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NONEXISTENCE OF NONTRIVIAL QUASI-EINSTEIN METRICS*

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Abstract

Let $(M^n, g, e^{-f} dvol_g)$ be a smooth metric measure space of dimension n. In this note, we first prove a nonexistence result for M^n with the Bakry-Émery Ricci tensor is bounded from below. Then we show that $f \in L^\infty(M^n, e^{-f} dvol)$ and $|\nabla f| \in L^\infty(M^n, e^{-f} dvol)$ are equivalent for complete gradient shrinking Ricci solitons. Furthermore, we prove that there is no non-Einstein shrinking soliton when the normalized function $\tilde{f}$ is non-positive.

1. Introduction

Let $(M^n, g, e^{-f} dvol_g)$ be an n-dimensional smooth metric measure space, where M^n is a complete n-dimensional Riemannian manifold with metric g, $f \in C^\infty(M^n)$ is a real valued function, and $dvol_g$ is the Riemannian volume form on M^n. A natural extension of the Ricci tensor Ric to a smooth metric measure space is the m-Bakry-Émery Ricci tensor

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{m} df \otimes df \quad \text{for } 0 < m \leq \infty. \tag{1.1}$$

When $m = \infty$, we denote $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess}\, f$, the usual Bakry-Émery Ricci tensor. For the smooth metric measure space, there is a naturally associated Bakry-Émery Laplacian (also called f-Laplacian [17]), defined by

$$\Delta_f = \Delta - g(\nabla f, \nabla) := \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint in $L^2(M^n, g, e^{-f} dvol_g)$, where $\Delta = g^{ij} \nabla_i \nabla_j$ is the Laplace-Beltrami operator.

A smooth metric measure space $(M^n, g, e^{-f} dvol_g)$ is called an m-quasi-Einstein manifold (Correspondingly, g is called an m-quasi-Einstein metric) if it satisfies the equation

$$\mathrm{Ric}_f^m = \lambda g \tag{1.2}$$

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for some constant $\lambda \in \mathbf{R}$ (see [6]). As a matter of fact, equation (1.2) encloses the following three cases: when f is a constant, we take $m = 0$ and call the underlying Einstein manifold a trivial quasi-Einstein manifold; when $m = \infty$, it is exactly the gradient Ricci soliton (the usual quasi-Einstein) equation, namely

$$\mathrm{Ric}_f = \lambda g. \tag{1.3}$$

It is called shrinking, steady, or expanding, if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Ricci solitons play an important role in the theory of Ricci flow, we refer to [2] and the references therein for background and recent progress on Ricci solitons. When m is a positive integer, it corresponds to Einstein warped product and has been actively investigated (see [3], [4], [6], [9], [10], [12]).

Recently, using the approach of gradient estimate and conformal rescaling, Case [3] studied the nonexistence of m-quasi-Einstein metrics and proved

Theorem A ([3]). *Let (M^n, g) be a complete Riemannian manifold such that $\mathrm{Ric}_f^m \geq 0$ for some function f and $0 \leq m \leq \infty$, and suppose that $\Delta_f f = \phi(f)$, where $\phi : \mathbf{R} \to \mathbf{R}$ satisfies*

$$\phi'(t) + \frac{2}{n}\phi(t) \geq 0$$

for all $t \in \mathbf{R}$. Then M^n is Einstein and f is a constant.

For a complete Riemannian manifold with negative m-Bakry-Émery Ricci curvature (i.e., $\mathrm{Ric}_f^m \geq \lambda$ and $\lambda < 0$), Mastrolia and Rimoldi [12] proved that the above result still holds under the assumptions that $0 \leq m < \infty$. Unfortunately, the method used in the proof of Theorem A for $m = \infty$ is invalid in this case. Therefore, it is of interest to know whether the conclusion of Theorem A is true for $\mathrm{Ric}_f^m \geq \lambda$ $(\lambda < 0)$ and $m = \infty$. In this paper, using the gradient estimate method and the weak maximum principle, we first discuss the question and give an affirmative answer, more precisely, we have

Theorem 1.1. *Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_f \geq \lambda$ $(\lambda \leq 0)$. Assume that $\Delta_f f = \phi(f)$, where $\phi : \mathbf{R} \to \mathbf{R}$ satisfies*

$$\phi'(t) + \frac{2}{n}\phi(t) + \lambda \geq 0$$

for all $t \in \mathbf{R}$. Then M^n is Einstein and f is a constant.

Remark 1.2. We thank the referee for making us aware of the extra assumption "f is bounded" for Theorem 1.1 in our earlier version of this paper is not needed and for bringing the paper [1] to our attention.

For a complete gradient expanding Ricci soliton (M^n, g, f), Pigola, Rigoli, Rimoldi, and Setti [14] proved that $|\nabla f| \in L^\infty(M^n, e^{-f} dvol)$ implies M^n is Einstein and f is a constant. In [11], Ma and Chen studied the minimum point of the potential function f of expanding Ricci solitons with non-negative

Ricci curvature. In section 3 of this paper, we are concerned with the complete gradient shrinking Ricci solitons and obtain

THEOREM 1.3. *Let (M^n, g, f) be an n-dimensional complete gradient shrinking Ricci soliton. Then $f \in L^\infty(M^n, e^{-f} dvol)$ is equivalent to $|\nabla f| \in L^\infty(M^n, e^{-f} dvol)$ on M^n. In particular, when the normalized function $\tilde{f} \le 0$ (see (3.3) for $\tilde{f}$) and M^n has no boundary, M^n is Einstein and f is a constant.*

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. For this purpose, we first establish the following crucial estimate.

PROPOSITION 2.1. *Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_f \ge \lambda$ and $\lambda \le 0$. Assume that*

$$\Delta_f f = \phi(f), \tag{2.1}$$

where $\phi : \mathbf{R} \to \mathbf{R}$ satisfies

$$\phi'(t) + \frac{2}{n}\phi(t) + \lambda \ge 0 \tag{2.2}$$

for all $t \in \mathbf{R}$. Then for all $x \in M^n$ and $a > 1$ such that $B(x, a)$ is geodesically connected in M^n and the closure $\overline{B(x, a)}$ is compact,

$$|\nabla f|^2(x) \le \frac{12n}{a^2} + \frac{\alpha n}{a} - n\lambda, \tag{2.3}$$

where $\alpha = \max_{y \in \{y : d(x,y)=1\}}\{|\Delta_f r(y)|\}$.

In order to prove Proposition 2.1, we need the following several known results.

LEMMA 2.2 (Bochner-Weitzenböck formula [17]). *Let $f, u \in C^\infty(M^n)$. Then*

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \tag{2.4}$$

Remark 2.3. For gradient Ricci solitons, equation (1.3) together with (2.4) yields (see also [13])

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |\mathrm{Hess}\, f|^2 - \lambda |\nabla f|^2. \tag{2.5}$$

LEMMA 2.4 (Laplacian Comparison [17]). *Assume that (M^n, g) is an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_f(\nabla f, \nabla f) \ge \lambda$. Then given any minimal geodesic segment and $r_0 > 0$,*

$$\Delta_f(r) \le \Delta_f(r_0) - \lambda(r - r_0) \quad \text{for } r \ge r_0. \tag{2.6}$$

Equality holds for some $r > r_0$ *if and only if all the radial sectional curvatures are zero*, $\mathrm{Hess}\, r \equiv 0$, *and* $\partial_r^2 f \equiv \lambda$ *along the geodesic from* r_0 *to* r.

Now we are ready to prove Proposition 2.1.

Proof of Proposition 2.1. Since

$$|\mathrm{Hess}\, f|^2 \geq \frac{1}{n}(\Delta f)^2,$$

from the Bochner-Weitzenböck formula (2.4) we obtain

$$\begin{aligned} \frac{1}{2}\Delta_f |\nabla f|^2 &= |\mathrm{Hess}\, f|^2 + \langle \nabla f, \nabla(\Delta_f f)\rangle + \mathrm{Ric}_f(\nabla f, \nabla f) \\ &\geq |\mathrm{Hess}\, f|^2 + \phi'(f)|\nabla f|^2 + \lambda |\nabla f|^2 \\ &\geq \frac{(\Delta_f f + |\nabla f|^2)^2}{n} + (\phi'(f) + \lambda)|\nabla f|^2 \\ &= \frac{\phi^2(f) + |\nabla f|^4}{n} + \left(\phi'(f) + \frac{2\phi(f)}{n} + \lambda\right)|\nabla f|^2 \\ &\geq \frac{|\nabla f|^4}{n}, \end{aligned} \tag{2.7}$$

here we have used (2.1) in the second line and (2.2) in the last.

Let $r(y) = d(x, y)$ be the radial distance function. Now we consider the function

$$F(y) = (a^2 - r^2(y))^2 |\nabla f|^2(y)$$

defined on $\overline{B(x,a)}$. Since $F \geq 0$ and $F|_{\partial B(x,a)} \equiv 0$, there exists a point $y_0 \in B(x,a)$ such that $F(y)$ achieves its maximum at y_0. Using the method of support functions (see [15]), we can assume that y_0 lies outside of the cut locus of x, then F is smooth near y_0 and by the maximum principle, we have

$$\nabla F(y_0) = 0, \quad \Delta_f F(y_0) \leq 0. \tag{2.8}$$

In follows, all the computations are carried out at the point y_0. From (2.8), we have

$$\frac{\nabla |\nabla f|^2}{|\nabla f|^2} = \frac{2\nabla(r^2)}{a^2 - r^2}, \tag{2.9}$$

and

$$\frac{2|\nabla(r^2)|^2}{(a^2 - r^2)^2} - \frac{4\nabla(r^2)\cdot \nabla |\nabla f|^2}{(a^2 - r^2)|\nabla f|^2} - \frac{2\Delta_f(r^2)}{a^2 - r^2} + \frac{\Delta_f |\nabla f|^2}{|\nabla f|^2} \leq 0. \tag{2.10}$$

Substituting (2.9) into (2.10), we derive that

$$\frac{-6|\nabla(r^2)|^2}{(a^2-r^2)^2}-\frac{2\Delta_f(r^2)}{a^2-r^2}+\frac{\Delta_f|\nabla f|^2}{|\nabla f|^2}\le 0.$$

Using $|\nabla(r^2)|=2r|\nabla r|=2r$ and (2.7), we have

$$\frac{-24r^2}{(a^2-r^2)^2}-\frac{2\Delta_f(r^2)}{a^2-r^2}+\frac{2}{n}|\nabla f|^2\le 0.$$

From (2.6), we get

$$\Delta_f(r^2)=2+2r(\Delta_f(r_0)-\lambda(r-r_0))$$

for some $0<r_0<\min\{1,r/2\}$. Setting $\alpha=\max_{y\in\{y:d(x,y)=1\}}\{|\Delta_f(r(y))|\}$, we have

$$\Delta_f(r^2)\le 2+2r(\alpha-\lambda a).$$

Thus

$$\frac{-24r^2}{(a^2-r^2)^2}+\frac{2}{n}|\nabla f|^2-\frac{4+4r(\alpha-\lambda a)}{a^2-r^2}\le 0. \tag{2.11}$$

Multiplying (2.11) by $(a^2-r^2)^2$, we have

$$\begin{aligned}\frac{F}{n}&\le 12r^2+(a^2-r^2)(2+2r(\alpha-\lambda a))\\&=10r^2+2a^2+2(a^2r-r^3)(\alpha-\lambda a)\\&\le 12a^2+2(a^2r-r^3)(\alpha-\lambda a).\end{aligned} \tag{2.12}$$

Since

$$ra^2-r^3\le\frac{2\sqrt{3}a^3}{9}\quad\text{for } r>0,$$

we conclude from (2.12) that

$$\frac{F}{n}\le 12a^2+\alpha-\lambda a.$$

Therefore

$$\sup_{B(x,a)}(a^2-r^2)^2|\nabla f|^2\le 12na^2+n(\alpha-\lambda a)a^3.$$

In particular,

$$a^4|\nabla f|^2(x)\le 12na^2+n(\alpha-\lambda a)a^3.$$

This proves Proposition 2.1. □

Proof of Theorem 1.1. From (2.3) in Proposition 2.1, we see that $\sup_{M^n}|\nabla f|^2 < +\infty$. By the weak maximum principle in [14], there exists a sequence $\{x_m\}$ such that,

$$|\nabla f|^2(x_m) \geq \sup_{M^n}|\nabla f|^2 - \frac{1}{m},$$

and

$$\Delta_f|\nabla f|^2(x_m) \leq \frac{1}{m}.$$

It follows from (2.7) that

$$\frac{1}{2m} \geq \frac{1}{2}\Delta_f|\nabla f|^2(x_m) \geq \frac{|\nabla f|^4}{n}(x_m) \geq 0.$$

Letting $m \to +\infty$ we see that

$$\sup_{M^n}|\nabla f|^4 = 0,$$

which completes the proof of Theorem 1.1. □

Remark 2.5. When $\lambda = 0$, we obtain the Case's result in [3] for $m = \infty$ by a different method.

3. The nonexistence of gradient shrinking Ricci solitons

In this section, we are going to prove Theorem 1.3. First of all, we recall the following useful results. Note that Lemma 3.1 is a special case of a more general result on complete ancient solutions of the Ricci flow, due to B. L. Chen [5] (see also [14], [18]).

LEMMA 3.1. *Let (M^n, g, f) be a complete gradient shrinking Ricci soliton with scalar curvature R. Then $R \geq 0$.*

LEMMA 3.2 ([8]). *Let (M^n, g, f) be a complete gradient Ricci soliton with scalar curvature R. Then*

$$R + |\nabla f|^2 = 2\lambda f + C_1, \tag{3.1}$$

where C_1 is a constant.

Remark 3.3. For gradient shrinking Ricci solitons, (3.1) together with Lemma 3.1 yields that f is bounded from below. In the case of steady, Wu [16] proved that $\limsup_{y \in B_r(x), r \to \infty}|\nabla f|^2 = C_1$.

Proof of Theorem 1.3. Taking the trace of (1.3), we get

$$R + \Delta f = n\lambda. \tag{3.2}$$

Taking the difference of (3.2) and (3.1), we have

$$\Delta_f f = n\lambda - 2\lambda f - C_1 = -2\lambda\left(f + \frac{C_1}{2\lambda} - \frac{n}{2}\right) := -2\lambda\tilde{f}. \tag{3.3}$$

The function $\tilde{f}$ is called a normalized soliton potential function.

Assume that (M^n, g, f) is shrinking. By (3.1) and Lemma 3.1, we see that $f \in L^\infty(M^n, e^{-f}dvol)$ implies $|\nabla f| \in L^\infty(M^n, e^{-f}dvol)$. Conversely, supposing $|\nabla f| \in L^\infty(M^n, e^{-f}dvol)$, we know from Fernández-López and García-Río's result (see Theorem 1 in [7]) that (M^n, g) must be compact, we thus get $f \in L^\infty(M^n, e^{-f}dvol)$. In particular, if $\tilde{f} \le 0$, equation (3.3) gives that

$$2\lambda f \le n\lambda - C_1 \tag{3.4}$$

and

$$\Delta_f f = -2\lambda\tilde{f} \ge 0. \tag{3.5}$$

Therefore, (M^n, g, f) is compact and f is f-subharmonic, since M^n has no boundary, we conclude that (M^n, g, f) is Einstein and f is a constant. □

Remark 3.4. As is shown in Theorem 1.3, the boundedness (from above) of f and $|\nabla f|$ on complete gradient shrinking Ricci solitons are equivalent.

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References

[1] K. Brighton, A Liouville-type theorem for smooth metric measure spaces, arXiv:1006.0751v2 [math.DG], 2010.

[2] H. D. Cao, Recent progress on Ricci solitons, Adv. Lect. Math. **11** (2010), 1–38.

[3] J. S. Case, The nonexistence of quasi-Einstein metrics, Pacific J. Math. **248** (2010), 277–284.

[4] J. S. Case, Conformally warped manifolds and quasi-Einstein metrics, arXiv:1011.2723v1 [math.DG], 2010.

[5] B. L. Chen, Strong uniqueness of the Ricci flow, J. Differential. Geom. **82** (2009), 363–382.

[6] J. S. Case, Y. Shu and G. Wei, Rigidity of quasi-Einstein metrics, Differential Geom. Appl. **29** (2011), 93–100.

[7] M. Fernández-López and E. García-Río, A remark on compact Ricci solitons, Math. Ann. **340** (2008), 893–896.

[8] R. S. Hamilton, The formation of singularities in the Ricci flow, Surv. Differ. Geom. **2** (1995), 7–136.

[9] C. X. He, P. Petersen and W. Wylie, On the classification of warped product Einstein metrics, arXiv:1010.5488v1 [math.DG], 2010.

[10] D. S. Kim and Y. H. Kim, Compact Einstein warped product spaces with nonpositive scalar curvature, Proc. Amer. Math. Soc. **131** (2003), 2573–2576.

[11] L. Ma and D. Z. Chen, Remarks on complete non-compact gradient Ricci expanding solitons, Kodai Math. J. **33** (2010), 173–181.
[12] P. Mastrolia and M. Rimoldi, Some triviality results for quasi-Einstein manifolds and Einstein warped products, arXiv:1011.0903v1 [math.DG], 2010.
[13] P. Petersen and W. Wylie, Rigidity of gradient Ricci solitons, Pacific J. Math. **241** (2009), 329–345.
[14] S. Pigola, M. Rigoli, M. Rimoldi and A. G. Setti, Remarks on non-compact gradient Ricci solitons, Math. Z. **268** (2011), 777–790.
[15] R. M. Schoen and S. T. Yau, Lectures on differential geometry, Cambridge, MA: International Press, 1994.
[16] P. Wu, On the potential function of gradient steady Ricci solitons, J. Geom. Anal., doi:10.1007/s12220-011-9243-7, 2011.
[17] G. F. Wei and W. Wylie, Comparison geometry for the Bakry-Émery Ricci tensor, J. Differential Geom. **83(2)** (2009), 377–405.
[18] Z. H. Zhang, On the completeness of gradient Ricci solitons, Proc. Amer. Math. Soc. **137** (2009), 2755–2759.

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