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FORCED HYPERBOLIC MEAN CURVATURE FLOW

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Abstract

In this paper, we investigate two hyperbolic flows obtained by adding forcing terms in direction of the position vector to the hyperbolic mean curvature flows in [5, 8]. For the first hyperbolic flow, as in [8], by using support function, we reduce it to a hyperbolic Monge-Ampère equation successfully, leading to the short-time existence of the flow by the standard theory of hyperbolic partial differential equation. If the initial velocity is non-negative and the coefficient function of the forcing term is non-positive, we also show that there exists a class of initial velocities such that the solution of the flow exists only on a finite time interval $[0, T_{\max})$, and the solution converges to a point or shocks and other propagating discontinuities are generated when $t \to T_{\max}$. These generalize the corresponding results in [8]. For the second hyperbolic flow, as in [5], we can prove the system of partial differential equations related to the flow is strictly hyperbolic, which leads to the short-time existence of the smooth solution of the flow, and also the uniqueness. We also derive nonlinear wave equations satisfied by some intrinsic geometric quantities of the evolving hypersurface under this hyperbolic flow. These generalize the corresponding results in [5].

1. Introduction

Generally, we refer to a hyperbolic flow whose main driving factor is mean curvature as the hyperbolic mean curvature flow (HMCF). In [14], Rostein, Brandon and Novick-Cohen studied a hyperbolic mean curvature flow of interfaces and gave a crystalline algorithm for the motion of closed convex polygonal curves. In [17], Yau has suggested hyperbolic mean curvature flow can be used to model a vibrating membrane or the motion of a surface. It seems necessary to study the hyperbolic mean curvature flow because of these applications.

To our knowledge, few kinds of hyperbolic mean curvature flow have been studied and also few results of these hyperbolic mean curvature flows have been obtained, see [5, 8, 9] for instance. Now we want to show the motivation

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why we consider the hyperbolic mean curvature flows (1.3) and (1.4) below in this paper. Actually, it is inspired by the similar situation in the mean curvature flow. More precisely, Ecker and Huisken [3] considered the problem that a hypersuface M_0 immersed in R^{n+1} evolves by a family of smooth immersions $X(\cdot,t) : M_0 \to R^{n+1}$ as follows

$$\begin{cases} \dfrac{\partial}{\partial t} X(x,t) = H(x,t)\vec{N}(x,t), & \forall x \in M_0,\ \forall t > 0 \\ X(\cdot,0) = M_0, \end{cases} \tag{1.1}$$

where $H(x,t)$ and $\vec{N}(x,t)$ are the mean curvature and unit inner normal vector of the hypersurface $M_t = X(M_0,t) = X_t(M_0)$, respectively. If additionally the initial hypersurface M_0 is a locally Lipschitz continuous entire graph over a hyperplane in R^{n+1}, they have proved that the classical mean curvature flow (1.1) exists for all the time $t \in [0,\infty)$, moreover, each $X(\cdot,t)$ is also an entire graph. Fortunately, by using a similar way, Mao, Li and Wu [12] proved that if the above initial hypersurface, a locally Lipschitz continuous entire graph in R^{n+1}, evolves along the following curvature flow

$$\begin{cases} \dfrac{\partial}{\partial t} X(x,t) = H(x,t)\vec{N}(x,t) + \tilde{c}(t)X(x,t), & \forall x \in M_0,\ \forall t > 0 \\ X(\cdot,0) = M_0, \end{cases} \tag{1.2}$$

where $\tilde{c}(t)$ is a bounded nonnegative continuous function, and $H(x,t)$ and $\vec{N}(x,t)$ have the same meanings as those in the flow (1.1), then the curvature flow (1.2) has long time existence solutions, and each each $X(\cdot,t)$ is also an entire graph. This generalizes part of results of Ecker and Huisken, since if $\tilde{c}(t) = 0$ in (1.2), then this flow degenerates into the classical mean curvature flow (1.1). Similarly, if $\tilde{c}(t)$ is a bounded continuous function, for a strictly convex compact hyersurface in R^{n+1} evolving along the curvature flow of the form (1.2), Li, Mao and Wu [11] proved a similar conclusion as in [10] by mainly using the methods shown in [10] and [7].

Since we could get these nice results if we add a forcing term in direction of the position vector to the classical mean curvature flow, we guess maybe it would also work if we add this kind of forcing term to the hyperbolic mean curvature flows introduced in [5] and [8] respectively. This process of adding the forcing term lets us consider the following two initial value problems.

First, we consider a family of closed plane curves $F : S^1 \times [0,T) \to R^2$ which satisfies the following evolution equation

$$\begin{cases} \dfrac{\partial^2 F}{\partial t^2}(u,t) = k(u,t)\vec{N}(u,t) - \nabla\rho + c(t)F(u,t), & \forall (u,t) \in S^1 \times [0,T) \\ F(u,0) = F_0(u), \\ \dfrac{\partial F}{\partial t}(u,0) = f(u)\vec{N}_0, \end{cases} \tag{1.3}$$

where $k(u,t)$ and $\vec{N}(u,t)$ are the curvature and unit inner normal vector of the plane curve $F(u,t)$ respectively, $f(u) \in C^\infty(S^1)$ is the initial normal velocity, and $\vec{N}_0$ is the unit inner normal vector of the smooth strictly convex plane curve $F_0(u)$. Besides, $c(t)$ is a bounded continuous function on the interval $[0,T)$ and $\nabla\rho$ is given by

$$\nabla\rho := \left[\left(\frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t}\right) + c(t)(F, \vec{T})\right]\vec{T}(u,t),$$

where $(\cdot\,,\cdot)$ denotes the standard Euclidean metric in R^2, and $\vec{T}$, s denote the unit tangent vector of the plane curve $F(u,t)$ and the arc-length parameter, respectively.

Fortunately, we can prove the following main results for this flow.

THEOREM 1.1 (Local existence and uniqueness). *For the hyperbolic flow* (1.3), *there exists a positive constant* $T_1 > 0$ *and a family of strictly closed curves* $F(\cdot,t)$ *with* $t \in [0,T_1)$ *such that each* $F(\cdot,t)$ *is its solution.*

THEOREM 1.2. *For the hyperbolic flow* (1.3), *if additionally* $c(t)$ *is non-positive and the initial velocity* $f(u)$ *is non-negative, there exists a class of the initial velocities such that its solution exists only on a finite time interval* $[0,T_{\max})$. *Moreover, when* $t \to T_{\max}$, *one of the following must be true*

(I) *the solution* $F(\cdot,t)$ *converges to a single point, or equivalently, the curvature of the limit curve becomes unbounded*;

(II) *the curvature* $k(\cdot,t)$ *of the curve* $F(\cdot,t)$ *is discontinuous so that the solution converges to a piecewise smooth curve, which implies shocks and propagating discontinuities may be generated within the hyperbolic flow* (1.3).

Second, we consider that an n-dimensional smooth manifold $\mathscr{M}$ evolves by a family of smooth hypersurface immersions $X(\cdot,t) : \mathscr{M} \to R^{n+1}$ in R^{n+1} as follows

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} X(x,t) = H(x,t)\vec{N}(x,t) + c_1(t)X(x,t), \quad \forall x \in \mathscr{M},\ \forall t > 0 \\ X(x,0) = X_0(x), \\ \dfrac{\partial X}{\partial t}(x,0) = X_1(x), \end{cases} \tag{1.4}$$

where $\vec{N}(x,t)$ is the unit inner normal vector of the hypersurface $\mathscr{M}_t = X(\mathscr{M},t) = X_t(\mathscr{M})$, X_0 is a smooth hypersurface immersion of $\mathscr{M}$ into R^{n+1}, $X_1(x)$ is a smooth vector-valued function on $\mathscr{M}$, and $c_1(t)$ is a bounded continuous function.

For this flow, we can prove the following result.

THEOREM 1.3 (Local existence and uniqueness). *For the hyperbolic flow* (1.4), *if additionally* $\mathscr{M}$ *is compact, then there exists a positive constant* $T_2 > 0$ *such that the initial value problem* (1.4) *has a unique smooth solution* $X(x,t)$ *on* $\mathscr{M} \times [0,T_2)$.

The paper is organized as follows. In Section 2, the notion of support function of $F(u,t)$ will be introduced, which is used to derive a hyperbolic Monge-Ampère equation leading to the local existence and uniqueness of the hyperbolic flow (1.3). An example and some properties of the evolving curve have been studied in Section 3. Theorem 1.2 will be proved in Section 4. In Section 5, by using the standard existence theory of hyperbolic system of partial differential equations, we show the short-time existence Theorem 1.3 of the hyperbolic flow (1.4). Some exact solutions of the hyperbolic flow (1.4) will be studied in Section 6. The nonlinear wave equations of some geometric quantities of the hypersurface $X(\cdot,t)$ will be derived in Section 7.

2. Proof of Theorem 1.1

In this section, we will reparametrize the evolving curves so that the hyperbolic Monge-Ampère equation could be derived for the support function defined below. Reparametrizations can be done since for an evolving curve $F(\cdot,t)$ under the flow (1.3), the underlying physics should be independent of the choice of the parameter $u \in S^1$. However, before deriving the hyperbolic Monge-Ampère equation, the following definition in [9] is necessary.

Definition 2.1. A flow $F : S^1 \times [0,T) \to R^2$ evolves normally if and only if its tangential velocity vanishes.

We claim that our hyperbolic flow (1.3) is a normal flow, since

$$\frac{d}{dt}\left(\frac{\partial F}{\partial t},\frac{\partial F}{\partial s}\right) = -\left(\nabla\rho,\frac{\partial F}{\partial s}\right) + c(t)(F,\vec{T}) + \left(\frac{\partial F}{\partial t},\frac{\partial^2 F}{\partial t\partial s}\right) = 0,$$

and the initial velocity of the flow (1.3) is in the normal direction. Then we have

$$\frac{d}{dt}F(u,t) = \left(\frac{d}{dt}F(u,t),\vec{N}(u,t)\right)\vec{N}(u,t) := \sigma(u,t)\vec{N}(u,t). \tag{2.1}$$

By (1.3) and (2.1), we have

$$\frac{\partial\sigma}{\partial t} = k(u,t) + c(t)(F,\vec{N})(u,t), \quad \sigma\frac{\partial\sigma}{\partial s} = \left(\frac{\partial^2 F}{\partial s\partial t},\frac{\partial F}{\partial t}\right), \tag{2.2}$$

where $s = s(\cdot,t)$ is the arc-length parameter of the curve $F(\cdot,t) : S^1 \to R^2$. Obviously, by arc-length formula, we have

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}}\frac{\partial}{\partial u} = \frac{1}{\left|\frac{\partial F}{\partial u}\right|}\frac{\partial}{\partial u} := \frac{1}{v}\frac{\partial}{\partial u}, \tag{2.3}$$

here (x, y) is the cartesian coordinates of R^2. For the orthogonal frame field $\{\vec{N}, \vec{T}\}$ of R^2, by Frenet formula, we have

$$\frac{\partial \vec{T}}{\partial s} = k\vec{N}, \quad \frac{\partial \vec{N}}{\partial s} = -k\vec{T}. \tag{2.4}$$

Now, in order to give the notion of support function, we have to use the unit out normal angel, denoted by θ, of a closed convex curve $F : S^1 \times [0, T) \to R^2$ w.r.t the cartesian coordinates of R^2. Then

$$\vec{N} = (-\cos\theta, -\sin\theta), \quad \vec{T} = (-\sin\theta, \cos\theta),$$

correspondingly, we have $\dfrac{\partial\theta}{\partial s} = k$ and

$$\frac{\partial \vec{N}}{\partial t} = -\frac{\partial\theta}{\partial t}\vec{T}, \quad \frac{\partial \vec{T}}{\partial t} = \frac{\partial\theta}{\partial t}\vec{N}. \tag{2.5}$$

Lemma 2.2. *The derivative of v with respect to t is* $\dfrac{\partial v}{\partial t} = -k\sigma v$.

Proof. By using (2.1), (2.3), and (2.4), as in [4], we calculate directly as follows

$$\begin{aligned}\frac{\partial}{\partial t}(v^2) &= 2\left(\frac{\partial F}{\partial u}, \frac{\partial^2 F}{\partial t \partial u}\right) = 2\left(\frac{\partial F}{\partial u}, \frac{\partial^2 F}{\partial u \partial t}\right) = 2\left(v\vec{T}, \frac{\partial}{\partial u}(\sigma\vec{N})\right)\\ &= 2\left(v\vec{T}, \frac{\partial\sigma}{\partial u}\vec{N} - k\sigma v\vec{T}\right) = -2v^2 k\sigma,\end{aligned}$$

which implies our lemma. □

Then, by using Lemma 2.2, we can obtain

$$\frac{\partial^2}{\partial t \partial s} = \frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial}{\partial u}\right) = k\sigma\frac{1}{v}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial u}\frac{\partial}{\partial t} = k\sigma\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s \partial t},$$

which implies

$$\frac{\partial \vec{T}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial s}\right)\vec{N} = \frac{\partial\sigma}{\partial s}\vec{N}.$$

Combining this equality with (2.1) yields $\dfrac{\partial\theta}{\partial t} = \dfrac{\partial\sigma}{\partial s}$.

Assume $F(u, t) : S^1 \times [0, T) \to R^2$ is a family of convex curves satisfying the flow (1.3). Now, as in [18], we will use the normal angel to reparametrize the evolving curve $F(\cdot, t)$, and then give the notion of support function which is used to derive the local existence of the flow (1.3). Set

$$\tilde{F}(\theta, \tau) = F(u(\theta, \tau), t(\theta, \tau)), \tag{2.6}$$

where $t(\theta,\tau)=\tau$. We claim that under the parametrization (2.6), $\vec{N}$ and $\vec{T}$ are independent of the parameter τ. In fact, by the chain rule we have

$$0=\frac{\partial\theta}{\partial\tau}=\frac{\partial\theta}{\partial u}\frac{\partial u}{\partial\tau}+\frac{\partial\theta}{\partial t},$$

which implies

$$\frac{\partial\theta}{\partial t}=-\frac{\partial\theta}{\partial u}\frac{\partial u}{\partial\tau}=-\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial\tau}=-kv\frac{\partial u}{\partial\tau}.$$

Therefore,

$$\frac{\partial\vec{T}}{\partial\tau}=\frac{\partial\vec{T}}{\partial t}+\frac{\partial\vec{T}}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial\tau}=\left(\frac{\partial\theta}{\partial t}+kv\frac{\partial u}{\partial\tau}\right)\vec{N}=0.$$

Similarly, we have $\dfrac{\partial\vec{N}}{\partial\tau}=-\left(\dfrac{\partial\theta}{\partial t}+kv\dfrac{\partial u}{\partial\tau}\right)\vec{T}=0$, then our claim follows.

Define the support function of the evolving curve $\tilde{F}(\theta,\tau)=(x(\theta,\tau),y(\theta,\tau))$ as follows

$$S(\theta,\tau)=(\tilde{F}(\theta,\tau),-\vec{N})=x(\theta,\tau)\cos\theta+y(\theta,\tau)\sin\theta,$$

consequently,

$$S_\theta(\theta,\tau)=-x(\theta,\tau)\sin\theta+y(\theta,\tau)\cos\theta=(\tilde{F}(\theta,\tau),\vec{T}).$$

Therefore, we have

$$\begin{cases}x(\theta,\tau)=S\cos\theta-S_\theta\sin\theta,\\ y(\theta,\tau)=S\sin\theta+S_\theta\cos\theta,\end{cases}\tag{2.7}$$

which implies the curve $\tilde{F}(\theta,\tau)$ can be represented by the support function. Then we have

$$S_{\theta\theta}+S=-x_\theta\sin\theta+y_\theta\cos\theta=\left(\frac{\partial\tilde{F}}{\partial\theta},\vec{T}\right)=\left(\frac{\partial\tilde{F}}{\partial s}\frac{\partial s}{\partial\theta},\vec{T}\right)=\frac{1}{k},\tag{2.8}$$

since the evolving curve $\tilde{F}(\theta,\tau)=F(u(\theta,\tau),t(\theta,\tau))$ is strictly convex, (2.8) makes sense.

On the other hand, since $\vec{N}$ and $\vec{T}$ are independent of the parameter τ, together with (2.1) and (2.6), we have

$$S_\tau=\left(\frac{\partial\tilde{F}}{\partial\tau},-\vec{N}\right)=\left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial\tau}+\frac{\partial F}{\partial t},\vec{N}\right)=\left(\frac{\partial F}{\partial t},-\vec{N}\right)=-\sigma(u,t),\tag{2.9}$$

furthermore, by the chain rule we obtain

$$\begin{aligned}S_{\tau\tau}&=\left(\frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial\tau^2}+\frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial\tau}\right)^2+2\frac{\partial^2 F}{\partial u\partial\tau}\frac{\partial u}{\partial\tau}+\frac{\partial^2 F}{\partial t^2},-\vec{N}\right)\\&=\left(\frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial\tau}\right)^2+\frac{\partial^2 F}{\partial u\partial\tau}\frac{\partial u}{\partial\tau},-\vec{N}\right)+\left(\frac{\partial^2 F}{\partial u\partial\tau}\frac{\partial u}{\partial\tau}+\frac{\partial^2 F}{\partial t^2},-\vec{N}\right)\end{aligned}$$

$$= \left(\left(\frac{\partial F}{\partial u} \right)_\tau, -\vec{N} \right) \frac{\partial u}{\partial \tau} + \left(\frac{\partial^2 F}{\partial u \partial \tau} \frac{\partial u}{\partial \tau}, -\vec{N} \right) - k - c(\tau)(F, \vec{N})$$

$$= \left(\frac{\partial^2 F}{\partial u \partial \tau} \frac{\partial u}{\partial \tau}, -\vec{N} \right) - k + c(\tau) S(\theta, \tau).$$

Since $F(u,t) : S^1 \times [0,T) \to R^2$ is a normal flow, which implies

$$\left(\frac{\partial F}{\partial t}, \vec{T} \right)(u,t) \equiv 0,$$

for all $t \in [0,T)$. By straightforward computation, we have

$$S_{\theta\tau} = \left(\frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \theta}, -\vec{N} \right) = \frac{1}{kv} \left(\frac{\partial^2 F}{\partial u \partial t}, -\vec{N} \right),$$

and

$$S_{\tau\theta} = \left(\frac{\partial \tilde{F}}{\partial \tau}, \vec{T} \right) = \left(\frac{\partial F}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial t}, \vec{T} \right) = v \frac{\partial u}{\partial \tau}.$$

Hence, the support function $S(\theta, \tau)$ satisfies

$$S_{\tau\tau} = \left(\frac{\partial^2 F}{\partial u \partial \tau} \frac{\partial u}{\partial \tau}, -\vec{N} \right) - k + c(\tau) S(\theta, \tau) = kv \frac{\partial u}{\partial \tau} S_{\theta\tau} - k + c(\tau) S(\theta, \tau)$$

$$= k(S_{\theta\tau}^2 - 1) + c(\tau) S,$$

combining this equality with (2.8) yields

$$S_{\tau\tau} = \frac{S_{\theta\tau}^2 - 1}{S_{\theta\theta} + S} + c(\tau) S, \quad \forall (\theta, \tau) \in S^1 \times [0,T). \tag{2.10}$$

Then it follows from (1.3), (2.6), (2.10) that

$$\begin{cases} S S_{\tau\tau} - c(\tau) S S_{\theta\theta} + (S_{\tau\tau} S_{\theta\theta} - S_{\theta\tau}^2) + 1 - c(\tau) S^2 = 0, \\ S(\theta, 0) = (F_0, -\vec{N}) = h(\theta), \\ S_\tau(\theta, 0) = -\tilde{f}(\theta) = -f(u(\theta, 0)), \end{cases} \tag{2.11}$$

where $h(\theta)$ and $\tilde{f}(\theta)$ are the support functions of the initial curve $F_0(u(\theta))$ and the initial velocity of this initial curve, respectively.

Now, we want to use the conclusion of the hyperbolic Monge-Ampère equation to get the short-time existence of the flow (1.3). Actually, for an unknown function $z(\theta, \tau)$ with two variables θ, τ, its Monge-Ampère equation has the form

$$A + B z_{\tau\tau} + C z_{\tau\theta} + D z_{\theta\theta} + E(z_{\tau\tau} z_{\theta\theta} - z_{\theta\tau}^2) = 0, \tag{2.12}$$

here the coefficients A, B, C, D, E depend on τ, θ, z, z_τ, z_θ. (2.12) is said to be τ-hyperbolic for S, if $\triangle^2(\tau, \theta, z, z_\tau, z_\theta) := C^2 - 4BD + 4AE > 0$ and $z_{\theta\theta} + B(\tau, \theta,$

$z, z_\tau, z_\theta) \neq 0$. We also need to require the τ-hyperbolicity at the initial time, in fact, if we rewrite the initial values as $z(\theta, 0) = z_0(\theta)$, $z_\tau(\theta, 0) = z_1(\theta)$ for the unknown function $z(\theta, \tau)$, $\theta \in [0, 2\pi]$, then the corresponding τ-hyperbolic condition is given as follows

$$\triangle^2(0, \theta, z_0, z_1, z_0') = (C^2 - 4BD + 4AE)|_{t=0} > 0,$$
$$z_0'' + B(0, \theta, z_0, z_1, z_0') \neq 0,$$

where $z_0' = \dfrac{dz_0}{d\theta}$, $z_0'' = \dfrac{d^2 z_0}{d\theta^2}$.

It is easy to check that (2.11) is a hyperbolic Monge-Ampère equation. In fact, for (2.11),

$$A = 1 - c(\tau)S^2, \quad B = S, \quad C = 0, \quad D = -c(\tau)S, \quad E = 1,$$

then we have

$$\begin{aligned}\triangle^2(\tau, \theta, S, S_\tau, S_\theta) &= C^2 - 4BD + 4AE = 0^2 - 4S \times (-c(\tau)S) + 4(1 - c(\tau)S^2) \times 1 \\ &= 4 > 0,\end{aligned}$$

and

$$S_{\theta\theta} + B(\tau, \theta, S, S_\tau, S_\theta) = S_{\theta\theta} + S = \frac{1}{k} \neq 0.$$

Furthermore, if at least $h(\theta) \in C^3([0, 2\pi])$ and $\tilde{f}(\theta) \in C^2([0, 2\pi])$, then we have

$$\triangle^2(0, \theta, h, \tilde{f}, h_\theta) = 4 > 0,$$

and

$$h_{\theta\theta} + B(0, \theta, h, \tilde{f}, h_\theta) \neq 0,$$

which implies (2.11) is also τ-hyperbolic at $\tau = 0$. Hence, (2.11) is a hyperbolic Monge-Ampère equation.

Then by the standard theory of hyperbolic equations (e.g., [6, 16]), Theorem 1.1 concerning the local existence and uniqueness of the solution of the hyperbolic flow (1.3) follows.

3. Some properties of the flow (1.3)

First, we would like to give an example so that we could understand the hyperbolic flow (1.3) deeply, however, first we need the following lemma.

LEMMA 3.1. *Consider the initial value problem*

$$\begin{cases} r_{tt} = -\dfrac{c_0}{r} + \bar{c}(t)r \\ r(0) = r_0 > 0, \quad r_t(0) = r_1, \end{cases} \tag{3.1}$$

where c_0 is a positive constant and $\bar{c}(t)$ is a non-positive bounded continuous function. For arbitrary initial data $r_0 > 0$, if the initial velocity $r_1 \le 0$, then the solution $r = r(t)$ decreases and attains its zero point at time t_0 (in particular, when $r_1 = 0$, we have $t_0 \le \sqrt{\frac{\pi}{2c_0}} r_0$, equality holds iff $\bar{c}(t) = 0$); if the initial velocity is positive, then the solution r increases first and then decreases and attains its zero point at a finite time.

Proof. The proof is similar with the arguments in [5, 14]. The discussion is divided into two cases.

Case (I). The initial velocity is non-positive, i.e. $r_1 \le 0$.

Assume $r(t) > 0$ for all the time $t > 0$. Then by (3.1) we have $r_{tt} = -\frac{c_0}{r} + \bar{c}(t) r < 0$, then by monotonicity $r_t(t) < r_t(0) = r_1 \le 0$ for all $t > 0$. Hence, there exists a time t_0 such that $r(t_0) = 0$, which is contradict with our assumption. Moreover, when the initial velocity vanishes, i.e. $r_t(0) = r_1 = 0$, let c^+ be the bound of the function $\bar{c}(t)$, i.e. $|\bar{c}(t)| \le c^+$ for all $t > 0$, obviously, multiplying both sides of $r_{tt} = -\frac{c_0}{r} + \bar{c}(t) r$ by r_t, integrating from 0 to $t < t_0$, and applying the condition $r_t(0) = r_1 = 0$ yield

$$c_0 \ln \frac{r_0}{r} \le \frac{r_t^2}{2} \le c_0 \ln \frac{r_0}{r} + \frac{c^+}{2}(r_0^2 - r(t)^2). \tag{3.2}$$

Integrating both sides of (3.2) on the interval $[0, t_0]$ and using the condition $r(t_0) = 0$ yield

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-u^2}\, du \ge \int_0^{t_0} \frac{\sqrt{2c_0}}{2r_0}\, dt \ge \int_0^\infty e^{-u^2}\, du - \frac{\sqrt{c^+}}{2} \int_0^{t_0} \sqrt{\frac{r_0^2 - r(t)^2}{\ln \frac{r_0}{r}}}\, r_0^{-1}\, dt,$$

where $u = \sqrt{\ln \frac{r_0}{r}}$. Therefore, we obtain

$$\sqrt{\frac{\pi}{2c_0}} r_0 - \frac{A r_0}{\sqrt{2c_0}} \le t_0 \le \sqrt{\frac{\pi}{2c_0}} r_0, \tag{3.3}$$

where

$$A = \sqrt{c^+} \int_0^{t_0} \sqrt{\frac{r_0^2 - r(t)^2}{\ln \frac{r_0}{r}}}\, r_0^{-1}\, dt.$$

Obviously, equalities in (3.3) hold simultaneously if and only if $c^+ = 0$, which implies $\bar{c}(t) = 0$, in this case, $t_0 = \sqrt{\frac{\pi}{2c_0}} r_0$, which is a conclusion in [5] for $c_0 = 1$.

CASE (II). The initial velocity is positive, i.e. $r_1 > 0$. From (3.1), we have

$$r_t^2(t) = -2c_0(\ln r(t) - \ln r_0) + r_1^2 + 2\int_0^t \bar{c}(s) r r_t(s)\, ds. \tag{3.4}$$

Assume r increases all the time, i.e. $r_t > 0$ for all the time $t > 0$. Since $r \geq r_0 > 0$, $r_t > 0$ and $\bar{c}(t)$ is non-positive, then from (3.4) we obtain

$$r_t^2(t) \leq -2c_0 \ln \frac{r}{r_0} + r_1^2,$$

which implies

$$r_0 \leq r(t) \leq e^{r_1^2/(2c_0)} r_0. \tag{3.5}$$

On the other hand, under our assumption, we have

$$-\frac{c_0}{r} - c^+ r \leq r_{tt} \leq -\frac{c_0}{r},$$

combining this relation with (3.5) results in

$$B(r_0) \leq r_{tt} \leq -e^{-r_1^2/(2c_0)} \frac{c_0}{r_0},$$

where

$$B(r_0) = \min\left\{ -\frac{c_0}{r_0} - c^+ r_0, -e^{-r_1^2/(2c_0)} \frac{c_0}{r_0} - c^+ e^{r_1^2/(2c_0)} r_0 \right\} < 0.$$

Thus the curve r_t can be bounded by two straight lines $r_t = B(r_0)t + r_1$ and $r_t = -e^{-r_1^2/(2c_0)} \frac{c_0}{r_0} t + r_1$, which implies r_t must be negative for $t > \frac{r_1 r_0 e^{r_1^2/(2c_0)}}{c_0}$. This is contradict with our assumption. Hence, r_t will change sign and become negative at certain finite time, which implies there exist a finite time t_1 such that $r_t(t_1) = 0$. Now, if we assume $r(t) > 0$ for all the time $t > 0$, then as in Case (I), we can prove that $r(t)$ attains its zero point at a finite time $t_2 > t_1$. Thus in this case $r(t)$ increases first and then decreases and attains its zero point at a finite time. Our conclusion follows by the above arguments. □

Example 3.2. Suppose $c(t)$ in the hyperbolic mean curvature flow (1.3) is also non-positive, and $F(\cdot, t)$ in (1.3) is a family of round circles with radius $r(t)$ centered at the origin. More precisely,

$$F(u, t) = r(t)(\cos u, \sin u), \quad r(0) > 0,$$

without loss of generality, we can also choose $u = s$ to be the arc-length parameter of the curve $F(\cdot, t)$. Then the curvature $k(\cdot, t)$ of the evolving curve

$F(\cdot, t)$ is $\dfrac{1}{r(t)}$, moreover, $\nabla\rho = 0$. Substituting these into (1.3) yields

$$\begin{cases} r_{tt} = -\dfrac{1}{r} + c(t)r, \\ r(0) = r_0 > 0, \quad r_t = r_1. \end{cases} \tag{3.6}$$

By Lemma 3.1, we know if the initial velocity $r_1 \leq 0$, then the flow (1.3) shrinks and converges to a single point at a finite time t_0 (in particular, when $r_1 = 0$, $t_0 \leq \sqrt{\dfrac{\pi}{2}} r_0$, equality holds iff $c(t) = 0$); if the initial velocity is positive, then the flow (1.3) expands first and shrinks and converges to a single point at a finite time. One can also interpret this phenomenon by physical principle as in [5, 8].

Remark 3.3. From this example, we know the necessity of the non-positivity of the bounded continuous function $c(t)$ if we want to get the convergence of the hyperbolic flow (1.3). That is the motivation why we add the condition $c(t)$ is non-positive in Theorem 1.2 to try to get the convergence.

Inspired by Chou's basic idea [1] for proving the convergence of the curve shortening flow, by using the maximum principle of the second order hyperbolic partial differential equations shown in [13], we could get the following conclusions as proposition 3.1 and proposition of preserving convexity in [8]. This is true, since, comparing with the evolution equations in the proofs of proposition 3.1 and proposition of preserving convexity in [8], one can easily check that the corresponding evolution equations of the difference of the support functions and the curvature function under the flow (1.3) only have extra first order terms $c(t)w$ and $-c(t)k$ respectively, moreover, these first order terms have no influence on the usage of the maximum principle.

Proposition 3.4 (Containment principle). *Suppose F_1 and $F_2 : S^1 \times [0, T_1) \to R^2$ are convex solutions of* (1.3). *If $F_2(\cdot, 0)$ lies in the domain enclosed by $F_1(\cdot, 0)$ and $f_2(u) \geq f_1(u)$, then $F_2(\cdot, t)$ is contained in the domain enclosed by $F_1(\cdot, t)$ for all $t \in [0, T_1)$.*

Proposition 3.5 (Preserving convexity). *Let k_0 be the mean curvature of the initial curve F_0, and let $\eta = \min_{\theta \in [0, 2\pi]} k_0(\theta)$. Then, for a C^4-solution of* (2.11), *one has*

$$k(\theta, t) \geq \eta := \min_{\theta \in [0, 2\pi]} k_0(\theta), \quad \textit{for } t \in [0, T_{\max}),\ \theta \in [0, 2\pi], \tag{3.7}$$

where $k(\theta, t)$ is the mean curvature of the evolving curve $F(\cdot, t)$, and $[0, T_{\max})$ is the maximal time interval of the solution $F(\cdot, t)$ of (1.3).

4. Convergence

In this section, we want to get the convergence of the hyperbolic flow (1.3). We assume that $c(t)$ is non-positive and initial velocity $f(u)$ is non-negative. In order to get the convergence, the following lemma is needed.

LEMMA 4.1. *The arclength $\mathfrak{L}(t)$ of the evolving closed curve $F(\cdot, t)$ of the flow* (1.3) *satisfies*

$$\frac{d\mathfrak{L}(t)}{dt} = -\int_0^{2\pi} \tilde{\sigma}(\theta, t)\, d\theta,$$

and

$$\frac{d^2\mathfrak{L}(t)}{dt^2} = \int_0^{2\pi} \left[\left(\frac{\partial \tilde{\sigma}}{\partial \theta}\right)^2 k - k + c(t)S \right] d\theta,$$

where $\tilde{\sigma}(\theta, t) = \tilde{\sigma}(\theta, \tau) = \sigma(u, t)$, the change of variables from (u, t) to (θ, τ) satisfies (2.6).

Proof. The convention of using t for time variable is used here. In addition, by straightforward computation, we have

$$\frac{d\mathfrak{L}(t)}{dt} = \frac{d}{dt}\int_{S^1} v(u, t)\, du = \int_{S^1} \frac{d}{dt} v(u, t)\, du = -\int_{S^1} k\sigma v\, du = -\int_0^{2\pi} \tilde{\sigma}\, d\theta,$$

and

$$\begin{aligned}\frac{d^2\mathfrak{L}(t)}{dt^2} &= -\int_0^{2\pi} \frac{\partial}{\partial t}(\tilde{\sigma}(\theta, t))\, d\theta = \int_0^{2\pi} [(S_{\theta t}^2 - 1)k + c(t)S]\, d\theta \\ &= \int_0^{2\pi} \left[\left(\left(\frac{\partial \tilde{\sigma}}{\partial \theta}\right)^2 - 1 \right) k + c(t)S \right] d\theta,\end{aligned}$$

here $v(u, t)$ is defined in (2.3), $u \in S^1$, and the fact $\frac{\partial}{\partial t} v(u, t) = -k\sigma v$ is shown in Lemma 2.2. Therefore, our proof is completed. □

Proof of Theorem 1.2. Let $[0, T_{\max})$ be the maximal time interval for the solution $F(\cdot, t)$ of the flow (1.3) with F_0 and f as initial curve and the initial velocity, respectively. We divide the proof into five steps.

Step 1. Preserving convexity

By Proposition 3.5, we know the evolving curve $F(S^1, t)$ remains strictly convex and the curvature of $F(S^1, t)$ has a uniformly positive lower bound $\min_{\theta \in [0, 2\pi]} k_0(\theta)$ on $S^1 \times [0, T_{\max})$.

Step 2. Short-time existence

Without loss of generality, we can assume that the origin o of R^2 is in the exterior of the domain enclosed by the initial curve F_0. Enclose the initial curve

F_0 by a large enough round circle γ_0 centered at o, and then let this circle evolve under the flow (1.3) with the initial velocity $\min_{u\in S^1} f(u)$ to get a solution $\gamma(\cdot,t)$. From Example 3.2, we know that the solution $\gamma(\cdot,t)$ exists only at a finite time interval $[0,T_0)$, and $\gamma(\cdot,t)$ shrinks into a point as $t\to T_0$. By Proposition 3.4, we know that $F(\cdot,t)$ is always enclosed by $\gamma(\cdot,t)$ for all $t\in[0,T_0)$. Therefore, we have that the solution $F(\cdot,t)$ must become singular at some time $T_{\max}\le T_0$.

Step 3. Hausdorff convergence

As in [1, 8, 18], we also want to use a classical result, Blaschke Selection Theorem, in convex geometry (c.f. [15]).

(Blaschke Selection Theorem). *Let $\{K_j\}$ be a sequence of convex sets which are contained in a bounded set. Then there exists a subsequence $\{K_{jk}\}$ and a convex set K such that K_{jk} converges to K in the Hausdorff metric.*

The round circle γ_0 in Step 2 is shrinking under the flow (1.3), since the normal initial velocity f is non-negative, this conclusion can be easily obtained from Lemma 3.1. Since for every time $t\in[0,T_{\max})$, $F(\cdot,t)$ is enclosed by $\gamma(\cdot,t)$, we have every convex set $K_{F(\cdot,t)}$ enclosed by $F(\cdot,t)$ is contained in a bounded set K_{γ_0} enclosed by γ_0. Thus, by Blaschke Selection Theorem, we can directly conclude that $F(\cdot,t)$ converges to a (maybe degenerate and nonsmooth) weakly convex curve $F(\cdot,T_{\max})$ in the Hausdorff metric.

Step 4. Length of evolving curve

We claim that there exists a finite time $\overline{T}\le\infty$ such that $\mathfrak{L}(\overline{T})=0$.

As Step 2, we can easily find a round circle $\overline{\gamma}_0$ centered at the origin o and enclosed by the convex initial curve F_0, and then let this circle evolve under the flow (1.3) with the initial velocity $\max_{u\in S^1} f(u)$ to get a solution $\overline{\gamma}(\cdot,t)$. From Example 3.2, we know that the solution $\overline{\gamma}(\cdot,t)$ exists only at a finite time interval $[0,\overline{T}_0)$ with $\overline{T}_0\le T_{\max}$, and $\overline{\gamma}(\cdot,t)$ shrinks into a point as $t\to\overline{T}_0$. By Proposition 3.4, we know that $F(\cdot,t)$ always encloses $\overline{\gamma}(\cdot,t)$ for all $t\in[0,\overline{T}_0)$. Thus we know that the support function $S(\theta,t)$ is nonnegative on the time interval $[0,\overline{T}_0)$, and we can also conclude that $\tilde{\sigma}(\theta,t)=\sigma(u,t)$ is also nonnegative on the interval $[0,\overline{T}_0)$, since

$$\frac{\partial\sigma}{\partial t}=k(u,t)+c(t)(F,\vec{N})(u,t)>0 \tag{4.1}$$

and $\sigma(u,0)=f(u)\ge 0$. The expression (4.1) holds since k has a uniformly positive lower bound, $c(t)$ is non-positive, and $(F,\vec{N})=-S\le 0$ on the time interval $[0,\overline{T}_0)$. Hence, we have

$$\frac{d\mathfrak{L}(t)}{dt}=-\int_0^{2\pi}\tilde{\sigma}\,d\theta<0, \tag{4.2}$$

on the time interval $[0,\overline{T}_0)$.

On the other hand, since $\sigma(u,t)>\sigma(u,0)$ for all $t\in(0,\overline{T}_0)$, which implies

$$\tilde{\sigma}(\theta,t)=\sigma(u,t)>\tilde{\sigma}(\theta,0)=\sigma(u,0),\quad \text{for all } t\in(0,\overline{T}_0),$$

so we have

$$\frac{\partial \tilde{\sigma}}{\partial t}(\theta, t) > 0, \tag{4.3}$$

for all $t \in (0, \overline{T}_0)$. Combining (4.3) with the truth

$$\frac{\partial \sigma}{\partial t}(u, t) = \frac{\partial \tilde{\sigma}}{\partial \theta}(\theta, t) \cdot \frac{\partial \theta}{\partial t} + \frac{\partial \tilde{\sigma}}{\partial t}(\theta, t) = \frac{\partial \tilde{\sigma}}{\partial \theta} \cdot \frac{\partial \sigma}{\partial s} + \frac{\partial \tilde{\sigma}}{\partial t}(\theta, t) = \left(\frac{\partial \tilde{\sigma}}{\partial \theta}\right)^2 (\theta, t) \cdot \frac{\partial \theta}{\partial s} + \frac{\partial \tilde{\sigma}}{\partial t}(\theta, t)$$

yields

$$\frac{\partial \tilde{\sigma}}{\partial t} = k\left[1 - \left(\frac{\partial \tilde{\sigma}}{\partial \theta}\right)^2\right] - c(t)S > 0,$$

which indicates

$$\frac{d^2 \mathfrak{L}(t)}{dt^2} = \int_0^{2\pi} \left[\left(\left(\frac{\partial \tilde{\sigma}}{\partial t}\right)^2 - 1\right)k + c(t)S\right] d\theta < 0 \tag{4.4}$$

on the time interval $(0, \overline{T}_0)$.

Then our claim follows from the facts $\mathfrak{L}(0) > 0$, (4.2) and (4.4).

Step 5. Convergence

This step is the same as the step 4 of the proof of theorem 4.1 in [8]. Our proof is finished. □

5. Short time existence of the flow (1.4)

In this section, we would like to give the short time existence of the solution of the hyperbolic mean curvature flow (1.4) by using the method shown in [5].

Now, consider the hyperbolic flow (1.4), additionally, we assume $\mathscr{M}$ is a compact Riemannian manifold. Endow the n-dimensional smooth compact manifold $\mathscr{M}$ with a local coordinate system $\{x^i\}$, $1 \le i \le n$. Denote by $\{g_{ij}\}$ and $\{h_{ij}\}$ the induced metric and the second fundamental form on $\mathscr{M}$ respectively, then the mean curvature is given by

$$H = g^{ij} h_{ij},$$

where (g^{ij}) is the inverse of the metric matrix (g_{ij}).

As the mean curvature flow (MCF) case, here we want to use a trick of DeTurck [2] to show that the evolution equation

$$\frac{\partial^2}{\partial t^2} X(x, t) = H(x, t)\vec{N}(x, t) + c_1(t)X(x, t), \tag{5.1}$$

in (1.4) is strictly hyperbolic, then we can use the standard existence theory of the hyperbolic equations to get the short-time existence of our flow (1.4). However, first we would like to rewrite (5.1) in terms of the coordinate components.

Denote by ∇ and $\triangle$ the Riemannian connection and the Beltrami-Laplace operator on $\mathscr{M}$ decided by the induced metric $\{g_{ij}\}$, respectively. Let $(\cdot,\cdot)$ be the standard Euclidean metric of R^{n+1}. Recall that in this case the Gauss-Weingarten relations of submanifold can be rewritten as follows

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma_{ij}^k \frac{\partial X}{\partial x^k} + h_{ij}\vec{\boldsymbol{n}}, \quad \frac{\partial \vec{\boldsymbol{n}}}{\partial x^j} = -h_{jl} g^{lm} \frac{\partial X}{\partial x^m}, \tag{5.2}$$

where $\vec{\boldsymbol{n}}$ is the unit inward normal vector field on $\mathscr{M}$, and Γ_{ij}^k is the Christoffel symbol of the Riemannian connection ∇, moreover, $\Gamma_{ij}^k = g^{kl}\left(\frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial X}{\partial x^l}\right)$. Therefore, we have

$$\triangle X = g^{ij}\nabla_i \nabla_j X = g^{ij}\left(\frac{\partial^2 X}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X}{\partial x^k}\right) = g^{ij} h_{ij}\vec{\boldsymbol{n}} = H\vec{\boldsymbol{n}},$$

which implies the evolution equation (5.1) can be equivalently rewritten as

$$\frac{\partial^2 X}{\partial t^2} = g^{ij}\frac{\partial^2 X}{\partial x^i \partial x^j} - g^{ij} g^{kl}\left(\frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial X}{\partial x^l}\right)\frac{\partial X}{\partial x^k} + c_1(t)X. \tag{5.3}$$

However, it is easy to see (5.3) is not strictly hyperbolic, since the Laplacian is taken in the induced metric which changes with $X(\cdot, t)$, and this adds extra terms to the symbol. One could get the detailed explanation in the chapter 2 of [18].

Now, we need to use the trick of DeTurck, modifying the flow (1.4) through a diffeomorphism of $\mathscr{M}$, to construct a strictly hyperbolic equation, leading to the short-time existence. Suppose $\overline{X}(x,t)$ is a solution of equation (5.1) (or equivalently (5.3)) and $\phi_t : \mathscr{M} \to \mathscr{M}$ is a family of diffeomorphisms of $\mathscr{M}$. Let

$$X(x,t) = \phi_t^* \overline{X}(x,t), \tag{5.4}$$

where ϕ_t^* is the pull-back operator of ϕ_t, and denote the diffeomorphism ϕ_t by

$$(y,t) = \phi_t(x,t) = \{y^1(x,t), y^2(x,t), \ldots, y^n(x,t)\}$$

in the local coordinates. In what follows, we need to show the existence of the the diffeomorphism ϕ_t, and the equation satisfied by $X(x,t)$ is strictly hyperbolic, which leads to the short-time existence of $X(x,t)$, together with the existence of ϕ_t and (5.4), we could obtain the short-time existence of $\overline{X}(x,t)$, which is assumed to be the solution of the flow (1.4). That is to say through this process we can get the short-time existence of the flow (1.4).

As in [5], consider the following initial value problem

$$\begin{cases} \dfrac{\partial^2 y^\alpha}{\partial t^2} = \dfrac{\partial y^\alpha}{\partial x^k}(g^{ij}(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k)), \\ y^\alpha(x,0) = x^\alpha, \quad y_t^\alpha(x,0) = 0, \end{cases} \tag{5.5}$$

where $\tilde{\Gamma}_{ij}^k$ is the Christoffel symbol related to the initial metric $\tilde{g}_{ij} = \left(\frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j}\right)(x,0)$. Since

$$\Gamma_{ij}^k = \frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^l}\frac{\partial x^k}{\partial y^\gamma}\bar{\Gamma}_{\alpha\beta}^\gamma + \frac{\partial x^k}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^j \partial x^l}, \tag{5.6}$$

which implies the initial problem (5.5) can be rewritten as

$$\begin{cases} \dfrac{\partial^2 y^\alpha}{\partial t^2} = g^{ij}\left(\dfrac{\partial^2 y^\alpha}{\partial x^j \partial x^l} + \dfrac{\partial y^\beta}{\partial x^j}\dfrac{\partial y^\gamma}{\partial x^l}\bar{\Gamma}_{\alpha\beta}^\gamma - \dfrac{\partial y^\alpha}{\partial x^j}\tilde{\Gamma}_{ij}^k\right), \\ y^\alpha(x,0) = x^\alpha, \quad y_t^\alpha(x,0) = 0, \end{cases}$$

which is an initial value problem for a strictly hyperbolic system. By the standard existence theory of a hyperbolic system, we know there must exist a family of diffeomorphisms ϕ_t satisfying the initial value problem (5.5).

On the other hand, by (5.6), we have

$$\begin{aligned} \triangle_{\bar{g}}\overline{X} &= \bar{g}^{\alpha\beta}\nabla_\alpha\nabla_\beta\overline{X} \\ &= g^{kl}\frac{\partial^2 X}{\partial x^k \partial x^l} + g^{kl}\frac{\partial y^\alpha}{\partial x^k}\frac{\partial y^\beta}{\partial x^l}\frac{\partial X}{\partial x^i}\frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} - g^{kl}\frac{\partial X}{\partial x^i}\left(\Gamma_{kl}^i - \frac{\partial x^i}{\partial y^\gamma}\frac{\partial^2 y^\gamma}{\partial x^k \partial x^l}\right) \\ &= g^{kl}\nabla_k\nabla_l X = \triangle_g X, \end{aligned}$$

and then

$$\begin{aligned} \frac{\partial^2 X}{\partial t^2} &= \frac{\partial^2 \overline{X}}{\partial y^\alpha \partial y^\alpha}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + 2\frac{\partial^2 \overline{X}}{\partial t \partial y^\beta}\frac{\partial y^\beta}{\partial t} + \frac{\partial^2 \overline{X}}{\partial t^2} + \frac{\partial \overline{X}}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial t^2} \\ &= \triangle_g X + c_1(t)\overline{X} + \frac{\partial X}{\partial y^\alpha}g^{ij}(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k) + \frac{\partial^2 \overline{X}}{\partial y^\alpha \partial y^\alpha}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + 2\frac{\partial^2 \overline{X}}{\partial t \partial y^\beta}\frac{\partial y^\beta}{\partial t} \\ &= g^{ij}\frac{\partial^2 X}{\partial x^i \partial x^j} - g^{ij}\tilde{\Gamma}_{ij}^k\frac{\partial X}{\partial x^k} + \frac{\partial^2 \overline{X}}{\partial y^\alpha \partial y^\alpha}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + 2\frac{\partial^2 \overline{X}}{\partial t \partial y^\beta}\frac{\partial y^\beta}{\partial t} + c_1(t)\overline{X}, \end{aligned}$$

which is strictly hyperbolic. Hence, by the standard existence theory of hyperbolic equations (see [6]), we could get the short-time existence of $X(x,t)$, then by what we have pointed out before this directly leads to the short-time existence of the solution, $\overline{X}(x,t)$, of the equation (5.1), which implies our local existence and uniqueness Theorem 1.3 naturally.

6. Examples

In this section, by using Lemma 3.1, we investigate the exact solution of examples given in [5], and find that we could get some similar results, which implies our hyperbolic flow (1.4) is meaningful.

Example 6.1. Suppose $c_1(t)$ in the hyperbolic flow (1.4) is non-positive. Now, consider a family of spheres

$$X(x,t) = r(t)(\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha),$$

where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\beta \in [0, 2\pi]$. By straightforward computation, we have the induced metric and the second fundamental form are

$$g_{11} = r^2, \quad g_{22} = r^2\cos^2\alpha, \quad g_{12} = g_{21} = 0,$$

and

$$h_{11} = r, \quad h_{22} = r\cos^2\alpha, \quad h_{12} = h_{21} = 0,$$

respectively. So, the mean curvature is

$$H = g^{ij}h_{ij} = \frac{2}{r}.$$

Additionally, the unit inward normal vector of each $F(\cdot,t)$ is $\vec{n} = -(\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha)$, hence our hyperbolic flow (1.4) becomes

$$\begin{cases} r_{tt} = -\dfrac{2}{r} + c_1(t)r \\ r(0) = r_0 > 0, \quad r_t(0) = r_1, \end{cases}$$

then by Lemma 3.1, we know for arbitrary $r(0) = r_0 > 0$, if the initial velocity $r_t(0) = r_1 > 0$, the evolving sphere will expand first and then shrink to a single point at a finite time; if the initial velocity $r_t(0) = r_1 \le 0$, the evolving sphere will shrink to a point directly at a finite time. One could also use the physical principle to interpret this phenomenon as in [5], which is very simple.

Example 6.2. Suppose $c_1(t)$ in the hyperbolic flow (1.4) is non-positive. Now, consider a family of round circles

$$X(x,t) = (r(t)\cos\alpha, r(t)\sin\alpha),$$

where $\alpha \in [0, 2\pi]$. It is easy to find that the mean curvature and the unit inward normal vector of each $X(\cdot,t)$ are $\frac{1}{r(t)}$ and $\vec{n} = -(\cos\alpha, \sin\alpha)$, respectively, then our hyperbolic flow (1.4) becomes

$$\begin{cases} r_{tt} = -\dfrac{1}{r} + c_1(t)r \\ r(0) = r_0 > 0, \quad r_t(0) = r_1, \end{cases}$$

then by Lemma 3.1, we know that the circles will shrink to a point at a finite time for arbitrary $r(0) > 0$ and the initial velocity r_1.

Remark 6.3. Comparing with the example 2 in [5], here we would like to point out the hyperbolic flow (1.4) does not have cylinder solution except $c_1(t) \equiv 0$. In fact, suppose the solution, $X(\cdot, t)$, of the flow (1.4) is a family of cylinders which takes form

$$X(x,t) = (r(t)\cos\alpha, r(t)\sin\alpha, \rho), \tag{6.1}$$

where $\alpha \in [0, 2\pi]$ and $\rho \in [0, \rho_0]$, then as before we could obtain $c_1(t)\rho = 0$ directly, which implies our claim here. Why the the hyperbolic flow (1.4) does not have cylinder solution of the form (6.1) if $c_1(t)$ dose not vanish? We think that is because the term $c_1(t)X(\cdot, t)$ not only has component perpendicular to ρ-axis, which lets the cylinder move toward ρ-axis vertically, but also has component parallel with ρ-axis, which leads to the moving of cylinder along the ρ-axis. This fact implies, after the initial time, the hyperbolic flow (1.4) will change the shape of the initial cylinder such that the evolving surface $X(\cdot, t)$ is not cylinder any more.

7. Evolution equations

In this section, we would like to give the evolution equations for some intrinsic quantities of the hypersurface $X(\cdot, t)$ under the hyperbolic mean curvature flow (1.4), which will be important for the future study, like convergence, on this flow. It is not difficult to derive them, since they just have slight changes comparing with the corresponding evolution equations in [5].

First, from [18], we have the following facts for hypersurface.

LEMMA 7.1. *Under the hyperbolic mean curvature flow* (1.4), *the following identities hold*

$$\triangle h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij}, \tag{7.1}$$

$$\triangle |A|^2 = 2 g^{ik} g^{jl} h_{kl} \nabla_i \nabla_j H + 2|\nabla A|^2 + 2H\,\mathrm{tr}(A^3) - 2|A|^4, \tag{7.2}$$

where

$$|A|^4 = g^{ij} g^{kl} h_{ik} h_{jl}, \quad \mathrm{tr}(A^3) = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}.$$

THEOREM 7.2. *Under the hyperbolic mean curvature flow* (1.4), *we have*

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2H h_{ij} + 2c_1(t) g_{ij} + 2\left(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j}\right), \tag{7.3}$$

$$\begin{aligned} \frac{\partial^2 \vec{n}}{\partial t^2} &= -g^{ij} \frac{\partial H}{\partial x^i} \frac{\partial X}{\partial x^j} + g^{ij}\left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^i}\right) \\ &\quad \times \left[2 g^{kl}\left(\frac{\partial X}{\partial x^j}, \frac{\partial^2 X}{\partial t \partial x^l}\right)\frac{\partial X}{\partial x^k} + g^{kl}\left(\frac{\partial X}{\partial x^l}, \frac{\partial^2 X}{\partial t \partial x^j}\right)\frac{\partial X}{\partial x^k} - \frac{\partial^2 X}{\partial t \partial x^j}\right], \end{aligned} \tag{7.4}$$

and

$$\frac{\partial^2 h_{ij}}{\partial t^2} = \triangle h_{ij} - 2Hh_{il}h_{mj}g^{lm} + |A|^2 h_{ij} + g^{kl}h_{ij}\left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k}\right)\left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^l}\right) - 2\frac{\partial \Gamma_{ij}^k}{\partial t}\left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k}\right) + c_1(t)h_{ij}. \tag{7.5}$$

Proof. By the definition of the induced metric and (5.2), we have

$$\begin{aligned}
\frac{\partial^2 g_{ij}}{\partial t^2} &= \left(\frac{\partial^3 X}{\partial t^2 \partial x^i}, \frac{\partial X}{\partial x^j}\right) + 2\left(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j}\right) + \left(\frac{\partial X}{\partial x^i}, \frac{\partial^3 X}{\partial t^2 \partial x^j}\right) \\
&= \left(\frac{\partial}{\partial x^i}(H\vec{n} + c_1(t)X), \frac{\partial X}{\partial x^j}\right) + 2\left(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j}\right) + \left(\frac{\partial X}{\partial x^i}, \frac{\partial}{\partial x^j}(H\vec{n} + c_1(t)X)\right) \\
&= H\left(-h_{ik}g^{kl}\frac{\partial X}{\partial x^l}, \frac{\partial X}{\partial x^j}\right) + 2c_1(t)\left(\frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j}\right) + 2\left(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j}\right) \\
&\quad + H\left(\frac{\partial X}{\partial x^i}, -h_{jk}g^{kl}\frac{\partial X}{\partial x^l}\right) \\
&= -2Hh_{ij} + 2c_1(t)g_{ij} + 2\left(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j}\right),
\end{aligned}$$

which finishes the proof of (7.3).

It is surprising that the evolution equation for the unit inward normal vector $\vec{n}$ under the flow (1.4) here has no difference with the one in [5], since in the process of deriving the evolution equation for $\vec{n}$, the only possible difference appears in the term

$$-\left(\vec{n}, \frac{\partial^3 X}{\partial t^2 \partial x^i}\right) g^{ij}\frac{\partial X}{\partial x^j} = -\left(\vec{n}, \frac{\partial}{\partial x^i}(H\vec{n} + c_1(t)X)\right) g^{ij}\frac{\partial X}{\partial x^j} = -g^{ij}\frac{\partial H}{\partial x^i}\frac{\partial X}{\partial x^j}.$$

However, this is the same with the case in [5], since the term

$$-\left(\vec{n}, \frac{\partial}{\partial x^i}(c_1(t)X)\right) g^{ij}\frac{\partial X}{\partial x^j}$$

vanishes. So, (7.4) follows according to the corresponding evolution equation in [5].

Actually, (7.5) is easy to be obtained by comparing with the proof of evolution equation (5.5) in [5], since, between our case and the case in [5], one could find that the processes of deriving the evolution equations only have slight difference. However, the deriving process in [5] is a little complicated, so we

would like to give the detailed steps here so that readers can note the difference clearly. By (5.2), we have

$$\frac{\partial h_{ij}}{\partial t} = \frac{\partial}{\partial t}\left(\vec{n}, \frac{\partial^2 X}{\partial x^i \partial x^j}\right) = \left(\frac{\partial \vec{n}}{\partial t}, \frac{\partial^2 X}{\partial x^i \partial x^j}\right) + \left(\vec{n}, \frac{\partial^3 X}{\partial t \partial x^i \partial x^j}\right),$$

furthermore,

$$\begin{aligned}
\frac{\partial^2 h_{ij}}{\partial t^2} &= \left(\frac{\partial^2 \vec{n}}{\partial t^2}, \frac{\partial^2 X}{\partial x^i \partial x^j}\right) + 2\left(\frac{\partial \vec{n}}{\partial t}, \frac{\partial^3 X}{\partial t \partial x^i \partial x^j}\right) + \left(\vec{n}, \frac{\partial^4 X}{\partial t^2 \partial x^i \partial x^j}\right) \\
&= -g^{kl}\left(\frac{\partial H}{\partial x^k}\frac{\partial X}{\partial x^l}, \frac{\partial^2 X}{\partial x^i \partial x^j}\right) - g^{kl}\left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k}\right)\left(\frac{\partial^2 X}{\partial t \partial x^l}, \frac{\partial^2 X}{\partial x^i \partial x^j}\right) \\
&\quad + g^{pq} g^{kl}\left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^p}\right)\left[\left(\frac{\partial X}{\partial x^l}, \frac{\partial^2 X}{\partial t \partial x^q}\right) + 2\left(\frac{\partial X}{\partial x^q}, \frac{\partial^2 X}{\partial t \partial x^l}\right)\right]\left(\frac{\partial X}{\partial x^k}, \frac{\partial^2 X}{\partial x^i \partial x^j}\right) \\
&\quad - 2g^{kl}\left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k}\right)\left(\frac{\partial X}{\partial x^l}, \frac{\partial^3 X}{\partial t \partial x^i \partial x^j}\right) + \left(\vec{n}, \frac{\partial}{\partial x^i \partial x^j}(H\vec{n} + c_1(t)X)\right),
\end{aligned}$$

then one could easily find that the difference between our case and the case in [5] appears from the last term

$$\left(\vec{n}, \frac{\partial}{\partial x^i \partial x^j}(H\vec{n} + c_1(t)X)\right),$$

which satisfies

$$\begin{aligned}
\left(\vec{n}, \frac{\partial}{\partial x^i \partial x^j}(H\vec{n} + c_1(t)X)\right) &= \left(\vec{n}, \frac{\partial}{\partial x^i}\left(\frac{\partial H}{\partial x^j}\vec{n} - Hh_{jk}g^{kl}\frac{\partial X}{\partial x^l} + c_1(t)\frac{\partial X}{\partial x^j}\right)\right) \\
&= \left(\vec{n}, \frac{\partial}{\partial x^i}\left(\frac{\partial H}{\partial x^j}\vec{n} - Hh_{jk}g^{kl}\frac{\partial X}{\partial x^l}\right)\right) + c_1(t)\left(\vec{n}, \frac{\partial^2 X}{\partial x^i \partial x^j}\right) \\
&= \left(\vec{n}, \frac{\partial}{\partial x^i}\left(\frac{\partial H}{\partial x^j}\vec{n} - Hh_{jk}g^{kl}\frac{\partial X}{\partial x^l}\right)\right) + c_1(t)h_{ij}.
\end{aligned}$$

Obviously, it will only produce an extra term $c_1(t)h_{ij}$ comparing with the evolution equation for the second fundamental form, (5.5), in [5]. So, the evolution equation (7.5) follows. □

At the end, by Lemma 7.1 and Theorem 7.2, we could derive the following evolution equations for the mean curvature and the square norm of the second fundamental form of the hypersurface $X(\cdot, t)$, which maybe play an important role in the future study, like convergence, of the hyperbolic flow (1.4) as the mean curvature flow case.

THEOREM 7.3. *Under the hyperbolic mean curvature flow* (1.4), *we have*

$$\begin{aligned}\frac{\partial^2 H}{\partial t^2} &= \triangle H + H|A|^2 - 2g^{ik}g^{jl}\left(\frac{\partial^2 X}{\partial t\partial x^k}, \frac{\partial^2 X}{\partial t\partial x^l}\right) \\ &\quad + Hg^{kl}\left(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^k}\right)\left(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^l}\right) - 2g^{ij}\frac{\partial \Gamma_{ij}^k}{\partial t}\left(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^k}\right) \\ &\quad + 2g^{ik}g^{jp}g^{lq}h_{ij}\frac{\partial_{g_{pq}}}{\partial t}\frac{\partial_{g_{kl}}}{\partial t} - 2g^{ik}g^{jl}\frac{\partial_{g_{kl}}}{\partial t}\frac{\partial_{h_{ij}}}{\partial t} - c_1(t)H,\end{aligned} \tag{7.6}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t^2}|A|^2 &= \triangle(|A|^2) - 2|\nabla A|^2 + 2|A|^4 + 2|A|^2 g^{pq}\left(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^p}\right)\left(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^q}\right) \\ &\quad + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{g_{mn}}{\partial t}\frac{h_{ik}}{\partial t} \\ &\quad - 4g^{im}g^{jn}g^{kl}h_{ik}h_{jl}\left(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^m}\right)\left(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^n}\right) \\ &\quad + 2g^{im}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl} \times (2g^{jp}g^{nq}g^{kl} + g^{jn}g^{kp}g^{lq}) \\ &\quad - 4g^{ij}g^{kl}h_{jl}\frac{\partial \Gamma_{ik}^p}{\partial t}\left(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^p}\right) - 2c_1(t)|A|^2.\end{aligned} \tag{7.7}$$

Proof. Here we do not give the detailed proof, since in [5] they have given the detailed and straightforward computation on how to derive the evolution equations. Moreover, in our case we find that if we want to get our theorem here, we only need to use the evolution equations (7.3) and (7.5) for the induced metric and the second fundamental form to replace the old ones in [5] in the computation. □

Remark 7.4. Here we want to point out an interesting truth. In [11, 12], we have proved

LEMMA ([11, 12]). *If the hypersuface* $X(\cdot, t)$ *of* R^{n+1} *satisfies the curvature flow of the form* (1.2), *then*

(1) $\frac{\partial}{\partial t}g_{ij} = -2Hh_{ij} + 2\tilde{c}(t)g_{ij}$,

(2) $\frac{\partial}{\partial t}\vec{v} = \nabla^i H \cdot \frac{\partial X}{\partial x^i}$,

(3) $\frac{\partial}{\partial t}h_{ij} = \triangle h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} + \tilde{c}(t)h_{ij}$,

(4) $\dfrac{\partial}{\partial t}H = \triangle H + |A|^2 H - \tilde{c}(t)H,$

(5) $\dfrac{\partial}{\partial t}|A|^2 = \triangle|A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2\tilde{c}(t)|A|^2,$

where $\vec{v}$ *denotes the unit outward normal vector of* $X(\cdot, t)$.

Comparing with those corresponding evolution equations derived by Huisken in [7], the extra terms are $2\tilde{c}(t)g_{ij}$, 0, $\tilde{c}(t)h_{ij}$, $-\tilde{c}(t)H$, and $-2\tilde{c}(t)|A|^2$, if we add a forcing term, $\tilde{c}(t)X$, to the evolution equation of the mean curvature flow in direction of the position vector. However, the surprising truth is that if we add this forcing term to the hyperbolic flow in [5], we find that no matter how complicated the evolution equations of the intrinsic quantities of the hypersurface $X(\cdot, t)$ under the hyperbolic flow (1.4) are, the evolution equations (7.3)–(7.7) also have the extra terms of the same forms as (1)–(5) by comparing with the corresponding evolution equations in [5].

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