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THE FEKETE-SZEGÖ PROBLEM FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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Abstract

By using Dziok-Srivastava operator a new subclass of analytic functions generalized k-parabolic starlike functions, denoted by $k - SP_{l,m}(\alpha_1; \gamma)$, is introduced. For this class the Fekete-Szegö problem is completely solved. Various known or new special cases of our results are also point out.

1. Introduction

Let $\mathscr{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic in the open unit disk $\mathbf{U} = \{z : z \in \mathbf{C} \text{ and } |z| < 1\}$.

Let $\mathscr{S}$ denote the class of functions $f \in \mathscr{A}$ which are univalent in $\mathbf{U}$. If f and g are analytic in $\mathbf{U}$, we say that f is subordinate to g, written symbolically as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbf{U})$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbf{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbf{U}$ such that $f(z) = g(w(z))$, $z \in \mathbf{U}$. In particular, if the function $g(z)$ is univalent in $\mathbf{U}$, then we have that $f(z) \prec g(z)$ $(z \in \mathbf{U})$ if and only if $f(0) = g(0)$ and $f(\mathbf{U}) \subseteq g(\mathbf{U})$.

A function $f \in \mathscr{A}$ is said to be in the class of k-uniformly convex functions of order γ, denoted by $k - UCV(\gamma)$ [3] if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \gamma, \tag{1.2}$$

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where $k \geq 0$, $\gamma \in [0,1)$ and it is said to be in the corresponding class k-parabolic starlike functions of order γ, denoted by $k - SP(\gamma)$ if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \tag{1.3}$$

where $k \geq 0$, $\gamma \in [0,1)$.

These classes generalize various other classes which are worthy to mention here. The class $k - UCV(0) = k - UCV$ is the class of k-uniformly convex functions [14] (also see [15] and [16]).

Using the Alexander type relation, we can obtain the class $k - SP(\gamma)$ in the following way:

$f \in k - UCV(\gamma) \Leftrightarrow zf' \in k - SP(\gamma)$. The classes $1 - UCV(0) = UCV$ and $1 - SP(0) = SP$, defined by Goodman [10] and Ronning [31], respectively.

Geometric Interpretation. It is known that $f \in k - UCV(\gamma)$ or $f \in k - SP(\gamma)$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ or $\frac{zf'(z)}{f(z)}$, respectively, takes all the values in the conic domain $\mathcal{R}_{k,\gamma}$ which is included in the right half plane given by

$$\mathcal{R}_{k,\gamma} := \{w = u + iv \in \mathbf{C} : u > k\sqrt{(u-1)^2 + v^2} + \gamma,\ k \geq 0 \text{ and } \gamma \in [0,1)\}. \tag{1.4}$$

Denote by $\mathcal{P}(P_{k,\gamma})$, $(k \geq 0, 0 \leq \gamma < 1)$ the family of functions p, such that $p \in \mathcal{P}$, where $\mathcal{P}$ denotes the well-known class of Caratheodory functions and $p \prec P_{k,\gamma}$ in $\mathbf{U}$. The function $P_{k,\gamma}$ maps the unit disk conformally onto the domain $\mathcal{R}_{k,\gamma}$ such that $1 \in \mathcal{R}_{k,\gamma}$ and $\partial\mathcal{R}_{k,\gamma}$ is a curve defined by the equality

$$\partial\mathcal{R}_{k,\gamma} := \{w = u + iv \in \mathbf{C} : u^2 = (k\sqrt{(u-1)^2 + v^2} + \gamma)^2,\quad k \geq 0 \text{ and } \gamma \in [0,1)\}. \tag{1.5}$$

From elementary computations we see that (1.5) represents conic sections symmetric about the real axis. Thus $\mathcal{R}_{k,\gamma}$ is an elliptic domain for $k > 1$, a parabolic domain for $k = 1$, a hyperbolic domain for $0 < k < 1$ and the right half plane $u > \gamma$, for $k = 0$.

The functions $P_{k,\gamma}$, which play the role of extremal functions of the class $\mathcal{P}(P_{k,\gamma})$, were obtained in [1], and for some unique $t \in (0,1)$, every positive number k can be expressed as

$$k = \cosh\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)} \tag{1.6}$$

where $\mathcal{K}$ is Legendre's complete elliptic integral of the first kind and $\mathcal{K}'$ is complementary integral of $\mathcal{K}$ (*for details see* [1], [22] and [26]).

For functions $f, g \in \mathcal{A}$, given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or *convolution*) of $f(z)$ and

$g(z)$ by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z) \quad (z \in \mathbf{U}).$$

For $\alpha_i, \beta_j \in \mathbf{C} - \{0, -1, -2, \ldots\}$ $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m)$, the generalized hypergeometric function ${}_lF_m(z)$ is defined by

$${}_lF_m(z) = {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}$$
$$(l \leq m+1, l, m \in \mathbf{N}_0 := \{0, 1, 2, \ldots\},\ z \in \mathbf{U})$$

where $(\lambda)_n$ is the Pochhammer symbol defined, for $\lambda \in \mathbf{C}$ and in terms of the Euler Γ-function, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1; & n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1); & n \in \mathbf{N} := \{1, 2, \ldots\}. \end{cases}$$

Let $H_{l,m} = (\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathscr{A} \to \mathscr{A}$ be the linear operator defined by

$$H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) = z\,{}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \tag{1.7}$$
$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}.$$

For simplicity, in the sequel, we will write $H_m^l(\alpha_1) f(z)$ instead of $H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z)$.

The linear operator $H_m^l(\alpha_1)$ is called the Dziok-Srivastava operator [6] (see also [7]) and it contains, amongst its special cases, various other operators introduced and studied by Hohlov [11], Carlson-Shaffer [4], Ruscheweyh [32], Noor [24] (also see [25]), Bernardi-Libera-Livingston ([2], [12], [13]) and Srivastava-Owa ([27], [28], [36]).

Now, by making use of $H_m^l(\alpha_1)$, we define new subclasses of functions in $\mathscr{A}$.

Definition 1.1. Let $\alpha_i, \beta_j \in \mathbf{C} - \{0, -1, -2, \ldots\}$ $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m)$, $k \geq 0$ and $0 \leq \gamma < 1$. We denote by

$$k - SP_{l,m}(\alpha_1; \gamma) := k - SP_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; \gamma)$$

the class of functions $f \in \mathscr{A}$ which satisfy the following condition

$$\Re\left\{\frac{z(H_m^l(\alpha_1) f(z))'}{H_m^l(\alpha_1) f(z)}\right\} > k\left|\frac{z(H_m^l(\alpha_1) f(z))'}{H_m^l(\alpha_1) f(z)} - 1\right| + \gamma \quad (z \in \mathbf{U}). \tag{1.8}$$

Note that $f \in k - SP_{l,m}(\alpha_1;\gamma)$ if and only if $H_m^l(\alpha_1)f \in k - SP(\gamma)$. Using the Alexander type relation, we define the class $k - UCV_{l,m}(\alpha_1;\gamma)$ as follows

$$f \in k - UCV_{l,m}(\alpha_1;\gamma) \text{ if and only if } zf' \in k - SP_{l,m}(\alpha_1;\gamma),$$

and also

$$k - UCV_{l,m}(\alpha_1;\gamma) \subseteq k - SP_{l,m}(\alpha_1;\gamma).$$

Geometric Interpretation. From (1.8) $f \in k - SP_{l,m}(\alpha_1;\gamma)$ if and only if $q(z) = \dfrac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)}$ take all the values in the conic domain $\mathcal{R}_{k,\gamma}$ given in (1.4) which is included in the right half plane.

Someone can find more information about uniformly convex functions and parabolic starlike in Gangadharan et al. [9], Kanas and Srivastava [17], Deniz et al. [5] and Orhan et al. [26].

The classical Fekete-Szegö inequality, presented by means of Loewner's method, for the coefficients of $f \in \mathcal{S}$ is that

$$|a_3 - \mu a_2^2| \leq 1 + 2\exp(-2\mu/(1-\mu)) \quad \text{for } 0 \leq \mu < 1.$$

As $\mu \to 1^-$, we have the elementary inequality $|a_3 - a_2^2| \leq 1$. Moreover, the coefficient functional

$$\mathbf{F}_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions f in the unit disk $\mathbf{U}$ plays an important role in function theory. For example, the quantity $a_3 - a_2^2$ represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of locally univalent functions f in $\mathbf{U}$. In the literature, there exists a large number of results about inequalities for $\mathbf{F}_\mu(f)$ corresponding to various subclasses of $\mathcal{S}$. The problem of maximizing the absolute value of the functional $\mathbf{F}_\mu(f)$ is called the Fekete-Szegö problem (see [8]). In [18], Koepf solved the Fekete-Szegö problem for close-to-convex functions and the largest real number μ for which $\mathbf{F}_\mu(f)$ is maximized by the Koebe function $z/(1-z)^2$ is $\mu = 1/3$, and later in [19] (see also [20]), this result was generalized for functions that are close-to-convex of order γ. In [29], Pfluger employed the variational method to give another treatment of the Fekete-Szegö inequality which includes a description of the image domains under extremal functions. Later, Pfluger [30] used Jenkin's method to show that

$$|\mathbf{F}_\mu(f)| \leq 1 + 2|\exp(-2\mu/(1-\mu))| \quad f \in \mathcal{S},$$

holds for complex μ such that $\Re(1/(1-\mu)) \geq 1$. The inequality is sharp if and only if μ is in a certain pear shaped subregion of the disk given by

$$\mu = 1 - (u + itv)/(u^2 + v^2), \quad -1 \leq t \leq 1,$$

where $u = 1 - \log(\cos\phi)$ and $v = \tan\phi - \phi$, $0 < \phi < \pi/2$. For different subclasses of $\mathscr{S}$, recently Fekete-Szegö problem has been investigated by many authors including (see [21], [22], [23], [26], [33], [34], [35]).

In this paper, we solve the Fekete-Szegö problem for functions in the class $k - SP_{l,m}(\alpha_1;\gamma)$ defined by using $H_m^l(\alpha_1)$. Consequences of the main results and their relevance to known results are also pointed out.

2. Preliminaries

In order to prove our results, we will need the following lemmas.

In [1] (also see [26]), we calculate the coefficients P_1 and P_2 from Taylor series expansion of the function $P_{k,\gamma}$ and give in Lemma 2.1 as follows.

LEMMA 2.1 (see [1] also [26]). *Let $0 \le k < \infty$ and $0 \le \gamma < 1$ be fixed and $P_{k,\gamma}$ be the Riemann map of $\mathbf{U}$ onto $\mathscr{R}_{k,\gamma}$, satisfying $P_{k,\gamma}(0) = 1$ and $P'_{k,\gamma}(0) > 0$. If*

$$P_{k,\gamma}(z) = 1 + P_1 z + P_2 z^2 + \cdots \quad (z \in \mathbf{U}) \tag{2.1}$$

then

$$P_1 = \begin{cases} \dfrac{2(1-\gamma)\mathscr{B}^2}{1-k^2}; & 0 \le k < 1, \\ \dfrac{8(1-\gamma)}{\pi^2}; & k = 1, \\ \dfrac{\pi^2(1-\gamma)}{4(k^2-1)\sqrt{t}(1+t)\mathscr{K}^2(t)}; & k > 1, \end{cases}$$

and

$$P_2 = \begin{cases} \dfrac{(\mathscr{B}^2+2)}{3}P_1; & 0 \le k < 1, \\ \dfrac{2}{3}P_1; & k = 1, \\ \dfrac{[4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2]}{24\sqrt{t}(1+t)\mathscr{K}^2(t)}P_1; & k > 1, \end{cases}$$

where

$$\mathscr{B} = \frac{2}{\pi}\arccos k \tag{2.2}$$

and $\mathscr{K}(t)$ is the complete elliptic integral of first kind.

LEMMA 2.2 (see [18]). *Let $h \in \mathscr{P}$ given by*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbf{U}). \tag{2.3}$$

Then

$$|c_n| \le 2 \quad (n \in \mathbf{N}), \quad |c_2 - c_1^2| \le 2 \quad \textit{and} \quad \left|c_2 - \frac{1}{2}c_1^2\right| \le 2 - \frac{1}{2}|c_1|^2. \tag{2.4}$$

3. Main results

In this section, we will give some upper bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$.

In order to prove our main results we have to recall the following.

Firstly, the following calculations will be used in the proofs of each of the Theorems 3.1–3.7. By geometric interpretation there exists a function w satisfying the conditions of the Schwarz lemma such that

$$\frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} = P_{k,\gamma}(w(z)) \quad (z \in \mathbf{U}), \tag{3.1}$$

where $P_{k,\gamma}$ is the function defined in Lemma 2.1.

Define the function h in $\mathscr{P}$ given by

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbf{U}).$$

It follows

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots$$

and

$$\begin{aligned} P_{k,\gamma}(w(z)) &= 1 + P_1\left\{\frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right\} \\ &\quad + P_2\left\{\frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right\}^2 + \cdots \\ &= 1 + \frac{P_1 c_1}{2}z + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)P_1 + \frac{1}{4}c_1^2 P_2\right\}z^2 + \cdots \end{aligned} \tag{3.2}$$

Thus, by using (3.1) and (3.2), we obtain

$$a_2 = \frac{\prod_{j=1}^{m} \beta_j}{\prod_{j=1}^{l} \alpha_j} \frac{P_1}{2} c_1 \tag{3.3}$$

and

(3.4)
$$a_3 = \frac{\prod_{j=1}^{m} \beta_j(\beta_j+1)}{\prod_{j=1}^{l} \alpha_j(\alpha_j+1)} \left[\frac{P_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{P_2 c_1^2}{4} + \frac{P_1^2 c_1^2}{4} \right].$$

Secondary, we introduce the following functions which will be used in the discussion of sharpness of our results.

Define the function $\mathscr{G}$ in $\mathbf{U}$ by

(3.5)
$$\mathscr{G}(z) = \frac{1}{z}\left[(z_l F_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)) * \left(z \exp\left(\int_0^z \frac{P_{k,\gamma}(\xi) - 1}{\xi}\, d\xi \right)\right)\right]$$

where $P_{k,\gamma}$ is the function defined in Lemma 2.1 and write the following extremal function in $k - SP_{l,m}(\alpha_1;\gamma)$ by

(3.6)
$$\psi(z,\theta,\tau) = (z_l F_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)) * z \exp\left(\int_0^z \left[P_{k,\gamma}\left(\frac{e^{i\theta}\xi(\xi+\tau)}{1+\tau\xi} \right) - 1 \right] \frac{d\xi}{\xi} \right).$$
$$(0 \le \theta \le 2\pi; 0 \le \tau \le 1)$$

Note that $\psi(z,0,1) = z\mathscr{G}(z)$ defined by (3.5) and $\psi(z,\theta,0)$ is an odd function.

THEOREM 3.1. *Let the function f given by* (1.1) *be in the class* $k - SP_{l,m}(\alpha_1;\gamma)$ $(0 \le \gamma < 1; 0 \le k < 1)$. *Then*

(3.7)
$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{4(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \left(\dfrac{(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu - \dfrac{1}{3} - \dfrac{(7-6\gamma-k^2)\mathscr{B}^2}{6(1-k^2)} \right); \\ \quad \mu \ge \sigma_1, \\ \dfrac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}; \quad \sigma_2 \le \mu \le \sigma_1, \\ \dfrac{4(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \left(\dfrac{(7-6\gamma-k^2)\mathscr{B}^2}{6(1-k^2)} + \dfrac{1}{3} - \dfrac{(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu \right); \\ \quad \mu \le \sigma_2, \end{cases}$$

where $\mathscr{B}$ *is given by* (2.2), *and*

$$\sigma_1 = \frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left(\frac{5(1-k^2)}{\mathscr{B}^2}+(7-6\gamma-k^2)\right), \tag{3.8}$$

$$\sigma_2 = \frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left((7-6\gamma-k^2)-\frac{1-k^2}{\mathscr{B}^2}\right). \tag{3.9}$$

Each of the estimates in (3.7) *is sharp for the function* $\psi(z,\theta,\tau)$ *given by* (3.6).

Proof. Putting the values of P_1 and P_2 for $0 \le k < 1$ from Lemma 2.1 in (3.3) and (3.4) we find that

$$a_2 = \frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{l}\alpha_j}c_1$$

and

$$a_3 = \frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}\left\{c_2-\frac{1}{6}\left(1-\frac{(7-6\gamma-k^2)\mathscr{B}^2}{1-k^2}\right)c_1^2\right\}.$$

An easy computation shows that

$$a_3-\mu a_2^2 = -\frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \times\left[\left(\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu +\frac{1}{3}\left(1-\frac{(7-6\gamma-k^2)\mathscr{B}^2}{(1-k^2)}\right)\right)c_1^2-2c_2\right]. \tag{3.10}$$

Thus, from (3.10) we obtain

$$|a_3-\mu a_2^2| \le \frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \times\left[\left|\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu -\frac{5}{3}-\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}\right||c_1^2|+2|c_1^2-c_2|\right]. \tag{3.11}$$

If $\mu \ge \sigma_1$, then by applying Lemma 2.2, we get

$$\begin{aligned}|a_3-\mu a_2^2| &\le \frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \\ &\quad\times\left[\left(\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu-\frac{5}{3}-\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}\right)4+4\right] \\ &= \frac{4(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \\ &\quad\times\left(\frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu-\frac{1}{3}-\frac{(7-6\gamma-k^2)\mathscr{B}^2}{6(1-k^2)}\right)\end{aligned} \tag{3.12}$$

which is the first part of assertion (3.7).

Next, if $\mu \leq \sigma_2$ then we rewrite (3.10) as

$$
\begin{aligned}
(3.13)\quad |a_3 - \mu a_2^2| &= \frac{(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l} \alpha_j(\alpha_j+1)} \\
&\times \left| \left(\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)} - \frac{1}{3} - \frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu \right) c_1^2 + 2c_2 \right| \\
&\leq \frac{(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l} \alpha_j(\alpha_j+1)} \\
&\times \left[\left(\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)} - \frac{1}{3} \right.\right. \\
&\left.\left. - \frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu \right) |c_1^2| + 2|c_2| \right].
\end{aligned}
$$

Applying Lemma 2.2 we have

$$
\begin{aligned}
&|a_3 - \mu a_2^2| \\
&\quad\leq \frac{(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l} \alpha_j(\alpha_j+1)} \\
&\quad\times \left[\left(\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)} - \frac{1}{3} - \frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu \right) 4 + 4 \right]
\end{aligned}
$$

$$= \frac{4(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{(1-k^2) \prod_{j=1}^{l} \alpha_j(\alpha_j+1)}$$

$$\times \left(\frac{(7-6\gamma-k^2)\mathscr{B}^2}{6(1-k^2)} + \frac{1}{3} - \frac{(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m} \beta_j}{(1-k^2) \prod_{j=1}^{m}(\beta_j+1) \prod_{j=1}^{l} \alpha_j} \mu \right)$$

which is the third part of assertion (3.7).

Finally from (3.10) we get

$$(3.14) \qquad |a_3 - \mu a_2^2|$$

$$= \frac{(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{2(1-k^2) \prod_{j=1}^{l} \alpha_j(\alpha_j+1)}$$

$$\times \left| \left(\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)} + \frac{2}{3} - \frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m} \beta_j}{(1-k^2) \prod_{j=1}^{m}(\beta_j+1) \prod_{j=1}^{m} \alpha_j} \mu \right) c_1^2 \right.$$

$$\left. + 2\left(c_2 - \frac{c_1^2}{2}\right) \right|$$

$$\leq \frac{(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{2(1-k^2) \prod_{j=1}^{l} \alpha_j(\alpha_j+1)}$$

$$\times \left[\left| \frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)} + \frac{2}{3} - \frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m} \beta_j}{(1-k^2) \prod_{j=1}^{m}(\beta_j+1) \prod_{j=1}^{m} \alpha_j} \mu \right| |c_1^2| \right.$$

$$\left. + 2\left|c_2 - \frac{c_1^2}{2}\right| \right].$$

We observe that $\sigma_2 \le \mu \le \sigma_1$ implies

$$\left| \frac{(7-6\gamma-\beta^2)\mathscr{B}^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{m}\alpha_j}\mu \right| \le 1.$$

Thus applying Lemma 2.2 to (3.14) we get

$$|a_3 - \mu a_2^2| \le \frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \tag{3.15}$$

which is the second part of assertion (3.7).

We now obtain sharpness of the estimates in (3.7).

If $\mu > \sigma_1$, equality holds in (3.7) if and only if equality holds in (3.12). This happens if and only if $|c_1| = 2$ and $|c_1^2 - c_2| = 2$. Thus $w(z) = z$. It follows that the extremal function is of the form $\psi(z, 0, 1)$ defined by (3.6) or one of its rotations.

If $\mu < \sigma_2$ then equality holds in (3.7) if and only if $|c_1| = 0$ and $|c_2| = 2$. Thus $w(z) = e^{i\theta}z^2$ and the extremal function is $\psi(z, 0, 1)$ or one of its rotations.

If $\mu = \sigma_2$, the equality holds if and only if $|c_2| = 2$. In this case, we have

$$h(z) = \frac{1+\tau}{2}\left(\frac{1+z}{1-z}\right) - \frac{1-\tau}{2}\left(\frac{1-z}{1+z}\right) \quad (0 < \tau < 1; z \in \mathbf{U}).$$

Therefore the extremal function f is $\psi(z, 0, \tau)$ or one of its rotations.

Similarly, $\mu = \sigma_1$ is equivalent to

$$\frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu - \frac{5}{3} - \frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)} = 0.$$

Thus the extremal function is $\psi(z, \pi, \tau)$ or one of its rotations.

Finally if $\sigma_2 \le \mu \le \sigma_1$, then equality holds if $|c_1| = 0$ and $|c_2| = 2$. Equivalently, we have

$$h(z) = \frac{1+\tau z^2}{1-\tau z^2} \quad (0 \le \tau \le 1; z \in \mathbf{U}).$$

Therefore the extremal function f is $\psi(z,0,0)$ or one of its rotations.
The proof of Theorem 3.1 is now completed. □

THEOREM 3.2. *Let the function f given by* (1.1) *be in the class* $k-SP_{l,m}(\alpha_1;\gamma)$ $(0\le\gamma<1;k=1)$. *Then*

(3.16)
$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{16(1-\gamma)\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\pi^2\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}\left(\dfrac{4(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{\pi^2\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu-\dfrac{1}{3}-\dfrac{4(1-\gamma)}{\pi^2}\right); & \mu\ge\delta_1,\\[2ex] \dfrac{8(1-\gamma)\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\pi^2\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}; & \delta_2\le\mu\le\delta_1,\\[2ex] \dfrac{16(1-\gamma)\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\pi^2\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}\left(\dfrac{4(1-\gamma)}{\pi^2}+\dfrac{1}{3}-\dfrac{4(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{\pi^2\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu\right); & \mu\le\delta_2, \end{cases}$$

where

$$\delta_1=\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left(1+\frac{5\pi^2}{24(1-\gamma)}\right), \tag{3.17}$$

$$\delta_2=\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left(1-\frac{\pi^2}{24(1-\gamma)}\right). \tag{3.18}$$

Each of the estimates in (3.16) *is sharp for the function $\psi(z,\theta,\tau)$ given by* (3.6).

Proof. Putting the values of P_1 and P_2 for $k=1$ from Lemma 2.1 in (3.3) and (3.4) we find that

$$a_2 = \frac{4(1-\gamma)\prod_{j=1}^{m}\beta_j}{\pi^2\prod_{j=1}^{l}\alpha_j}c_1$$

and

$$a_3 = \frac{4(1-\gamma)\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\pi^2\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}\left\{c_2 - \frac{1}{6}\left(1-\frac{24(1-\gamma)}{\pi^2}\right)c_1^2\right\}.$$

An easy computation shows that

$$\begin{aligned}(3.19)\quad &|a_3-\mu a_2^2| \\ &= \frac{2(1-\gamma)\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\pi^2\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \\ &\times\left|\left(\frac{8(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{\pi^2\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu+\frac{1}{3}-\frac{8(1-\gamma)}{\pi^2}\right)c_1^2-2c_2\right| \\ &\le \frac{2(1-\gamma)\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\pi^2\prod_{j=1}^{l}\alpha_j(\alpha_j+1)} \\ &\times\left[\left|\frac{8(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{\pi^2\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu-\frac{5}{3}-\frac{8(1-\gamma)}{\pi^2}\right||c_1^2|+2|c_1^2-c_2|\right].\end{aligned}$$

To complete the proof of Theorem 3.2, we follow the same steps as in the proof of Theorem 3.1. Therefore, we choose to omit the details involved. □

THEOREM 3.3. *Let the function f given by (1.1) be in the class $k-SP_{l,m}(\alpha_1;\gamma)$ $(0\le\gamma<1; 1<k<\infty)$ and let t be the unique positive number in the open interval $(0,1)$ defined by (1.6). Then*

(3.20)

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{P_1 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{\prod_{j=1}^{l} \alpha_j(\alpha_j+1)} \left(\dfrac{P_1 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m} \beta_j}{\prod_{j=1}^{m}(\beta_j+1) \prod_{j=1}^{l} \alpha_j} \mu - P_1 - \dfrac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathscr{K}^2(t)} \right); \\ \qquad \mu \ge \rho_1, \\ \dfrac{P_1 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{\prod_{j=1}^{l} \alpha_j(\alpha_j+1)}; \quad \rho_2 \le \mu \le \rho_1, \\ \dfrac{P_1 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{\prod_{j=1}^{l} \alpha_j(\alpha_j+1)} \left(\dfrac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathscr{K}^2(t)} + P_1 - \dfrac{P_1 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m} \beta_j}{\prod_{j=1}^{m}(\beta_j+1) \prod_{j=1}^{l} \alpha_j} \mu \right); \\ \qquad \mu \le \rho_2, \end{cases}$$

where $\mathscr{K}(t)$ *is the complete elliptic integral of the first kind,* P_1 *is given by* (2.1), *and*

$$\rho_1 = \frac{\prod_{j=1}^{m}(\beta_j+1) \prod_{j=1}^{l} \alpha_j}{P_1 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m} \beta_j} \left(1 + P_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathscr{K}^2(t)} \right), \tag{3.21}$$

$$\rho_2 = \frac{\prod_{j=1}^{m}(\beta_j+1) \prod_{j=1}^{l} \alpha_j}{P_1 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m} \beta_j} \left(P_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathscr{K}^2(t)} - 1 \right). \tag{3.22}$$

Each of the estimates in (3.20) *is sharp for the function* $\psi(z, \theta, \tau)$ *given by* (3.6).

Proof. Putting the values of P_1 and P_2 for $1 < k < \infty$ from Lemma 2.1 in (3.3) and (3.4) we obtain

$$a_2 = \frac{P_1 \prod_{j=1}^{m} \beta_j}{2 \prod_{j=1}^{l} \alpha_j} c_1,$$

$$a_3 = \frac{P_1 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{2\prod_{j=1}^{l} \alpha_j(\alpha_j+1)} \left\{ c_2 - \frac{1}{2}\left(1 - P_1 - \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(t+1)\mathscr{K}^2(t)}\right) c_1^2 \right\}$$

and

$$(3.23) \qquad |a_3 - \mu a_2^2| \le \frac{P_1 \prod_{j=1}^{m} \beta_j(\beta_j+1)}{4\prod_{j=1}^{l} \alpha_j(\alpha_j+1)} \times \left[\left| \frac{P_1 \prod_{j=1}^{l}(\alpha_j+1) \prod_{j=1}^{m} \beta_j}{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j} \mu - 1 - P_1 - \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(t+1)\mathscr{K}^2(t)} \right| |c_1^2| + 2|c_1^2 - c_2| \right].$$

To complete the proof of Theorem 3.3, we follow the same steps as in the proof of Theorem 3.1. Therefore, we choose to omit the details involved. □

Remark 3.4. For special values of the parameters

$((l = 2, m = 1, \alpha_1 = \beta_1 = 1, \alpha_2 = 2)$ and $(l = 2, m = 1, \alpha_1 = \alpha_2 = \beta_1 = 1))$

in Theorem 3.1–3.3, we obtain new results for the classes $k - UCV(\gamma)$ or $k - SP(\gamma)$.

THEOREM 3.5. *Let the function f given by* (1.1) *be in the class* $k - SP_{l,m}(\alpha_1; \gamma)$ $(0 \le \gamma < 1; 0 \le k < 1)$. *Then*

$$(3.24) \qquad |a_3 - \mu a_2^2| + \left[\mu - \frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j} \left((7 - 6\gamma - k^2) - \frac{1-k^2}{\mathscr{B}^2} \right) \right] |a_2^2| \le \frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}, \quad (\sigma_2 \le \mu \le \sigma_3)$$

and

$$(3.25)\quad |a_3-\mu a_2^2| + \left[\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left((7-6\gamma-k^2)+\frac{5(1-k^2)}{\mathscr{B}^2}\right)-\mu\right]|a_2^2|$$

$$\le \frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}, \quad (\sigma_3\le\mu\le\sigma_1)$$

where $\mathscr{B}$, σ_1 *and* σ_2 *are given by* (2.2), (3.8) *and* (3.9), *respectively, and*

$$(3.26)\qquad \sigma_3 = \frac{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left(2+\frac{(7-6\gamma-k^2)\mathscr{B}^2}{(1-k^2)}\right).$$

Proof. Suppose that $0\le k<1$ and $\sigma_2\le\mu\le\sigma_3$. Using (3.14) for $|a_3-\mu a_2^2|$ and (3.3) for $|a_2|$ we have

$$|a_3-\mu a_2^2| + \left[\mu-\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left((7-6\gamma-k^2)-\frac{1-k^2}{\mathscr{B}^2}\right)\right]|a_2^2|$$

$$\le \frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}$$

$$\times\left[\left|\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}+\frac{2}{3}-\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu\right||c_1^2|+2\left|c_2-\frac{c_1^2}{2}\right|\right]$$

$$+\left[\mu-\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left((7-6\gamma-k^2)-\frac{1-k^2}{\mathscr{B}^2}\right)\right]$$

$$\times\left(\frac{(1-\gamma)^2\mathscr{B}^4\left(\prod_{j=1}^{m}\beta_j\right)^2}{(1-k^2)^2\left(\prod_{j=1}^{l}\alpha_j\right)^2}\right)|c_1|^2$$

$$=\frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}$$

$$\times\left[2\left|c_2-\frac{c_1^2}{2}\right|+\left|\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}+\frac{2}{3}-\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu\right||c_1^2|\right.$$

$$\left.+\left(\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu-\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}+\frac{1}{3}\right)|c_1|^2\right].$$

Note that, since $\mu\le\sigma_3$

$$\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}+\frac{2}{3}-\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu\ge 0.$$

Thus, from Lemma 2.2 we have

$$|a_3-\mu a_2^2|+\left\{\mu-\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left[(7-6\gamma-k^2)-\frac{1-k^2}{\mathscr{B}^2}\right]\right\}|a_2^2|$$

$$\le\frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}\left\{2\left|c_2-\frac{c_1^2}{2}\right|+|c_1^2|\right\}\le\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)},$$

which proves (3.24). Similarly, for the value of μ given in (3.25), we have

$$|a_3-\mu a_2^2|+\left[\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left((7-6\gamma-k^2)+\frac{5(1-k^2)}{\mathscr{B}^2}\right)-\mu\right]|a_2^2|$$

$$\leq\frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}$$

$$\times\left[\left|\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}+\frac{2}{3}-\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu\right||c_1^2|+2\left|c_2-\frac{c_1^2}{2}\right|\right]$$

$$+\left[\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left((7-6\gamma-k^2)+\frac{5(1-k^2)}{\mathscr{B}^2}\right)-\mu\right]$$

$$\times\left(\frac{(1-\gamma)^2\mathscr{B}^4\left(\prod_{j=1}^{m}\beta_j\right)^2}{(1-k^2)^2\left(\prod_{j=1}^{l}\alpha_j\right)^2}\right)|c_1|^2$$

$$=\frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}$$

$$\times\left[2\left|c_2-\frac{c_1^2}{2}\right|+\left|\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu-\frac{2}{3}-\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}\right||c_1^2|\right.$$

$$\left.+\left(\frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)}+\frac{5}{3}-\frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu\right)|c_1|^2\right].$$

Since $\mu \geq \sigma_3$,

$$\frac{2(1-\gamma)\mathscr{B}^2 \prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}{(1-k^2)\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}\mu - \frac{2}{3} - \frac{(7-6\gamma-k^2)\mathscr{B}^2}{3(1-k^2)} \geq 0$$

and from Lemma 2.2, we get

$$|a_3-\mu a_2^2| + \left[\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{6(1-\gamma)\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left((7-6\gamma-k^2)+\frac{5(1-k^2)}{\mathscr{B}^2}\right)-\mu\right]|a_2^2|$$

$$\leq \frac{(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{2(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}\left\{2\left|c_2-\frac{c_1^2}{2}\right|+|c_1^2|\right\} \leq \frac{2(1-\gamma)\mathscr{B}^2\prod_{j=1}^{m}\beta_j(\beta_j+1)}{(1-k^2)\prod_{j=1}^{l}\alpha_j(\alpha_j+1)},$$

which proves (3.25). The proof of Theorem 3.5 is thus completed. □

THEOREM 3.6. *Let the function f given by* (1.1) *be in the class* $k-SP_{l,m}(\alpha_1;\gamma)$ $(0 \leq \gamma < 1; k = 1)$. *Then*

$$|a_3-\mu a_2^2| + \left[\mu - \frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left(1-\frac{\pi^2}{24(1-\gamma)}\right)\right]|a_2^2| \tag{3.27}$$

$$\leq \frac{8(1-\gamma)\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\pi^2\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}, \quad (\delta_2 \leq \mu \leq \delta_3)$$

and

$$|a_3-\mu a_2^2| + \left[\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left(\frac{5\pi^2}{24(1-\gamma)}+1\right)-\mu\right]|a_2^2| \tag{3.28}$$

$$\leq \frac{8(1-\gamma)\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\pi^2\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}, \quad (\delta_3 \leq \mu \leq \delta_1)$$

where δ_1 *and* δ_2 *are given as before by* (3.17) *and* (3.18), *respectively, and*

$$\delta_3 = \frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left(1+\frac{\pi^2}{12(1-\gamma)}\right). \tag{3.29}$$

Theorem 3.7. *Let the function* f *given by* (1.1) *be in the class* $k-SP_{l,m}(\alpha_1;\gamma)$ $(0 \le \gamma < 1; 1 < k < \infty)$ *and let* t *be the unique positive number in the open interval* $(0,1)$ *defined by* (1.6). *Then*

$$|a_3-\mu a_2^2| + \left[\mu - \frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{P_1\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j} \times \left(P_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathscr{K}^2(t)} - 1\right)\right]|a_2^2| \le \frac{P_1\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}, \quad (\rho_2 \le \mu \le \rho_3) \tag{3.30}$$

and

$$|a_3-\mu a_2^2| + \left[\frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{P_1\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j} \times \left(1 + P_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathscr{K}^2(t)}\right) - \mu\right]|a_2^2| \le \frac{P_1\prod_{j=1}^{m}\beta_j(\beta_j+1)}{\prod_{j=1}^{l}\alpha_j(\alpha_j+1)}, \quad (\rho_3 \le \mu \le \rho_1) \tag{3.31}$$

where $\mathscr{K}(t)$ *is the complete elliptic integral of the first kind*, P_1, ρ_1 *and* ρ_2 *are given by* (2.1), (3.21) *and* (3.22), *respectively*, *and*

$$\rho_3 = \frac{\prod_{j=1}^{m}(\beta_j+1)\prod_{j=1}^{l}\alpha_j}{P_1\prod_{j=1}^{l}(\alpha_j+1)\prod_{j=1}^{m}\beta_j}\left(P_1 + \frac{4\mathscr{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}(1+t)\mathscr{K}^2(t)}\right). \tag{3.32}$$

Proofs of the Theorems 3.6, 3.7. The proofs of Theorems 3.6, 3.7 are similar to the proof of Theorem 3.5, except for some obvious changes. Therefore, we omitted the details.

The following particular cases can be pointed out.

Remark 3.8. Some known results can be obtained as particular cases of Theorem 3.1–3.7. For example:

(i) Taking $l = 2$, $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 2 - \lambda$ $(0 \le \lambda < 1)$ and $\gamma = 0$ in all our work, we obtain all results of Mishra and Gochhayat [22].

(ii) Taking $l = 2$, $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = \eta + 1$ $(\eta \in \mathbf{N}_0 := \mathbf{N} \cup \{0\})$ and $\gamma = 0$ in all our work, we obtain all results of Mishra and Gochhayat [23].

(iii) Taking $l = 2$, $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 2 - \lambda$ $(0 \le \lambda < 1)$ and $\gamma = 0$ in Theorems 3.2 and 3.6 we get the results obtained by Srivastava and Mishra [33].

(iv) Taking $l = 2$, $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 1$ and $\gamma = 0$ in Theorem 3.2 we obtain a result due to Ma and Minda [21].

(v) Taking $l = 2$, $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 2 - \lambda$ $(0 \le \lambda < 1)$ and $k = \gamma = 0$ in Theorem 3.1 we obtain a result due to Srivastava, Mishra and Das [34].

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