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# $\delta(2)$-IDEAL NULL 2-TYPE HYPERSURFACES OF EUCLIDEAN SPACE ARE SPHERICAL CYLINDERS

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**Abstract**

We prove that a null 2-type hypersurface in the Euclidean $(n+1)$-space is an open portion of a spherical cylinder $S^{n-1} \times \mathbf{R}$ if and only if it is $\delta(2)$-ideal.

## 1. Introduction

Let $M$ be a Riemannian $n$-manifold isometrically immersed in the Euclidean $m$-space $\mathbf{E}^m$. Denote by $\Delta$ the Laplacian of $M$. Then the position vector $\mathbf{x}$ and the mean curvature vector $H$ of $M$ in $\mathbf{E}^m$ are related by Beltrami's formula:

$$\Delta \mathbf{x} = -nH, \tag{1.1}$$

which implies the well-known result: A submanifold $M$ in $\mathbf{E}^m$ is minimal if and only if all coordinate functions of $\mathbf{E}^m$, restricted to $M$, are harmonic functions, i.e.,

$$\Delta \mathbf{x} = 0. \tag{1.2}$$

In other words, minimal submanifolds of $\mathbf{E}^m$ are constructed by eigenfunctions of the Laplacian $\Delta$ with eigenvalue zero. It is well-known that there are many minimal submanifolds in $\mathbf{E}^m$. In particular, according to the famous Douglas and Rado's solutions to the Plateau problem, there are ample examples of minimal surfaces in $\mathbf{E}^3$ (see for instance [14, 15]).

On the other hand, it is easy to verify that circular cylinders in $\mathbf{E}^3$ are constructed from both harmonic functions and eigenfunctions of $\Delta$ with a single nonzero eigenvalue, say $\lambda$. Thus the position vector $\mathbf{x}$ of such a surface admits the following simple spectral resolution:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_q, \quad \Delta \mathbf{x}_0 = 0, \quad \Delta \mathbf{x}_q = \lambda \mathbf{x}_q, \tag{1.3}$$

for some non-constant maps $\mathbf{x}_0$ and $\mathbf{x}_q$.

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In terms of finite type theory, a Euclidean submanifold is said to be of *null* 2-*type* if its position vector admits the spectral decomposition (1.3). Similarly, a Euclidean submanifold is said to be of 1-type if its position vector satisfies $\mathbf{x} = \mathbf{x}_q$ with $\Delta \mathbf{x}_q = \lambda \mathbf{x}_q$ for some non-constant vector function $\mathbf{x}_q$ (cf. [1, 2, 4, 7, 9]). According to a well-known result of Takahashi, a 1-type submanifold of a Euclidean space $\mathbf{E}^m$ is either a minimal submanifold of $\mathbf{E}^m$ or a minimal submanifold of a hypersphere of $\mathbf{E}^m$ (see [17]).

The study of finite type submanifolds treats an interesting question:

*To what extent is the geometric structure of a submanifold determined by a simple analytic information, that is, by the spectral resolution of the immersion*?

In particular, due to the simplicity of null 2-type submanifolds, it is very natural and interesting to ask the following geometric question (cf. [4, Problem 12]):

"*Determine all submanifolds of Euclidean spaces which are of null* 2-*type. In particular, classify null* 2-*type hypersurfaces in Euclidean spaces*."

So far, very few results are known concerning this problem. The first result was obtained in [2], in which the first author proved that a surface in $\mathbf{E}^3$ is of null 2-type if and only if it is an open portion of a circular cylinder $S^1 \times \mathbf{R}$. By applying the method in [2], it was shown in [12] that a null 2-type Euclidean hypersurface in $\mathbf{E}^{n+1}$ with at most two distinct principal curvatures is a spherical cylinder $S^p \times \mathbf{E}^{n-p}$ in $\mathbf{E}^{n+1}$. Moreover, it was proved in [13] that every null 2-type hypersurface in $\mathbf{E}^4$ has nonzero constant mean curvature and constant scalar curvature. Also, there exist several results on null 2-type surfaces of high codimension in Euclidean spaces under additional assumptions obtained by U. Dursun, S. J. Li, Y. H. Kim, H. S. Lue, A. Ferrández, and P. Lucas among others (cf. [4, 11] for details).

Let $M$ be a Riemannian $n$-manifold. Denote by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1, \dots, e_n$ of the tangent space $T_pM$, the scalar curvature $\tau$ at $p$ is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let $L$ be a subspace of $T_pM$ of dimension $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of $L$. We define the scalar curvature $\tau(L)$ of the $r$-plane section $L$ by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

For an integer $k \geq 1$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered $k$-tuples $(n_1, \dots, n_k)$ of integers $\geq 2$ satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. For each integer $n \geq 3$, we put $\mathcal{S}(n) = \bigcup_{k \geq 1} \mathcal{S}(n,k)$.

For each $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the Riemannian $\delta$-invariant $\delta(n_1, \ldots, n_k)$ was introduced by the first author as (cf. [5, 7, 9])

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where $L_1, \ldots, L_k$ run over all $k$ mutually orthogonal subspaces of $T_pM$ such that $\dim L_j = n_j,\ j = 1, \ldots, k$.

The $\delta$-curvatures are very different in nature from the "classical" scalar and Ricci curvatures; simply due to the fact that both scalar and Ricci curvatures are the "total sum" of sectional curvatures on a Riemannian manifold. In contrast, the $\delta$-curvature invariants are obtained from the scalar curvature by throwing away a certain amount of sectional curvatures.

In [5, 6] the author proved that, for any $n$-dimensional submanifold of $\mathbf{E}^m$ and any $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the $\delta$-invariants $\delta(n_1, \ldots, n_k)$ are related to the squared mean curvature $\|H\|^2$ of the submanifold by the following general optimal inequality:

$$\delta(n_1, \ldots, n_k) \leq \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)} \|H\|^2. \tag{1.4}$$

A submanifold of $\mathbf{E}^m$ is called $\delta(n_1, \ldots, n_k)$-*ideal* if it satisfies the equality case of (1.4) identically. Roughly speaking, an ideal immersion is an immersion which produces the least possible amount of tension from the ambient space. Such submanifolds have many interesting properties and have been studied by many geometers in recent years (see [8, 9] for details).

In this paper we determine all null 2-type hypersurfaces in $\mathbf{E}^{n+1}$ which are $\delta(2)$-ideal. More precisely, we prove the following theorem which provides a connection between the $\delta(2)$-invariant and finite type submanifolds.

THEOREM 1. *A null* 2*-type hypersurface in the Euclidean* $(n+1)$*-space* $\mathbf{E}^{n+1}$ *is an open portion of a spherical cylinder* $S^{n-1} \times \mathbf{R} \subset \mathbf{E}^{n+1}$ *if and only if it is* $\delta(2)$*-ideal.*

## 2. Preliminaries

Let $\mathbf{x} : M \to \mathbf{E}^m$ be an isometric immersion of a Riemannian $n$-manifold $M$ into the Euclidean $m$-space $\mathbf{E}^m$. Denote by $\langle\, , \rangle$ the inner product associated with the standard metric on $\mathbf{E}^m$.

Let $\nabla$ and $\tilde{\nabla}$ be the Levi-Civita connection on $M$ and $\mathbf{E}^m$, respectively. Then the formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields $X$, $Y$ tangent to $M$ and $\xi$ normal to $M$, where $h$, $A$ and $D$ are the second fundamental form, the shape operator and the normal connection.

The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle. \tag{2.3}$$

The mean curvature vector is given by

$$H = \left(\frac{1}{n}\right) \operatorname{trace} h. \tag{2.4}$$

The equations of Gauss and Codazzi are given respectively by

$$\langle R(X, Y)Z, W\rangle = \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \tag{2.5}$$

$$(\overline{\nabla}_X h)(Y, Z) = (\overline{\nabla}_Y h)(X, Z), \tag{2.6}$$

for $X$, $Y$, $Z$, $W$ tangent to $M$, where $R$ is the Riemann curvature tensor of $M$ and $\overline{\nabla}h$ is defined by

$$(\overline{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.7}$$

## 3. Basics on null 2-type hypersurfaces

For an isometric immersion $\mathbf{x} : M \to \mathbf{E}^m$ of a Riemannian $n$-manifold $M$ into $\mathbf{E}^m$, we have (cf. [1, 9])

$$\Delta H = \Delta^D H + \sum_{i=1}^{n} h(A_H e_i, e_i) + (\Delta H)^T, \tag{3.1}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal tangent frame,

$$(\Delta H)^T = \left(\frac{n}{2}\right)\nabla\langle H, H\rangle + 2 \operatorname{trace} A_{DH} \tag{3.2}$$

is the tangential component of $\Delta H$ and $\Delta^D$ denotes the Laplacian associated with the normal connection $D$, i.e.

$$\Delta^D H = \sum_{i=1}^{n} D_{\nabla_{e_i} e_i} H - \sum_{i=1}^{n} D_{e_i} D_{e_i} H.$$

In particular, if $m = n + 1$, then (3.1) and (3.2) reduce respectively to

$$\Delta H = (\Delta\alpha + \alpha\|A\|^2)e_{n+1} + (\Delta H)^T, \tag{3.3}$$

$$(\Delta H)^T = n\alpha\nabla\alpha + 2A(\nabla\alpha), \tag{3.4}$$

where $e_{n+1}$ is a unit normal vector field, $H = \alpha e_{n+1}$ and $A = A_{e_{n+1}}$.

If $M$ is a null 2-type hypersurface of $\mathbf{E}^{n+1}$, it follows from (1.1) and (1.2) that

$$\Delta H = \lambda H. \tag{3.5}$$

By combining (3.3), (3.4) and (3.5) we obtain

$$\Delta\alpha + \alpha\|A\|^2 = \lambda\alpha, \tag{3.6}$$

$$A(\nabla\alpha) = -\left(\frac{n\alpha}{2}\right)\nabla\alpha. \tag{3.7}$$

Consequently, $\nabla\alpha$ is an eigenvector of $A$ whenever $\nabla\alpha \neq 0$.

We have the following.

Lemma 3.1. *Let $M$ be a submanifold of a Euclidean space. Then the mean curvature vector $H$ of $M$ satisfies $\Delta H = \lambda H$ for some real number $\lambda$ if and only if $M$ is one of the following submanifolds*:

(1) *a biharmonic submanifold, i.e.*, $\Delta H = 0$;
(2) *a* 1-*type submanifold*;
(3) *a null* 2-*type submanifold.*

This lemma can be found in [4, page 216] and in [9, page 140].

Lemma 3.2. *A non-spherical hypersurface of the Euclidean* 4-*space* $\mathbf{E}^4$ *is of null* 2-*type if and only if it has nonzero constant mean curvature and constant scalar curvature.*

*Proof.* Follows immediately from Lemma 4.1 and the main result of [13]. □

We also have the following result from [10, Theorem 1].

Proposition 1. *Every* 2-*type hypersurface of a Euclidean space is of null* 2-*type if it has constant mean curvature.*

The proof of this proposition was based on (3.6) and (3.7).

## 4. Proof of Theorem 1

First we prove the following lemma which extends Lemma 3.2 to hypersurfaces in Euclidean spaces with arbitrary dimension.

Lemma 4.1. *A non-spherical hypersurface of* $\mathbf{E}^{n+1}$ *with nonzero constant mean curvature is of null* 2-*type if and only if it has constant scalar curvature.*

*Proof.* Let $M$ be a non-spherical hypersurface of $\mathbf{E}^{n+1}$ with nonzero constant mean curvature $\alpha$. Then we find from (3.4) that $(\Delta H)^T = 0$. Thus it follows from (3.3) that

$$\Delta H = \|A\|^2 H.$$

On the other hand, from the following well-known relation:

$$n^2\alpha^2 = \|A\|^2 + 2\tau,$$

we know that the squared norm $\|A\|^2$ of $A$ is constant if and only if the scalar curvature $\tau$ is also constant. In this case, $\Delta H = \lambda H$ holds with $\lambda = \|A\|^2 \neq 0$. Since $\alpha \neq 0$ and $M$ is non-spherical and non-biharmonic, $M$ is of null 2-type according to Lemma 3.1. □

Now, we return to the proof of Theorem 1. Let us assume that $M$ is a null 2-type hypersurface of $\mathbf{E}^{n+1}(n \geq 2)$ which is $\delta(2)$-ideal. If $n = 2$, Theorem 1 follows from the main result of [2]. Thus, we may assume that $n \geq 3$. Since $M$ is assumed to be $\delta(2)$-ideal, Lemma 3.2 of [3] implies that there exists a local orthonormal frame field $\{e_1, \dots, e_n, e_{n+1}\}$ such that the shape operator with respect to this frame field takes the form:

$$A = \begin{pmatrix} \eta & 0 & 0 & \cdots & 0 \\ 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & \eta+\mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \eta+\mu \end{pmatrix} \tag{4.1}$$

for some functions $\eta$ and $\mu$.

*Case* (1): *M has constant mean curvature* $\alpha$. Since $M$ is non-minimal, $\alpha \neq 0$. Thus, by Lemma 4.1, $M$ has constant scalar curvature. Hence, it follows from (4.1) that $\eta$ and $\mu$ are both constant. Therefore, $M$ is an open portion of a spherical cylinder $S^k \times \mathbf{E}^{n-k}$ for some $k \in \{0, \dots, n\}$ according to [16]. Consequently, the principal curvatures of $M$ are given by a nonzero real number $r$ (repeated $k$ times) and 0 (repeated $n - k$ times). After comparing these with (4.1), we conclude that the only possibilities are either $\eta = 0$ or $\mu = 0$. Consequently, $M$ is an open portion of $S^{n-1} \times \mathbf{R}$.

*Case* (2): *M has non-constant mean curvature* $\alpha$. It follows from (3.7) that $\nabla\alpha$ is an eigenvector of $A$ with eigenvalue $-\frac{n\alpha}{2}$. Thus, one of the following three cases must occur:

(a) $\eta = -\frac{n\alpha}{2}$ and $e_2\alpha = \cdots = e_n\alpha = 0$;

(b) $\mu = -\frac{n\alpha}{2}$ and $e_1\alpha = e_3\alpha = \cdots = e_n\alpha = 0$; or

(c) $\eta + \mu = -\frac{n\alpha}{2}$ and $e_1\alpha = \cdots = e_{n-1}\alpha = 0$.

*Case* (2.a): $\eta = -\frac{n\alpha}{2}$ *and* $e_2\alpha = \cdots = e_n\alpha = 0$. Since $n\alpha = (n-1)(\eta + \mu)$ holds, we find

$$\eta = -\frac{n\alpha}{2}, \quad \mu = \frac{n(n+1)\alpha}{2(n-1)}, \quad \eta + \mu = \frac{n\alpha}{n-1}. \tag{4.2}$$

Thus the second fundamental form $h$ satisfies

$$\begin{aligned} &h(e_1, e_1) = -\frac{n\alpha}{2} e_{n+1}, \\ &h(e_2, e_2) = \frac{n(n+1)\alpha}{2(n-1)} e_{n+1}, \\ &h(e_j, e_j) = \frac{n\alpha}{n-1} e_{n+1}, \quad j = 3, \dots, n, \\ &h(e_i, e_k) = 0, \quad 1 \le i \ne k \le n. \end{aligned} \tag{4.3}$$

Let $\omega_i^k$ $(i, k = 1, \dots, n)$ be the connection forms of $M$ defined by

$$\nabla_X e_i = \sum_{k=1}^{n} \omega_i^k(X) e_k, \quad i = 1, \dots, n. \tag{4.4}$$

Then we have

$$\omega_i^k = -\omega_k^i, \quad i, k = 1, \dots, n.$$

It follows from (4.3), (4.4) and Codazzi's equation that

$$\nabla_{e_1} e_1 = 0, \tag{4.5}$$

$$\omega_2^j(e_1) = \left(\frac{n+1}{1-n}\right)\omega_1^j(e_2) = \left(\frac{2n}{n-1}\right)\omega_2^1(e_j), \quad j = 3, \dots, n, \tag{4.6}$$

$$e_1\alpha = \left(\frac{n+1}{2}\right)\alpha\omega_j^1(e_j), \quad j = 3, \dots, n, \tag{4.7}$$

$$e_1\alpha = \left(\frac{2n}{n+1}\right)\alpha\omega_2^1(e_2), \tag{4.8}$$

$$\omega_j^2(e_k) = 0, \quad 2 \le j, k \le n, \tag{4.9}$$

$$\omega_j^1(e_k) = 0, \quad 3 \le j \ne k \le n. \tag{4.10}$$

Now, after applying (4.4)–(4.10) we conclude that the Levi-Civita connection $\nabla$ of $M$ satisfies

$$\nabla_{e_1} e_1 = 0, \tag{4.11}$$

$$\nabla_{e_1} e_2 = \sum_{k=3}^{n} \omega_2^k(e_1) e_k, \tag{4.12}$$

$$\nabla_{e_1} e_j = \omega_j^2(e_1)e_2 + \sum_{k=3}^{n} \omega_j^k(e_1)e_k, \tag{4.13}$$

$$\nabla_{e_2} e_1 = -\left(\frac{n+1}{2n}\right)\frac{e_1\alpha}{\alpha}e_2 + \sum_{k=3}^{n} \omega_1^k(e_2)e_k, \tag{4.14}$$

$$\nabla_{e_2} e_2 = \left(\frac{n+1}{2n}\right)\frac{e_1\alpha}{\alpha}e_1, \tag{4.15}$$

$$\nabla_{e_2} e_j = \omega_j^1(e_2)e_1 + \sum_{k=3}^{n} \omega_j^k(e_2)e_k, \tag{4.16}$$

$$\nabla_{e_j} e_1 = \omega_1^2(e_j)e_2 - \left(\frac{2}{n+1}\right)\frac{e_1\alpha}{\alpha}e_j, \tag{4.17}$$

$$\nabla_{e_j} e_2 = \omega_2^1(e_j)e_1, \tag{4.18}$$

$$\nabla_{e_j} e_j = \left(\frac{2}{n+1}\right)\frac{e_1\alpha}{\alpha}e_1 + \sum_{k=3}^{n} \omega_j^k(e_j)e_k, \tag{4.19}$$

for $j = 3, \dots, n$.

By using (4.3), (4.11)–(4.18) and Gauss' equation, we find from $R(e_j, e_1; e_1, e_j)$ for $j \in \{2, \dots, n\}$ that

$$e_1\left(\frac{e_1\alpha}{\alpha}\right) = \left(\frac{2}{n+1}\right)\frac{(e_1\alpha)^2}{\alpha^2} - \frac{\alpha^2 n^2(n+1)}{4(n-1)} + \frac{2n(n+1)}{1-n}(\omega_2^1(e_j))^2. \tag{4.20}$$

Similarly, we derive from $R(e_j, e_2; e_2, e_j)$ that

$$(\omega_2^1(e_j))^2 = \frac{(e_1\alpha)^2(n+1)}{4\alpha^2 n^2} + \frac{\alpha^2 n(n+1)^2}{8(n-1)^2}. \tag{4.21}$$

Combining (4.20) and (4.21) gives

$$e_1\left(\frac{e_1\alpha}{\alpha}\right) = \frac{(e_1\alpha)^2(1+7n-n^2+n^3)}{2\alpha^2 n(1-n^2)} + \frac{\alpha^2 n^2(n+1)(n^2+1)}{2(1-n)^3}. \tag{4.22}$$

From $R(e_1, e_2; e_2, e_1)$ and (4.21) we derive that

$$e_1\left(\frac{e_1\alpha}{\alpha}\right) = \left(\frac{n+1}{2n}\right)\frac{(e_1\alpha)^2}{\alpha^2} + \frac{\alpha^2 n^3}{2(1-n)} + \frac{16n^3}{(n-1)(n+1)^2}\sum_{j=3}^{n}(\omega_2^1(e_j))^2. \tag{4.23}$$

After combining (4.21) and (4.23), we find

$$e_1\left(\frac{e_1\alpha}{\alpha}\right) = \frac{(e_1\alpha)^2(9n^3 - 15n^2 - n - 1)}{2\alpha^2 n(n^2 - 1)} + \frac{\alpha^2 n^3(1 + 6n - 3n^2)}{2(1 - n)^3}. \tag{4.24}$$

By combining (4.22) and (4.24), we obtain

$$(e_1\alpha)^2 = -\frac{n^2(4n^3 + 3n^2 - 2n - 1)}{2(n - 1)^2(5n - 3)}\alpha^4, \tag{4.25}$$

which cannot happen for $n \geq 3$ unless $\alpha = 0$. Consequently, this case cannot happen.

*Case* (2.b): $\mu = -\frac{n\alpha}{2}$ *and* $e_1\alpha = e_3\alpha = \cdots = e_n\alpha = 0$. By applying the same argument as case (2.a), we conclude that this case is impossible.

*Case* (2.c): $\eta + \mu = -\frac{n\alpha}{2}$ *and* $e_1\alpha = \cdots = e_{n-1}\alpha = 0$. In this case, the second fundamental form $h$ satisfies

$$\begin{aligned} h(e_1, e_1) &= -\left(\frac{n\alpha}{2} + \mu\right)e_{n+1}, \\ h(e_2, e_2) &= \mu e_{n+1}, \\ h(e_j, e_j) &= -\frac{n\alpha}{2}e_{n+1}, \quad j = 3, \dots, n, \\ h(e_i, e_k) &= 0, \quad 1 \leq i \neq k \leq n. \end{aligned} \tag{4.26}$$

Since trace $h = n\alpha e_{n+1}$, it follows from (4.26) that $n = -1$. Consequently, this case is also impossible.

Conversely, suppose that $M$ is a spherical cylinder $S^{n-1} \times \mathbf{R}$ in $\mathbf{E}^{n+1}$. Let us choose an orthonormal frame $e_1, \dots, e_n$ such that $e_1$ is tangent to the second factor $\mathbf{R}$ and $e_2, \dots, e_n$ tangent to the first factor $S^{n-1}$. Then the shape operator $A$ satisfies (4.1) with $\eta = 0$ and $\mu \neq 0$. Hence, $M$ is a $\delta(2)$-ideal hypersurface according to Lemma 3.2 of [3]. Obviously, $M$ is of null 2-type by Lemma 4.1. This completes the proof of the theorem.

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