A \begin{pmatrix} D_{0,2} \\ N_{0,2} \\ S_{0,2}^1 \\ \vdots \\ S_{0,2}^{n_s} \end{pmatrix} = A \begin{pmatrix} D_{a_s^{\alpha}=1,1} \\ N_{a_s^{\alpha}=1,1} \\ S_{a_s^{\alpha}=1,1}^1 \\ \vdots \\ S_{a_s^{\alpha}=1,1}^{n_s} \end{pmatrix} = A \begin{pmatrix} D_{a_s^{\alpha}=2,0} \\ N_{a_s^{\alpha}=2,0} \\ S_{a_s^{\alpha}=2,0}^1 \\ \vdots \\ S_{a_s^{\alpha}=2,0}^{n_s} \end{pmatrix} = 0 \tag{88}$$

for

$$A = \begin{pmatrix} \bar{N}_{0,0} & -\bar{D}_{0,0} & 2i\bar{S}_{0,0}^1 & \cdots & 2i\bar{S}_{0,0}^{n_s} \\ \bar{N}_{0,1} & -\bar{D}_{0,1} & 2i\bar{S}_{0,1}^1 & \cdots & 2i\bar{S}_{0,1}^{n_s} \\ \bar{N}_{a_s^1=1,0} & -\bar{D}_{a_s^1=1,0} & 2i\bar{S}_{a_s^1=1,0}^1 & \cdots & 2i\bar{S}_{a_s^1=1,0}^{n_s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{N}_{a_s^{n_s}=1,0} & -\bar{D}_{a_s^{n_s}=1,0} & 2i\bar{S}_{a_s^1=1,0}^1 & \cdots & 2i\bar{S}_{a_s^{n_s}=1,0}^{n_s} \end{pmatrix}. \tag{89}$$

By substituting (86) and the constant terms in (81) into A, it is easy to show that the determinant of A equals to $-(2i)^{n_s} \bar{D}_{0,1}/D_{0,1} (\neq 0)$. Therefore we obtain the result of Claim 7 for $|a_s| + b = 2$. Assume that Claim 7 holds for $2 \leq |a_s| + b \leq k - 1$. Let

$$K_k^{\alpha} = \{(a_s, b) : |a_s| + b = k, a_s^{\alpha} \geq 1\}, \quad J^k = \{(a_s, b) : |a_s| + b = k, b \geq 1\}. \tag{90}$$

Pick up the terms of order $k - 1$ from (80) to (84). Then we obtain the matrix equations:

$$A \begin{pmatrix} \sum_{K_{\alpha}^k} D_{a_s,b} \left(\dfrac{\partial}{\partial z_s^{\alpha}}\right) (z_s)^{a_s} t^b \\ \sum_{K_{\alpha}^k} N_{a_s,b} \left(\dfrac{\partial}{\partial z_s^{\alpha}}\right) (z_s)^{a_s} t^b \\ \sum_{K_{\alpha}^k} S_{a_s,b}^1 \left(\dfrac{\partial}{\partial z_s^{\alpha}}\right) (z_s)^{a_s} t^b \\ \vdots \\ \sum_{K_{\alpha}^k} S_{a_s,b}^{n_s} \left(\dfrac{\partial}{\partial z_s^{\alpha}}\right) (z_s)^{a_s} t^b \end{pmatrix} = 0 \tag{91}$$

and

$$A\begin{pmatrix} \sum_{J^k} D_{a_s,b}(z_s)^{a_s} bt^{b-1} \\ \sum_{J^k} N_{a_s,b}(z_s)^{a_s} bt^{b-1} \\ \sum_{J^k} S^1_{a_s,b}(z_s)^{a_s} bt^{b-1} \\ \vdots \\ \sum_{J^k} S^{n_s}_{a_s,b}(z_s)^{a_s} bt^{b-1} \end{pmatrix} = 0 \tag{92}$$

for the matrix A above. Therefore Claim 7 holds for $(a_s, b) \in K^k_\alpha \cup J^k$. Since α moves in I_s, Claim 7 holds for all $|a_s| + b = k$. By induction, we conclude Claim 7. $\square$

Now we have proved that the Taylor expansions of components of f are the following:

$$f^\gamma_j = \sum_{\lambda=1}^{n_j} A^\gamma_{j:\lambda} z^\lambda_j \frac{1}{(\sum_{|a_s|+b\le 1} D_{a_s,b}(z_s)^{a_s}(t+ip)^b)^{1/m_j}}, \tag{93}$$

$$f^\alpha_s = \sum_{|a_s|+b\le 1} S^\alpha_{a_s,b}(z_s)^{a_s}(t+ip)^b \frac{1}{\sum_{|a_s|+b\le 1} D_{a_s,b}(z_s)^{a_s}(t+ip)^b}, \tag{94}$$

$$f_{n+1} = \sum_{|a_s|+b\le 1} N_{a_s,b}(z_s)^{a_s}(t+ip)^b \frac{1}{\sum_{|a_s|+b\le 1} D_{a_s,b}(z_s)^{a_s}(t+ip)^b}, \tag{95}$$

where the matrix $(A^\gamma_{j:\lambda})_{\gamma,\lambda}$ is a unitary matrix for each $j = 1, \dots, s-1$.

Next we shall obtain the relations of the coefficients of f. First, we pick up the constant terms in (80) and (81). They satisfy the following:

$$\begin{pmatrix} \overline{D}_{a^1_s=1,0} & -2i\overline{S}^1_{a^1_s=1,0} & \cdots & -2i\overline{S}^{n_s}_{a^1_s=1,0} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{D}_{a^{n_s}_s=1,0} & -2i\overline{S}^1_{a^{n_s}_s=1,0} & \cdots & -2i\overline{S}^{n_s}_{a^{n_s}_s=1,0} \\ \overline{D}_{0,1} & -2i\overline{S}^1_{0,1} & \cdots & -2i\overline{S}^{n_s}_{0,1} \end{pmatrix} \begin{pmatrix} N_{0,0} \\ S^1_{0,0} \\ \vdots \\ S^{n_s}_{0,0} \end{pmatrix} = \begin{pmatrix} D_{0,0}\overline{N}_{a^1_s=1,0} \\ \vdots \\ D_{0,0}\overline{N}_{a^{n_s}_s=1,0} \\ -1 + D_{0,0}\overline{N}_{0,1} \end{pmatrix}. \tag{96}$$

The determinant of the matrix in this equation does not vanish and therefore we get

$$(97)\qquad \begin{pmatrix} S^1_{0,0} \\ \vdots \\ S^{n_s}_{0,0} \end{pmatrix} = \begin{pmatrix} \dfrac{D_{0,0}S^1_{0,1}}{D_{0,1}} - \dfrac{1}{2i\bar{D}_{0,1}}(C^1_{s,1}\bar{D}_{a^1_s=1,0} + \cdots + C^1_{s,n_s}\bar{D}_{a^{n_s}_s=1,0}) \\ \vdots \\ \dfrac{D_{0,0}S^{n_s}_{0,1}}{D_{0,1}} - \dfrac{1}{2i\bar{D}_{0,1}}(C^{n_s}_{s,1}\bar{D}_{a^1_s=1,0} + \cdots + C^{n_s}_{s,n_s}\bar{D}_{a^{n_s}_s=1,0}) \end{pmatrix}$$

and

$$(98)\qquad \begin{aligned} N_{0,0} = {} & \frac{D_{0,0}\bar{N}_{0,1} - 1}{\bar{D}_{0,1}} + 2i|S_{0,1}|^2 \frac{D_{0,0}}{|D_{0,1}|^2} \\ & - \frac{1}{(\bar{D}_{0,1})^2}\{\bar{S}^1_{0,1}(C^1_{s,1}\bar{D}_{a^1_s=1,0} + \cdots + C^1_{s,n_s}\bar{D}_{a^{n_s}_s=1,0}) \\ & \qquad + \cdots + \bar{S}^{n_s}_{0,1}(C^{n_s}_{s,1}\bar{D}_{a^1_s=1,0} + \cdots + C^{n_s}_{s,n_s}\bar{D}_{a^{n_s}_s=1,0})\}. \end{aligned}$$

Claim 8. *The coefficients of f satisfy the relations*:

$$(99)\qquad \bar{N}_{0,1}D_{0,1} - N_{0,1}\bar{D}_{0,1} + 2i|S_{0,1}|^2 = 0,$$

$$(100)\qquad 2i(D_{0,0}\bar{D}_{0,1} - \bar{D}_{0,0}D_{0,1}) + |D_{a^1_s=1,0}|^2 + \cdots + |D_{a^{n_s}_s=1,0}|^2 = 0.$$

The first relation comes from the constant terms of (84). Substitute (93), (94) and (95) into (73) and the constant terms of the resulting equation is

$$(101)\qquad \bar{D}_{0,0}N_{0,0} - D_{0,0}\bar{N}_{0,0} = 2i|S_{0,0}|^2.$$

Together this with (97) and (98) implies the second relation.

In case of $D_{0,1} = 0$, by the suitable change of the argument, we get the coefficients relations,

$$(102)\qquad \begin{aligned} & S_{0,1} = 0, \quad D_{a^\alpha_s=1,0} = 0, \quad 1 - \bar{D}_{0,0}N_{0,1} = 0, \\ & \bar{D}_{0,0}N_{a^\alpha_s=1,0} - 2i\sum_{\gamma\in I_s}\bar{S}^\gamma_{0,0}S^\gamma_{a^\alpha_s=1,0} = 0. \end{aligned}$$

As a summary, the CR mapping under consideration has expansion (93), (94) and (95) with coefficients relations (86) and (97), (98), (99) and (100) or with relation (102).

4. Comparison of expansion and integers m_j and n_j

In this section, we compare our result with [MM10], and modification of the decomposition of holomorphic tangent bundle.

4.1. Specify the coefficients. If we put $D_{a_s,b}$, $S_{a_s,b}$ and $N_{a_s,b}$ and matrices $(C^{\gamma}_{s,\alpha})_{\gamma,\alpha}$ and $(A^{\gamma}_{j\lambda})_{\gamma,\lambda}$ to be special forms, our expansions become the ones in [MM10]. In case of $J = I$, we put

$$(103) \quad D_{0,1} = 1/r, \quad D_{0,0} = (t^0 + i|a_s|^2)/r, \quad S^{\alpha}_{0,1} = b^{\alpha}_s/r, \quad D_{a^{\alpha}_s=1,0} = -2\bar{a}^{\alpha}_s/ir,$$
$$(A^{\gamma}_{j,\lambda})_{\gamma,\lambda} = B_j, \quad (C^{\gamma}_{s,\alpha})_{\gamma,\alpha} = B_s, \quad \operatorname{Im} N_{0,1} = |b_s|^2/r.$$

In case J being an identity, we put

$$(104) \quad D_{0,0} = 1/N_{0,1} = 1/r, \quad D_{a_s,b} = 0 \quad \text{for other } a_s \text{ and } b,$$
$$(S^{\alpha}_{a^{\beta}_s=1,0})_{\alpha,\beta} = B_s,$$
$$S_{0,1} = 0, \quad S_{0,0} = B_s a_s + b_s/r, \quad N_{a_s=1,0} = 2ir\bar{a}_s + 2iB^t_s \cdot \bar{b}_s,$$
$$N_{0,0} = b_{n+1}/r + ra_{n+1} + 2iB_s a_s \cdot \bar{b}_s.$$

Here B^t_s is a transpose of a matrix B_s.

4.2. Modification of the decomposition. Next we make a remark about m_j and n_j. The analogous argument here works for some $m_j = 1$. If there exists j with $m_j = 1$, the z_j components of ψ and ϕ_a should be changed as $B_j z_{\sigma(j)}$ to $B_j z_{\sigma(j)} + b_j$ and z_j to $z_j + a_j$.

If there exists j with $n_j = 1$, then the subbundle $\mathscr{W}_j$ is empty for such j, and if all $n_j = 1$, our hypersurface M is a boundary of pseudoellipsoid. Therefore the mixed case, some $n_j = 1$ and the others $n_j > 1$, is considered here. If $n_1 = \cdots = n_{j_0-1} = 1$ and the rests are bigger than 1, then the decomposition (13) will be

$$(105) \quad T^{1,0}M = \mathscr{W}_{j_0} \oplus \cdots \oplus \mathscr{W}_{s-1} \oplus \mathscr{W}_s \oplus \mathscr{E}$$

and after small modification of our argument shows that the z_j component in ψ should be changed as $B_j z_j$ to $e^{i\theta_j} z_j$.

Part 2. Infinitesimal CR automorphism method

5. CR automorphism group and infinitesimal CR automorphism

For any $p \in M$, we let $\operatorname{Aut}(M,p)$ denote the set of germs H of biholomorphisms near p, with $H(M) \subset M$ and $H(p) = p$. A smooth real vector field X defined in a neighborhood of $p \in M$ is called an infinitesimal CR automorphism if the one-parameter group of transformation $\exp tX$ is a CR diffeomorphism. We denote by $\operatorname{hol}(M,p)$ the Lie algebra generated by the infinitesimal CR automorphisms. Then M. S. Baouendi, P. Ebenfelt, L. P. Rothschild proved the correspondence between $\operatorname{Aut}(M,p)$ and $\operatorname{hol}(M,p)$. In the next theorem, for definition of holomorphically nondegeneracy and minimality and their properties, refer to the book [BER99]. Note that the boundary of a generalized ellipsoid M under consideration satisfies the condition of their theorem.

THEOREM 5.1 ([BER98]). *Let M be a real analytic, holomorphically non-degenerate, generic submanifold of $\mathbf{C}^N$ which is minimal at some point. For all $p \in M$, there exist a unique topology on the group* $\mathrm{Aut}(M, p)$ *with respect to which it is a Lie group whose Lie algebra is* $\mathrm{hol}(M, p)$.

In what follows, we omit the reference point. This theorem implies that, in order to classify CR automorphisms, we need to classify $\mathrm{hol}(M)$ and compute the one-parameter group of transformations of them. To obtain the explicit form of $X \in \mathrm{hol}(M)$, we need the following characterization, whose proof appears in [BER99].

THEOREM 5.2. *Let $M \subset \mathbf{C}^N$ be a smooth generic submanifold, and let X be a smooth real vector field on M defined in an open subset $U \subset M$. Then X is an infinitesimal CR automorphism of M if and only if*

$$X = \sum_{j=1}^{N} \left(a_j \frac{\partial}{\partial z_j} + \bar{a}_j \frac{\partial}{\partial \bar{z}_j} \right), \tag{106}$$

where the a_j, $j = 1, \dots, N$ are CR functions in U.

Using this characterization, we can determine the coefficients of X.

LEMMA 5.1. *Let*

$$X = 2\,\mathrm{Re}\left(\sum_{j=1}^{s} \sum_{\alpha \in I_j} g_j^{\alpha}(z, z_{n+1}) \frac{\partial}{\partial z_j^{\alpha}} + g_{n+1}(z, z_{n+1}) \frac{\partial}{\partial z_{n+1}} \right) \tag{107}$$

be an infinitesimal CR automorphism of the boundary of a generalized ellipsoid. Then coefficients have the following expansions:

$$g_j^{\alpha}(z, z_{n+1}) = \frac{1}{m_j}(a z_{n+1} + b) z_j^{\alpha} + \sum_{\substack{\beta \in I_j \\ \alpha \neq \beta}} a_{j,\beta}^{\alpha} z_j^{\beta}, \tag{108}$$

$$g_s^{\alpha}(z, z_{n+1}) = c^{\alpha} + (a z_{n+1} + b) z_s^{\alpha} + \sum_{\substack{\beta \in I_s \\ \alpha \neq \beta}} a_{s,\beta}^{\alpha} z_s^{\beta}, \tag{109}$$

$$g_{n+1}(z, z_{n+1}) = a(z_{n+1})^2 + (b + \bar{b}) z_{n+1} + d + 2i \sum_{\beta \in I_s} \bar{c}^{\beta} z_s^{\beta}, \tag{110}$$

where $a_{j,\beta}^{\alpha} + \bar{a}_{j,\alpha}^{\beta} = 0$ for $j = 1, \dots, s$, $\alpha, \beta \in I_j$, $a, d \in \mathbf{R}$, and $c^{\beta}, b \in \mathbf{C}$.

Proof. Write

$$g_j^\alpha(z, z_{n+1}) = \sum_{|a| \geq 0} g_{j,a}^\alpha(z_{n+1})(z)^a, \quad j = 1, \dots, s, \tag{111}$$

$$g_{n+1}(z, z_{n+1}) = \sum_{|a| \geq 0} g_{n+1,a}(z_{n+1})(z)^a. \tag{112}$$

Here $g_{j,a}^\alpha(z_{n+1})$ and $g_{n+1,a}(z_{n+1})$ are holomorphic functions of z_{n+1} and $a = (a_1, \dots, a_s) \in (\mathbf{Z}_{\geq 0})^n$, $a_j = (a_j^1, \dots, a_j^{n_j}) \in (\mathbf{Z}_{\geq 0})^{n_j}$ are multi-indices.

Applying the defining equation $-\mathrm{Im}\, z_{n+1} + p(z, \bar{z}) = 0$ to X and restrict it to M, we have

$$\begin{aligned} &\sum_{j=1}^{s-1} \sum_{\alpha \in I_j} g_j^\alpha(z, t + ip) m_j |z_j|^{2(m_j - 1)} \bar{z}_j^\alpha + \sum_{\alpha \in I_s} g_s^\alpha(z, t + ip) \bar{z}_s^\alpha \\ &\quad - \frac{1}{2i} g_{n+1}(z, t + ip) + \sum_{j=1}^{s-1} \sum_{\alpha \in I_j} \bar{g}_j^\alpha(\bar{z}, t - ip) m_j |z_j|^{2(m_j - 1)} z_j^\alpha \\ &\quad + \sum_{\alpha \in I_s} \bar{g}_s^\alpha(\bar{z}, t - ip) z_s^\alpha + \frac{1}{2i} \bar{g}_{n+1}(\bar{z}, t - ip) = 0. \end{aligned} \tag{113}$$

Setting $\bar{z} = 0$ in (113) gives

$$-\frac{1}{2i} g_{n+1}(z, t) + \frac{1}{2i} \bar{g}_{n+1}(0, t) + \sum_{\alpha \in I_s} \bar{g}_s(0, t) z_s^\alpha = 0. \tag{114}$$

Setting $z = 0$ gives

$$g_{n+1,0}(t) = \bar{g}_{n+1,0}(t). \tag{115}$$

Substitute this into the equation (114) to obtain

$$g_{n+1}(z, t) = g_{n+1,0}(t) + 2i \sum_{\alpha \in I_s} \bar{g}_{s,0}^\alpha(t) z_s^\alpha. \tag{116}$$

Substituting (115) and (116) into (113),

$$\begin{aligned} &\sum_{j=1}^{s-1} \sum_{\alpha \in I_j} m_j |z_j|^{2(m_j - 1)} \{ g_j^\alpha(z, t + ip) \bar{z}_j^\alpha + \bar{g}_j^\alpha(\bar{z}, t - ip) z_j^\alpha \} \\ &\quad + \sum_{\alpha \in I_s} g_s^\alpha(z, t + ip) \bar{z}_s^\alpha + \sum_{\alpha \in I_s} \bar{g}_s^\alpha(\bar{z}, t - ip) z_s^\alpha \\ &\quad - \frac{1}{2i} \left\{ g_{n+1,0}(t + ip) + 2i \sum_{\alpha \in I_s} \bar{g}_{s,0}^\alpha(t + ip) z_s^\alpha \right\} \\ &\quad + \frac{1}{2i} \left\{ g_{n+1,0}(t - ip) - 2i \sum_{\alpha \in I_s} g_{s,0}^\alpha(t - ip) \bar{z}_s^\alpha \right\} = 0. \end{aligned} \tag{117}$$

For simplicity, we write $\partial/\partial z = \partial_z$. Apply $\partial_{z_s^\alpha}\partial_{\bar{z}_s^\beta}$ and $\partial_{z_j^\alpha}\partial_{z_k^\gamma}\partial_{\bar{z}_s^\beta}$ to (117) and evaluate the resulting equations at $z = \bar{z} = 0$. These give

(118) $$\delta_\beta^\alpha g_{n+1,0}{}'(t) = g^\beta_{s,a_s^\alpha=1}(t) + \bar{g}^\alpha_{s,a_s^\beta=1}(t),$$

(119) $$g^\beta_{s,a_s^\alpha=a_s^\gamma=1}(t) = 2i\delta_\alpha^\beta \bar{g}^\gamma_{s,0}{}'(t) = 2i\delta_\gamma^\beta \bar{g}^\alpha_{s,0}{}'(t) = i(\delta_\alpha^\beta \bar{g}^\gamma_{s,0}{}'(t) + \delta_\gamma^\beta \bar{g}^\alpha_{s,0}{}'(t)),$$

(120) $$g^\beta_{s,a_k^\gamma=a_j^\alpha=1}(t) = 0 \quad j,k \neq s.$$

Similarly, applying $\partial_{z_{j_1}^{\alpha_1}}\partial_{z_{j_2}^{\alpha_2}}\cdots\partial_{z_{j_n}^{\alpha_n}}\partial_{\bar{z}_s^\beta}$ $(n \geq 3)$ to (117), we obtain the relation

(121) $$g^\beta_{s,a}(t) = 0$$

for $|a| \geq 3$. Substitute these relations obtained above into (117), it becomes

(122) $$\sum_{j=1}^{s-1}\sum_{\alpha\in I_j} m_j|z_j|^{2(m_j-1)}\{g_j^\alpha(z,t+ip)\bar{z}_j^\alpha + \bar{g}_j^\alpha(z,t-ip)z_j^\alpha\}$$
$$+\frac{1}{2i}(g_{n+1,0}(t-ip) - g_{n+1,0}(t+ip))$$
$$+\sum_{\alpha\in I_s}\bar{z}_s^\alpha\Bigg\{g^\alpha_{s,0}(t+ip) - g^\alpha_{s,0}(t-ip) + 2i(z_s^\alpha)^2\bar{g}^\alpha_{s,0}{}'(t-ip) + \sum_{\beta\in I_s} g^\alpha_{s,a_s^\beta=1}(t+ip)z_s^\beta\Bigg\}$$
$$+\sum_{\alpha\in I_s} z_s^\alpha\Bigg\{\bar{g}^\alpha_{s,0}(t-ip) - \bar{g}^\alpha_{s,0}(t+ip) - 2i(\bar{z}_s^\alpha)^2 g^\alpha_{s,0}{}'(t+ip) + \sum_{\beta\in I_s}\bar{g}^\alpha_{s,a_s^\beta=1}(t-ip)\bar{z}_s^\beta\Bigg\} = 0.$$

Expand $g^\alpha_{s,a}$ and $\bar{g}^\alpha_{s,a}$ in the above equation in a Taylor series about u, and pick up the terms with homogeneous degree 3, 4, 6 in z_s and $\bar{z}_s$. Then we obtain

(123) $$g^\alpha_{s,0} = c_s^\alpha = \text{constant},$$

(124) $$g^\alpha_{s,a_s^\beta=1}{}'(t) = \bar{g}^\beta_{s,a_s^\alpha=1}{}'(t) \quad \text{for } \alpha \neq \beta,$$

(125) $$g^\alpha_{s,a_s^\alpha=1}{}'(t) \quad \text{is a real valued function,}$$

(126) $$\sum_{\beta\in I_s}\bar{z}_s^\beta\sum_{\alpha\in I_s} g^\beta_{s,a_s^\alpha=1}{}''(t)z_s^\alpha + \sum_{\beta\in I_s} z_s^\beta\sum_{\alpha\in I_s}\bar{g}^\beta_{s,a_s^\alpha=1}{}''(t)\bar{z}_s^\alpha - \frac{1}{3}g_{n+1,0}{}'''(t)|z_s|^2 = 0.$$

Combining these with (118), (119) and (120), we conclude that g_s^α and g_{n+1} have the following forms,

$$g_s^\alpha(z, z_{n+1}) = c^\alpha + (az_{n+1} + b)z_s^\alpha + \sum_{\substack{\beta \in I_s \\ \alpha \neq \beta}} a_{s,\beta}^\alpha z_s^\beta, \tag{127}$$

$$g_{n+1}(z, z_{n+1}) = a(z_{n+1})^2 + (b + \bar{b})z_{n+1} + d + 2i \sum_{\beta \in I_s} \bar{c}^\beta z_s^\beta, \tag{128}$$

where $a_{s,\beta}^\alpha + \bar{a}_{s,\alpha}^\beta = 0$ and $a, d \in \mathbf{R}$, $b, c^\beta \in \mathbf{C}$. These are the conclusion of g_s^α and g_{n+1}.

Substitute these into (113) and the resulting equation becomes

$$\sum_{j=1}^{s-1} \sum_{\alpha \in I_j} m_j |z_j|^{2(m_j - 1)} \{\bar{z}_j^\alpha g_j^\alpha(z, t + ip) + z_j^\alpha \bar{g}_j^\alpha(\bar{z}, t - ip)\} \tag{129}$$
$$+ (2at + b + \bar{b})(|z_s|^2 - p) = 0.$$

Evaluating $z = (0, \dots, 0, z_j, 0, \dots, 0)$, $\bar{z} = (0, \dots, 0, \bar{z}_j, 0, \dots, 0)$, then we have

$$m_j \sum_{\alpha \in I_j} \{\bar{z}_j^\alpha g_j^\alpha(z_j, t + i|z_j|^{2m_j}) + z_j^\alpha \bar{g}_j^\alpha(\bar{z}_j, t - i|z_j|^{2m_j})\} = (2at + b + \bar{b})|z_j|^2. \tag{130}$$

Evaluating $z_j = 0$ and $\bar{z}_j = (0, \dots, 0, \bar{z}_j^\alpha, 0, \dots, 0)$ we get $g_{j,0}^\alpha(t) = 0$. Applying $\partial_{z_j^\alpha} \partial_{\bar{z}_j^\beta}$ to (130) and evaluating $z_j = \bar{z}_j = 0$, we get

$$g_{j, a_j^\beta = 1}^\alpha(t) + \bar{g}_{j, a_j^\alpha = 1}^\beta(t) = 0, \quad \text{for } \alpha \neq \beta, \tag{131}$$

$$g_{j, a_j^\alpha = 1}^\alpha(t) + \bar{g}_{j, a_j^\alpha = 1}^\alpha(t) = \frac{1}{m_j}(2at + b + \bar{b}). \tag{132}$$

The terms in (130) with $\bar{z}_j^\alpha z_j^{\beta_1} z_j^{\beta_2} \cdots z_j^{\beta_n}$ $(n \geq 2)$ appear only from $\bar{z}_j^\alpha g_j^\alpha(z_j, t + i|z_j|^{2m_j})$ and they must be zero. It means that

$$g_{j,a}^\alpha(t) = 0 \tag{133}$$

for $|a| \geq 2$. Now $g_j^\alpha(z, z_{n+1})$ has an expansion

$$g_j^\alpha(z, z_{n+1}) = \sum_{\beta \in I_j} g_{j, a_j^\beta = 1}^\alpha(z_{n+1}) z_j^\beta. \tag{134}$$

Substitute this into (130) and the terms with $z_j^\beta \bar{z}_j^\alpha |z_j^\alpha|^{2m_j}$ in the resulting equation satisfy

$$g_{j, a_j^\beta = 1}^\alpha{}'(t) - \bar{g}_{j, a_j^\alpha = 1}^\beta{}'(t) = 0. \tag{135}$$

Combining this with the derivative of (131) if $\alpha \neq \beta$, and with (132) and its derivative if $\alpha = \beta$, we obtain

$$g_{j, a_j^\beta = 1}^\alpha(t) = a_{j,\beta}^\alpha = \text{constant if } \alpha \neq \beta, \tag{136}$$

$$g_{j, a_j^\alpha = 1}^\alpha(t) = \frac{1}{m_j}(at + b), \tag{137}$$

where $a_{j,\beta}^{\alpha} + \bar{a}_{j,\alpha}^{\beta} = 0$. Now we have obtained the expansion of $g_j^{\alpha}(z, z_{n+1})$ as

$$g_j^{\alpha}(z, z_{n+1}) = \frac{1}{m_j}(az_{n+1} + b)z_j^{\alpha} + \sum_{\substack{\beta \in I_j \\ \beta \neq \alpha}} a_{j,\beta}^{\alpha} z_j^{\beta}. \tag{138}$$

This completes the proof. □

6. One-parameter group of transformations of hol(M)

In this section, we construct one-parameter group of transformations generated by infinitesimal CR automorphisms, which generate all CR automorphisms of the boundary of a generalized ellipsoid.

By Lemma 5.1, we know the generators of hol(M). If we take the parameters in (108), (109) and (110) to be real, the generators of hol(M) become the following forms:

$$\begin{aligned} Y_1 = {} & \sum_{j=1}^{s-1} \sum_{\alpha \in I_j} \left(\frac{z_j^{\alpha} z_{n+1}}{m_j} \frac{\partial}{\partial z_j^{\alpha}} + \frac{\bar{z}_j^{\alpha} \bar{z}_{n+1}}{m_j} \frac{\partial}{\partial \bar{z}_j^{\alpha}} \right) \\ & + \sum_{\alpha \in I_s} \left(z_s^{\alpha} z_{n+1} \frac{\partial}{\partial z_s^{\alpha}} + \bar{z}_s^{\alpha} \bar{z}_{n+1} \frac{\partial}{\partial \bar{z}_s^{\alpha}} \right) + (z_{n+1})^2 \frac{\partial}{\partial z_{n+1}} + (\bar{z}_{n+1})^2 \frac{\partial}{\partial \bar{z}_{n+1}}, \end{aligned} \tag{139}$$

$$\begin{aligned} Y_2 = {} & \sum_{j=1}^{s-1} \sum_{\alpha \in I_j} \left(\frac{z_j^{\alpha}}{m_j} \frac{\partial}{\partial z_j^{\alpha}} + \frac{\bar{z}_j^{\alpha}}{m_j} \frac{\partial}{\partial \bar{z}_j^{\alpha}} \right) + \sum_{\alpha \in I_s} \left(z_s^{\alpha} \frac{\partial}{\partial z_s^{\alpha}} + \bar{z}_s^{\alpha} \frac{\partial}{\partial \bar{z}_s^{\alpha}} \right) \\ & + 2 \left(z_{n+1} \frac{\partial}{\partial z_{n+1}} + \bar{z}_{n+1} \frac{\partial}{\partial \bar{z}_{n+1}} \right), \end{aligned} \tag{140}$$

$$Y_{j,\beta}^{\alpha} = z_j^{\beta} \frac{\partial}{\partial z_j^{\alpha}} + \bar{z}_j^{\beta} \frac{\partial}{\partial \bar{z}_j^{\alpha}} - z_j^{\alpha} \frac{\partial}{\partial z_j^{\beta}} - \bar{z}_j^{\alpha} \frac{\partial}{\partial \bar{z}_j^{\beta}} \quad \text{for } j = 1, \ldots, s,\ \alpha, \beta \in I_j,\ \alpha < \beta, \tag{141}$$

$$Y_s^{\alpha} = \frac{\partial}{\partial z_s^{\alpha}} + \frac{\partial}{\partial \bar{z}_s^{\alpha}} + 2iz_s^{\alpha} \frac{\partial}{\partial z_{n+1}} - 2i\bar{z}_s^{\alpha} \frac{\partial}{\partial \bar{z}_{n+1}} \quad \text{for } \alpha \in I_s, \tag{142}$$

$$Y_3 = \frac{\partial}{\partial z_{n+1}} + \frac{\partial}{\partial \bar{z}_{n+1}}. \tag{143}$$

As explained, classification of infinitesimal CR automorphisms leads to that of CR automorphisms. Therefore we construct one parameter group of transformations generated by above vectors.

THEOREM 6.1. *One parameter group of transformations generated by the vectors in* hol(M) *can be classified into the following five types.*

$$\exp(\varepsilon Y_1)(z, z_{n+1}) = \left(\frac{z_1}{(1-\varepsilon z_{n+1})^{1/m_1}}, \dots, \frac{z_{s-1}}{(1-\varepsilon z_{n+1})^{1/m_{s-1}}}, \frac{z_s}{1-\varepsilon z_{n+1}}, \frac{z_{n+1}}{1-\varepsilon z_{n+1}}\right), \tag{144}$$

$$\exp(\varepsilon Y_2)(z, z_{n+1}) = ((e^{\varepsilon})^{1/m_1} z_1, \dots, (e^{\varepsilon})^{1/m_{s-1}} z_{s-1}, e^{\varepsilon} z_s, (e^{\varepsilon})^2 z_{n+1}), \tag{145}$$

$$\exp(\varepsilon Y_3)(z, z_{n+1}) = (z_1, \dots, z_{s-1}, z_s, z_{n+1} + \varepsilon), \tag{146}$$

$$\exp(\varepsilon_1 Y_s^1 + \cdots + \varepsilon_{n_s} Y_s^{n_s})(z, z_{n+1}) = (z_1, \dots, z_{s-1}, z_s + \varepsilon, z_{n+1} + 2iz_s \cdot \varepsilon + i|\varepsilon|^2), \tag{147}$$

$$\exp\left(\sum_{j=1}^{s} \sum_{\substack{\alpha,\beta \in I_j \\ \alpha > \beta}} \varepsilon_{j,\beta}^{\alpha} Y_{j,\beta}^{\alpha}\right)(z, z_{n+1}) = (B^1 z_1, \dots, B^{s-1} z_{s-1}, B^s z_s, z_{n+1}). \tag{148}$$

Here we denote, in (147), $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n_s})$ *and* $z_s \cdot \varepsilon$ *is an inner product of* z_s *and* ε. *The* B^j*'s are unitary matrices depending on* $\varepsilon_{j,\beta}^{\alpha}$.

Proof. Since we have

$$\begin{aligned}(Y_1)^l = \sum_{j=1}^{s-1} \sum_{\alpha \in I_j} &\left(\frac{1}{m_j}\right)\left(\frac{1}{m_j}+1\right)\cdots\left(\frac{1}{m_j}+l-1\right) \\ &\times \left\{ z_j^{\alpha}(z_{n+1})^l \frac{\partial}{\partial z_j^{\alpha}} + \bar{z}_j^{\alpha}(\bar{z}_{n+1})^l \frac{\partial}{\partial \bar{z}_j^{\alpha}} \right\} \\ &+ \sum_{\alpha \in I_s} l! \left\{ z_s^{\alpha}(z_{n+1})^l \frac{\partial}{\partial z_s^{\alpha}} + \bar{z}_s^{\alpha}(\bar{z}_{n+1})^l \frac{\partial}{\partial \bar{z}_s^{\alpha}} \right\} \\ &+ l! \left\{ (z_{n+1})^{l+1} \frac{\partial}{\partial z_{n+1}} + (\bar{z}_{n+1})^{l+1} \frac{\partial}{\partial \bar{z}_{n+1}} \right\},\end{aligned} \tag{149}$$

$$\begin{aligned}(Y_2)^l = \sum_{j=1}^{s-1} \sum_{\alpha \in I_j} &\left\{ \frac{z_j^{\alpha}}{m_j^l} \frac{\partial}{\partial z_j^{\alpha}} + \frac{\bar{z}_j^{\alpha}}{m_j^l} \frac{\partial}{\partial \bar{z}_j^{\alpha}} \right\} + \sum_{\alpha \in I_s} \left\{ z_s^{\alpha} \frac{\partial}{\partial z_s^{\alpha}} + \bar{z}_s^{\alpha} \frac{\partial}{\partial \bar{z}_s^{\alpha}} \right\} \\ &+ 2^l \left\{ z_{n+1} \frac{\partial}{\partial z_{n+1}} + \bar{z}_{n+1} \frac{\partial}{\partial \bar{z}_{n+1}} \right\},\end{aligned} \tag{150}$$

$$(Y_3)^2 = 0, \tag{151}$$

$$(Y_s^{\alpha})^2 = 2i\left(\frac{\partial}{\partial z_{n+1}} - \frac{\partial}{\partial \bar{z}_{n+1}}\right), \quad (Y_s^{\alpha})^3 = 0, \tag{152}$$

$$(Y_{j,\beta}^{\alpha})^{2l} = (-1)^{l+1}(Y_{j,\beta}^{\alpha})^2, \quad (Y_{j,\beta}^{\alpha})^{2l+1} = (-1)^l Y_{j,\beta}^{\alpha}, \tag{153}$$

the result follows. □

7. Comparison Theorem 1.1 with Theorem 6.1 and integers n_j and m_j

In this section, we compare parameters in Theorem 1.1 with ones in Theorem 6.1 and modification of the coefficients of Lemma 5.1 when integers m_j and n_j change.

7.1. Comparison of parameters. In [MM10], we do not know how to find each component I, δ_r, ψ and ϕ_a. But the following lemma shows that all components are obtained from infinitesimal CR automorphisms and those mappings are enough to construct all CR automorphisms by virtue of Theorem 5.1.

LEMMA 7.1. *Putting* $t^1 = t^0 = r = -1/\varepsilon$, $a_s = b_s = 0$, $B_j = id$ *in the mappings in Theorem* 1.1, *we have*

$$I = \delta_r^{-1} \circ \psi^{-1} \circ (\exp \varepsilon Y_1) \circ \phi_a^{-1}. \tag{154}$$

Putting $r = e^{\varepsilon}$ *in the mappings in Theorem* 1.1, *we have*

$$\delta_r = \exp(\varepsilon Y_2). \tag{155}$$

Putting $a_s = (\varepsilon_1, \ldots, \varepsilon_{n_s})$ *and* $t^0 = 0$ *in the mappings in Theorem* 1.1, *we have*

$$\phi_a = \exp(\varepsilon_1 Y_s^1 + \cdots + \varepsilon_{n_s} Y_s^{n_s}). \tag{156}$$

Putting $a_s = 0$, $a_{n+1} = \varepsilon$ *in the mappings in Theorem* 1.1, *we have*

$$\phi_a = \exp(\varepsilon Y_3). \tag{157}$$

Putting $b_{n+1} = 0$ *in the mappings in Theorem* 1.1, *we have*

$$\psi = \exp\left(\sum_{j=1}^{s} \sum_{\substack{\alpha,\beta \in I_j \\ \alpha > \beta}} \varepsilon_{j,\beta}^{\alpha} Y_{j,\beta}^{\alpha}\right). \tag{158}$$

The proof is straightforward. This lemma shows that all mappings in Theorem 1.1 can be obtained from one-parameter group of transformations.

7.2. Integers n_j and m_j. By modifying our arguments a little, our argument also works in the case that some n_j and m_j are equal to one. In these cases, coefficients of infinitesimal CR automorphisms change.

THEOREM 7.1. *Assume that* $m_{j_0} = \cdots = m_s = 1$. *Then the coefficients* g_j^{α} *and* g_{n+1} *of an infinitesimal CR automorphism in Lemma* 5.1 *can be expanded as the following*:

$$g_j^\alpha(z, z_{n+1}) = \frac{1}{m_j}(az_{n+1} + b)z_j^\alpha + \sum_{\substack{\beta \in I_j \\ \alpha \neq \beta}} a_{j,\beta}^\alpha z_j^\beta \quad \text{for } j = 1, \dots, j_0 - 1, \tag{159}$$

$$g_j^\alpha(z, z_{n+1}) = c_j^\alpha + (az_{n+1} + b)z_j^\alpha + \sum_{\substack{\beta \in I_j \\ \alpha \neq \beta}} a_{j,\beta}^\alpha z_j^\beta \quad \text{for } j = j_0, \dots, s, \tag{160}$$

$$\begin{aligned} g_{n+1}(z, z_{n+1}) &= a(z_{n+1})^2 + (b + \bar{b})z_{n+1} + d \\ &\quad + 2i\left\{\sum_{\alpha \in I_{j_0}} \bar{c}_{j_0}^\alpha z_{j_0}^\alpha + \cdots + \sum_{\alpha \in I_s} \bar{c}_s^\alpha z_s^\alpha\right\}. \end{aligned} \tag{161}$$

Here $a_{j,\beta}^\alpha + \bar{a}_{j,\alpha}^\beta = 0$ *and* $a, d \in \mathbf{R}$, $c_j^\alpha, b \in \mathbf{C}$.

If $n_j = 1$, *we need to remove the summation terms* $\sum_{\substack{\beta \in I_j \\ \alpha \neq \beta}} a_{j,\beta}^\alpha z_j^\beta$ *from* g_j^α.

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