

## DEFORMING TWO-DIMENSIONAL GRAPHS IN $R^4$ BY FORCED MEAN CURVATURE FLOW

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### Abstract

A surface  $\Sigma_0$  is a graph in  $R^4$  if there is a unit constant 2-form  $w$  in  $R^4$  such that  $\langle e_1 \wedge e_2, w \rangle \geq v_0 > 0$ , where  $\{e_1, e_2\}$  is an orthonormal frame on  $\Sigma_0$ . In this paper, we investigate a 2-dimensional surface  $\Sigma$  evolving along a mean curvature flow with a forcing term in direction of the position vector. If  $v_0 \geq \frac{1}{\sqrt{2}}$  holds on the initial graph  $\Sigma_0$  which is the immersion of the surface  $\Sigma$ , and the coefficient function of the forcing vector is nonnegative, then the forced mean curvature flow has a global solution, which generalizes part of the results of Chen-Li-Tian in [2].

### 1. Introduction

In the past three decades, the mean curvature flow and the forced mean curvature flows of hypersurfaces have been investigated deeply, and many nice results have been obtained. In contrast, very little about the higher codimensional mean curvature flow has been known, since in high codimensional case the complexity of the evolution equations of the intrinsic geometric quantities, such as all derivatives of the second fundamental form, the mean curvature and so on, increases the difficulty of doing estimates for those quantities. However, there still exist nice results about the higher codimensional mean curvature flow, see [1, 2, 5, 8, 10, 11] for example. In [2], by using some conclusions in [1, 3], Chen-Li-Tian proved that a 2-dimensional graph in  $R^4$  moving by the mean curvature flow has a global solution, and the corresponding scaled surfaces converge to a self-similar solution under some suitable condition therein.

Let  $F_0 : \Sigma^2 \rightarrow R^4$  be an immersion from a 2-dimensional oriented surface  $\Sigma^2$  to  $R^4$ . Denote by  $\Sigma_0 = F_0(\Sigma)$ , and we say that  $\Sigma_0$  is a graph if there is a unit constant 2-form  $w$  in  $R^4$  such that

$$\langle e_1 \wedge e_2, w \rangle \geq v_0 > 0,$$

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where  $\{e_1, e_2\}$  is an orthonormal frame on  $\Sigma_0$ . In this paper, we consider the surface  $\Sigma$  evolves along the forced mean curvature flow

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} F(x, t) = H(x, t) + c(t)F(x, t), & \forall x \in \Sigma, \forall t > 0 \\ F(\cdot, 0) = F_0, \end{cases}$$

where  $H(x, t)$  denotes the mean curvature vector of  $\Sigma_t = F(\Sigma, t)$  at  $F(x, t)$ , and  $c(t)$  is a bounded continuous function. In order to state our main result, we would like to introduce a notation here. Define

$$(1.2) \quad v(x, t) := \langle e_1(x, t) \wedge e_2(x, t), w \rangle,$$

where  $\{e_1(x, t), e_2(x, t)\}$  is an orthonormal frame on  $\Sigma_t$  at  $F(x, t)$ .

This forced mean curvature flow (1.1) has been investigated in [6, 7], where the initial submanifolds are an entire graph and a convex hypersurface respectively. However, the codimensions there are just 1, but here we want to discuss the higher codimensional case. In fact, we can prove the following.

**THEOREM 1.1.** *Suppose  $\Sigma^2$  is a 2-dimensional oriented surface evolving under the forced mean curvature flow (1.1) in  $R^4$ . If the graph  $\Sigma_0 = F_0(\Sigma)$  has bounded curvature,  $v(x, 0) \geq v_0 > \frac{1}{\sqrt{2}}$ , and additionally  $c(t)$  is a bounded nonnegative continuous function, then the flow (1.1) has a global solution.*

The paper is organized as follows. The geometric evolution equations of an  $m$ -dimensional immersed submanifold evolving along the forced mean curvature flow having the form as (1.1) in  $R^n$  are derived in the next section. Theorem 1.1 will be proved in the last section.

## 2. Geometric evolution equations

In this section, we derive the evolution equations for some geometric quantities. Given an immersion  $F_0 : M^m \rightarrow R^n$  from an  $m$ -dimensional submanifold to the Euclidean space  $R^n$  with the standard Euclidean metric  $\langle \cdot, \cdot \rangle$ . Consider a one-parameter family of smooth maps  $F_t = F(\cdot, t) : M \rightarrow R^n$  with corresponding images  $M_t = F_t(M)$  evolving along the forced mean curvature flow

$$(2.1) \quad \begin{cases} \frac{\partial}{\partial t} F(x, t) = H(x, t) + c(t)F(x, t), & \forall x \in M, \forall t > 0 \\ F(\cdot, 0) = F_0, \end{cases}$$

where  $H(x, t)$  denotes the mean curvature vector of  $M_t = F(M, t)$  at  $F(x, t)$ , and  $c(t)$  is a bounded continuous function. Denote by  $\Delta$  and  $\nabla$  the Laplace and gradient operators for the induced metric on  $M_t$ , respectively. For a normal

coordinate system  $\{x^1, \dots, x^m\}$  around a point  $p$  on  $M$ , the metric  $g$  on  $M_t$  induced by  $\langle \cdot, \cdot \rangle$  satisfies

$$g_{ij} = \langle \partial_i F, \partial_j F \rangle,$$

where  $\partial_i F$ ,  $1 \leq i \leq m$ , is the partial derivative with respect to the local coordinates. Choose a local field of orthonormal frame  $e_1, \dots, e_m, v_1, \dots, v_{n-m}$  at the point  $F(p, t)$  of  $R^n$  along  $M_t$  such that  $e_1, \dots, e_m$  are tangent vectors of  $M_t$  and  $v_1, \dots, v_{n-m}$  are in the normal bundle over  $M_t$ . We make use of the indices range,  $1 \leq i, j, k, \dots, \leq m$  and  $1 \leq \alpha, \beta, \gamma, \dots, \leq n - m$ . The Einstein summation convention that repeated indices are summed over is adopted in the rest of the article. Naturally, we can write

$$A = A^\alpha v_\alpha, \quad H = -H^\alpha v_\alpha,$$

where  $A^\alpha = (h_{ij}^\alpha)$  is a matrix with  $h_{ij}^\alpha$  the component of the second fundamental form, and  $H^\alpha = g^{ij} h_{ij}^\alpha = h_{ii}^\alpha$ . Then the squared norm of the second fundamental form should be

$$|A|^2 = \sum_\alpha |A^\alpha|^2 = g^{ij} g^{kl} h_{ik}^\alpha h_{jl}^\alpha = h_{ik}^\alpha h_{ik}^\alpha.$$

Here we point out that the matrix  $A^\alpha$  is symmetric, since we have

$$h_{ij}^\alpha = \langle \partial_i v_\alpha, \partial_j F \rangle = \langle \partial_j v_\alpha, \partial_i F \rangle = h_{ji}^\alpha$$

by the Weingarten equation (cf. [4, 9]).

In order to give the simple forms of the geometric evolution equations below, we need to introduce several notations. Define  $C_{i\beta}^\alpha := \langle v_\alpha, \nabla_i v_\beta \rangle$  and  $b_\alpha^\beta := \left\langle \frac{\partial v_\alpha}{\partial t}, v_\beta \right\rangle$ , obviously,  $C_{i\beta}^\alpha = -C_{i\alpha}^\beta$  and  $b_\alpha^\beta = -b_\beta^\alpha$ . Notice that  $b_\alpha^\beta$  vanishes for hypersurfaces. We first derive the evolution equations for the induced metric and the normal vector.

LEMMA 2.1. *Under the forced mean curvature flow (2.1), the induced metric and the normal vector satisfy*

$$(2.2) \quad \frac{\partial g_{ij}}{\partial t} = -2H^\alpha h_{ij}^\alpha + 2c(t)g_{ij},$$

and

$$(2.3) \quad \frac{\partial v_\alpha}{\partial t} = \nabla H^\alpha + H^\gamma C_{i\gamma}^\alpha e_i + b_\alpha^\beta v_\beta.$$

*Proof.* Using normal coordinate systems at  $x$  on  $M_t$  and at  $F(x, t)$  on  $R^n$ , together with the Gauss-Weingarten equations (cf. [4, 9])

$$\partial_i v_\alpha = h_{ij}^\alpha g^{jk} \partial_k F + C_{i\alpha}^\beta v_\beta,$$

we have

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= \frac{\partial}{\partial t} \langle \partial_i F, \partial_j F \rangle \\ &= -\langle \partial_i (H^\alpha v_\alpha - c(t)F), \partial_j F \rangle - \langle \partial_j (H^\alpha v_\alpha - c(t)F), \partial_i F \rangle \\ &= -2H^\alpha h_{ij}^\alpha + 2c(t)g_{ij},\end{aligned}$$

which finishes the proof of (2.2).

Using normal coordinate systems at  $x$  on  $M_t$  and at  $F(x, t)$  on  $R^n$ , and then translating the identity into normal frames, as the proof of lemma 2.2 in [1], we have

$$\begin{aligned}\frac{\partial v_\alpha}{\partial t} &= \left\langle \frac{\partial v_\alpha}{\partial t}, \partial_i F \right\rangle g^{ij} \partial_j F + b_\alpha^\beta v_\beta \\ &= -\langle v_\alpha, \partial_i (-H^\gamma v_\gamma + c(t)F) \rangle g^{ij} \partial_j F + b_\alpha^\beta v_\beta \\ &= \nabla H^\alpha + H^\gamma C_{i\gamma}^\alpha e_i + b_\alpha^\beta v_\beta,\end{aligned}$$

which finishes the proof of (2.3).  $\square$

Furthermore, we can prove the evolution equations for the second fundamental form, its squared norm, and also the squared norm of the mean curvature vector.

LEMMA 2.2. *Assume that the Christoffel symbols  $\Gamma_{ij}^k$  of the Levi-Civita connection of the induced metric are zero at a point  $p \in M_t$ , then under the forced mean curvature flow (2.1) we have*

$$\begin{aligned}\frac{\partial}{\partial t} h_{ij}^\alpha &= \nabla_i \nabla_j H^\alpha - H^\gamma h_{ji}^\gamma h_{il}^\alpha + H^\gamma C_{j\gamma}^\beta C_{i\beta}^\alpha + H^\gamma \nabla_i C_{j\gamma}^\alpha \\ &\quad + \nabla_j H^\beta C_{i\beta}^\alpha + \nabla_i H^\beta C_{j\beta}^\alpha - h_{ij}^\beta b_\beta^\alpha + c(t)h_{ij}^\alpha\end{aligned}$$

at  $p$ .

*Proof.* Under the normal coordinate systems as before and our assumption, by the Gauss-Weingarten equations (cf. [4, 9]), we have that at the point  $p$

$$h_{ij}^\alpha = -\langle \partial_{ij}^2 F, v_\alpha \rangle, \quad \partial_{ij}^2 F = -h_{ij}^\alpha v_\alpha,$$

and

$$\partial_i v_\alpha = h_{ij}^\alpha g^{jm} \partial_m F + C_{i\alpha}^\beta v_\beta.$$

Hence, we have

$$\begin{aligned}\frac{\partial}{\partial t} h_{ij}^\alpha &= -\frac{\partial}{\partial t} \langle \partial_{ij}^2 F, v_\alpha \rangle \\ &= \langle \partial_{ij}^2 (H^\alpha v_\alpha - c(t)F), v_\alpha \rangle - \langle \partial_{ij}^2 F, \nabla H^\alpha + H^\gamma C_{i\gamma}^\alpha e_i + b_\alpha^\beta v_\beta \rangle\end{aligned}$$

$$\begin{aligned}
 &= \nabla_{ij}^2 H^\alpha - H^\gamma h_{ji}^\gamma h_{il}^\alpha + H^\gamma C_{j\gamma}^\beta C_{i\beta}^\alpha + \nabla_i H^\beta C_{j\beta}^\alpha - h_{ij}^\beta b_\beta^\alpha + \langle \nabla_j H^\gamma C_{i\gamma}^\beta, v_\beta, v_\alpha \rangle \\
 &\quad + \langle \nabla_i H^\gamma C_{j\gamma}^\beta, v_\beta, v_\alpha \rangle + c(t)h_{ij}^\alpha,
 \end{aligned}$$

which implies our lemma. □

By lemma 2.4 in [1], we have the following lemma.

LEMMA 2.3. *Assume that the Christoffel symbols  $\Gamma_{ij}^k$  of the Levi-Civita connection of the induced metric are zero at a point  $p \in M_t$ , then at the point  $p$  we have*

$$\begin{aligned}
 \nabla_i \nabla_j H^\alpha &= \Delta h_{ij}^\alpha + \nabla_l (h_{ij}^\beta C_{l\beta}^\alpha) - \nabla_l (h_{jl}^\beta C_{i\beta}^\alpha) + h_{ii}^\beta h_{lm}^\beta h_{mj}^\alpha - H^\beta h_{im}^\beta h_{mj}^\alpha + h_{ij}^\beta h_{lm}^\beta h_{ml}^\alpha \\
 &\quad - h_{im}^\beta h_{jl}^\beta h_{ml}^\alpha + \nabla_i (h_{jl}^\beta C_{i\beta}^\alpha) - \nabla_i (H^\beta C_{j\beta}^\alpha).
 \end{aligned}$$

Combining Lemma 2.2 and Lemma 2.3 immediately yields the following.

LEMMA 2.4. *Under the forced mean curvature flow (2.1), the second fundamental form satisfies*

$$\begin{aligned}
 \left( \frac{\partial}{\partial t} - \Delta \right) h_{ij}^\alpha &= \nabla_l (h_{ij}^\beta C_{l\beta}^\alpha) + h_{il}^\beta h_{im}^\beta h_{mj}^\alpha - H^\beta (h_{im}^\beta h_{mj}^\alpha + h_{jl}^\beta h_{il}^\alpha) + h_{ij}^\beta h_{lm}^\beta h_{ml}^\alpha - h_{im}^\beta h_{jl}^\beta h_{ml}^\alpha \\
 &\quad + h_{jl}^\beta (\nabla_i C_{l\beta}^\alpha - \nabla_l C_{i\beta}^\alpha) + h_{jl}^\gamma C_{l\gamma}^\beta C_{i\beta}^\alpha - h_{ij}^\beta b_\beta^\alpha + c(t)h_{ij}^\alpha.
 \end{aligned}$$

Now, by using the previous lemmas, we can prove the main result of this section as follows.

LEMMA 2.5. *Under the forced mean curvature flow (2.1), we have*

$$\left( \frac{\partial}{\partial t} - \Delta \right) |A|^2 = -2|\tilde{\nabla}A|^2 + 2|A|^4 - 2c(t)|A|^2,$$

and

$$\left( \frac{\partial}{\partial t} - \Delta \right) |H|^2 = -2|\tilde{\nabla}H|^2 + 2|A|^2|H|^2 - 2c(t)|H|^2,$$

where  $\tilde{\nabla}$  is the covariant differentiation on  $\text{Hom}(TM_t \times TM_t, \text{Nor } M_t)$  determined by the covariant differentiation on  $TM_t$  and  $D$  on the normal bundle,  $D$  is the normal connection for the embedding  $M_t \subset R^n$  (cf. [9]).

### 3. Long time existence

Obviously, choose  $M^m = \Sigma^2$  to be an oriented 2-dimensional surface and  $n = 4$ , then the flow (2.1) coincides with the flow (1.1), which implies those

evolution equations derived in the last section also hold under the flow (1.1). In this section, we want to show the global existence of the smooth solution of the forced mean curvature flow (1.1).

As the hypersurface mean curvature flow case, we could get the short-time existence of the smooth solution of the flow (1.1) by the standard theory of parabolic partial differential equation. We state it as follows.

**THEOREM 3.1.** *Suppose that the initial surface  $\Sigma_0 = F_0(\Sigma)$  has bounded curvature, then there exists  $T > 0$  such that (1.1) has a smooth solution on the time interval  $[0, T)$ . If  $\max_{\Sigma_t} |A|^2$  is bounded near  $T$ , then the solution could be extended to  $[0, T + \varepsilon)$  for  $\varepsilon > 0$ .*

So, if we want to show the global existence of the smooth solution of the flow (1.1), it needs to show that  $\max_{\Sigma_t} |A|^2$  is bounded as  $t \rightarrow T$ . In order to get the boundness of  $\lim_{t \rightarrow T} \max_{\Sigma_t} |A|^2$ , we first derive a monotonicity formula as in [2, 3]. Define a function  $\rho = \rho(x, t)$  by

$$(3.1) \quad \rho(x, t) = 4\pi(t_0 - t)H(x, x_0, t) = \frac{1}{4\pi(t_0 - t)} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right), \quad t < t_0,$$

where  $H(x, x_0, t)$  is the backward heat kernel in  $R^4$ . We can prove the following.

**PROPOSITION 3.2.** *Suppose  $F$  satisfies the flow (1.1), and  $f(x, t)$  is a smooth function defined on  $\Sigma^2 \times R^+$ , then*

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Sigma_t} f\rho(F, t) d\mu_t &= \int_{\Sigma_t} \left(\frac{df}{dt} - \Delta f\right)\rho(F, t) d\mu_t - \int_{\Sigma_t} f\rho(F, t) \left|H + \frac{(F - x_0)^\perp}{2(t_0 - t)}\right|^2 d\mu_t \\ &\quad + 2 \int_{\Sigma_t} c(t)f\rho(F, t) d\mu_t - \int_{\Sigma_t} \frac{c(t)\langle F, F - x_0 \rangle}{2(t_0 - t)} f\rho(F, t) d\mu_t, \end{aligned}$$

where  $(F - x_0)^\perp$  denotes the projection of  $(F - x_0)$  onto the normal bundle of  $\Sigma_t$ .

*Proof.* By Lemma 2.1, we have

$$\frac{\partial}{\partial t} d\mu_t = \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial t} d\mu_t = -(|H|^2 - 2c(t)) d\mu_t,$$

so, it follows that

$$(3.2) \quad \begin{aligned} \frac{\partial}{\partial t} \int_{\Sigma_t} f\rho(F, t) d\mu_t &= \int_{\Sigma_t} \left[\left(\frac{\partial}{\partial t} - \Delta\right)f\right]\rho(F, t) d\mu_t + \int_{\Sigma_t} f\left(\frac{\partial}{\partial t} + \Delta\right)\rho(F, t) d\mu_t \\ &\quad - \int_{\Sigma_t} f\rho(F, t)|H|^2 d\mu_t + 2 \int_{\Sigma_t} c(t)f\rho(F, t) d\mu_t. \end{aligned}$$

By straightforward computation, we obtain

$$\frac{\partial}{\partial t} \rho(F, t) = \left[ \frac{1}{t_0 - t} - \frac{\langle H + c(t)F, F - x_0 \rangle}{2(t_0 - t)} - \frac{|F - x_0|^2}{4(t_0 - t)^2} \right] \rho(F, t)$$

and

$$\begin{aligned} \Delta \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right) &= \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right) \\ &\cdot \left[ \frac{|\langle F - x_0, \nabla F \rangle|^2}{4(t_0 - t)^2} - \frac{\langle F - x_0, \Delta F \rangle}{2(t_0 - t)} - \frac{|\nabla F|^2}{2(t_0 - t)} \right], \end{aligned}$$

together with the fact that  $|\nabla F|^2 = 2$  and  $\Delta F = H$  for the induced metric on  $\Sigma_t$ , the equality

$$(3.3) \quad \begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) \rho(F, t) &= - \left[ \frac{\langle F - x_0, H \rangle}{(t_0 - t)} + \frac{|(F - x_0)^\perp|^2}{4(t_0 - t)^2} \right] \rho(F, t) \\ &\quad - \frac{c(t) \langle F, F - x_0 \rangle}{2(t_0 - t)} \rho(F, t), \end{aligned}$$

holds. Our lemma follows by substituting (3.3) into (3.2). □

Then we can prove the following maximum principle.

**PROPOSITION 3.3.** *Suppose  $f(x, t)$  is a smooth function defined on  $\Sigma^2 \times R^+$ , which satisfies the inequality*

$$\frac{\partial f}{\partial t} - \Delta f \leq \vec{a} \cdot \nabla f$$

for some vector field  $\vec{a}$ , where  $\nabla$  and  $\Delta$  denote the tangential gradient and Laplacian on  $\Sigma_t$ . If  $a_0 = \sup_{\Sigma \times [0, t_1]} |\vec{a}| < \infty$  for some  $t_1 > 0$ , and in addition  $c(t)$  in (1.1) is nonnegative, then

$$\sup_{\Sigma_t} f \leq \sup_{\Sigma_0} f$$

for all  $t \in [0, t_1]$ .

*Proof.* Let  $k = \sup_{\Sigma_0} f$  and  $f_k = \max(f - k, 0)$ , as the proof of corollary 1.1 in [3], we have

$$\left(\frac{d}{dt} - \Delta\right) f_k^2 \leq \frac{1}{2} a_0^2 f_k^2.$$

Employing the monotonicity formula of Proposition 3.3 with  $f_k^2$  instead of  $f$ , and choosing  $x_0 = 0$  in (3.1) result in

$$(3.4) \quad \begin{aligned} \frac{d}{dt} \int_{\Sigma_t} f_k^2 \rho \, d\mu_t &\leq \frac{1}{2} a_0^2 \int_{\Sigma_t} f_k^2 \rho \, d\mu_t + 2 \int_{\Sigma_t} c(t) f_k^2 \rho \, d\mu_t \\ &\leq \left( \frac{1}{2} a_0^2 + 2c^+ \right) \int_{\Sigma_t} f_k^2 \rho \, d\mu_t, \end{aligned}$$

where  $c^+$  is the bound of the function  $c(t)$ . Then the desired result follows from (3.4) directly.  $\square$

Now, we want to show that if the initial image  $\Sigma_0$  is a graph in  $R^4$ , then under the forced mean curvature flow (1.1),  $\Sigma_t$  is also graph for  $0 \leq t < T$ , which will be used for deriving the boundness of  $\lim_{t \rightarrow T} \max_{\Sigma_t} |A|^2$  at the end of this section. In fact, as the proof of lemma 2.4 in [2], by using the evolution equation (2.3) for normal vectors, we have the following.

LEMMA 3.4. *Let  $w$  be a unit constant 2-form in  $R^4$ , and let  $v$  be defined as (1.2) with respect to an orthonormal frame  $\{e_1(x, t), e_2(x, t)\}$  of  $\Sigma_t = F(\Sigma, t)$  at  $F(x, t)$ . Then under the flow (1.1) we have*

$$\left( \frac{\partial}{\partial t} - \Delta \right) v = |A|^2 v - 2(h_{11}^1 h_{12}^2 - h_{11}^2 h_{12}^1 + h_{21}^1 h_{22}^2 - h_{21}^2 h_{22}^1) \langle v_1 \wedge v_2, w \rangle,$$

where as before  $\{v_1, v_2\}$  is an orthonormal frame for the normal bundle of  $\Sigma_t$ .

Then by Proposition 3.3 and Lemma 3.4, as the proof of proposition 2.5 in [2], we can prove the following conclusion.

PROPOSITION 3.5. *Let  $w$  be a unit constant 2-form in  $R^4$ . If  $v(x, 0) \geq v_0 > \frac{1}{\sqrt{2}}$  for all  $x \in \Sigma^2$ , and in addition  $c(t)$  in (1.1) is nonnegative for  $0 \leq t < T$ , then under the flow (1.1),  $v(x, t) \geq v_0$  holds for all  $t \in [0, T)$  and  $x \in \Sigma^2$ .*

*Proof of Theorem 1.1.* Let  $w$  be a unit constant 2-form in  $R^4$  with respect to which  $\Sigma_0$  is a graph. Consider the functions  $u_1 = \langle e_1 \wedge e_2, w + *w \rangle$  and  $u_2 = \langle e_1 \wedge e_2, w - *w \rangle$ . As the proof of theorem 2.6 in [2], by Lemma 3.4 and Proposition 3.5, we have

$$(3.5) \quad u_i(x, t) \geq u_i(x, 0) \geq v_0 - \frac{1}{\sqrt{2}} > 0, \quad i = 1, 2.$$

Moreover, let  $u = u_1 u_2$ , then we have

$$(3.6) \quad \left( \frac{\partial}{\partial t} - \Delta \right) u = 2|A|^2 u - 2\nabla u_1 \cdot \nabla u_2 = 2|A|^2 u - 2 \frac{\nabla u_1}{u_1} \cdot \nabla u + 2 \frac{|\nabla u_1|^2 u}{u_1^2}.$$

Define  $\phi = \frac{|A|^2}{u}$ , by Lemma 2.5 and (3.6), we have



$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\phi &= \frac{1}{u}\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 - \frac{|A|^2}{u^2}\left(\frac{\partial}{\partial t} - \Delta\right)u + 2\nabla|A|^2 \cdot \frac{\nabla u}{u} - 2|A|^2 \cdot \frac{|\nabla u|^2}{u^3} \\
&\leq \nabla\phi \cdot \frac{\nabla u}{u} - 2c(t)\phi \\
&\leq \nabla\phi \cdot \frac{\nabla u}{u},
\end{aligned}$$

together with (3.5) and Proposition 3.3, it follows that  $\sup_{\Sigma_t}|A|^2 \leq \sup_{\Sigma_0}|A|^2 < \infty$ . Then by Theorem 3.1, we have  $T = \infty$ , which implies Theorem 1.1.  $\square$

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