

A SOLUTION TO AN AMBARZUMYAN PROBLEM ON TREES

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Abstract

We consider the Neumann Sturm-Liouville problem defined on trees such that the ratios of lengths of edges are not necessarily rational. It is shown that the potential function of the Sturm-Liouville operator must be zero if the spectrum is equal to that for zero potential. This extends previous results and gives an Ambarzumyan theorem for the Neumann Sturm-Liouville problem on trees. To prove this, we compute approximated eigenvalues for zero potential by using a generalized pigeon hole argument, and make use of recursive formulas for characteristic functions.

1. Introduction

Recently there is a lot of interest in the study of Sturm-Liouville problem on graphs (see e.g. [2, 11, 17, 12, 13, 18]). On one hand, the problem is a natural extension of the classical Sturm-Liouville problem on an interval. On the other hand, it has a number of applications in thin domains, networks, spider webs, curvature-driven motion with triple junctions and even nanostructures (see [6, 13, 9, 10, 21]). This Sturm-Liouville problem defined on a graph was called a *quantum graph* by Gutkin and Smilansky [7]. In [14], Kuchment and Post studied the spectral properties of quantum graphs on graphene as well as carbon nanotubes. The interested reader may consult [6] for excellent surveys and lists of references.

When the quantum graph is a compact metric tree (a connected graph with no cycles), the spectrum consists of point spectra only. We are interested in an inverse spectral problem for certain quantum graphs. Namely, we study the problem of a Neumann Sturm-Liouville operator defined on a metric tree $\Gamma = \{V, E\}$, where $V = \{\mathbf{v}_j : j = 0, \dots, J\}$ is the set of all vertices and $E = \{\gamma_i : i = 1, \dots, I\}$ is the set of all edges with lengths $a_i \in (0, \infty)$. We let $\partial\Gamma$ be the set of all pendant (boundary) vertices. Any edge connected to a pendant vertex is called a boundary edge. For any internal vertex \mathbf{v} , we let $I(\mathbf{v})$ be the set of all indices i such that the edge γ_i is incident to \mathbf{v} . The degree of \mathbf{v} is defined as $|I(\mathbf{v})|$. We also choose an arbitrary internal vertex \mathbf{v}_0 to be the root. All edges are directed towards the root \mathbf{v}_0 and away from the pendant

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vertices. Thus we may assign local coordinates $[0, a_i]$ to each edge γ_i , such that the local coordinate increases as the distance to the root decreases. So all pendant vertices have local coordinate 0, while the root \mathbf{v}_0 has local coordinate a_i corresponding to the edge γ_i connected to it. For each internal vertex \mathbf{v} other than the root, there is an outgoing edge which gets closer to the root. All the other edges in $I(\mathbf{v})$ are incoming edges. Therefore this Neumann Sturm-Liouville problem can be expressed as a system of equations for y_i 's given by

$$(1.1) \quad -y_i'' + q_i(x)y_i = \lambda y_i \quad 0 < x < a_i, \quad i = 1, 2, \dots, I,$$

where each q_i is a real L^1 function defined on γ_i ($i = 1, \dots, I$), subject to the following boundary and matching conditions:

- (A1) $y_i'(0) = 0$ whenever γ_i is a boundary edge.
- (A2) The continuity and Kirchhoff conditions are satisfied at every internal vertex \mathbf{v} (other than \mathbf{v}_0) as

$$y_k(0) = y_i(a_i), \quad \text{and} \quad y_k'(0) = \sum_{i \neq k} y_i'(a_i),$$

where γ_k is an outgoing edge while all the other edges $i \in I(\mathbf{v}) \setminus \{k\}$ are incoming edges.

- (A3) At the root \mathbf{v}_0 , all the connecting edges are incoming. Hence the continuity and Kirchhoff conditions are defined as: for any $i, k \in I(\mathbf{v}_0)$,

$$y_k(a_k) = y_i(a_i), \quad \text{and} \quad \sum_{i \in I(\mathbf{v}_0)} y_i'(a_i) = 0,$$

The above formulation of the Sturm-Liouville problem on Γ is essentially the same as in [15]. Note that the Neumann boundary conditions (A1) can be viewed as a special case of (A2), for the degree of a pendant vertex is 1, hence the continuity condition is empty and right-hand side of the Kirchhoff condition vanishes.

A real number λ is said to be an eigenvalue of the above problem if it has a nontrivial solution (y_1, \dots, y_I) . We write $Q = (q_1, \dots, q_I)$, and let $\sigma(Q)$ be the set of eigenvalues for the vector potential function Q . In particular, $\sigma(\mathbf{0})$ denotes the set of eigenvalues for $Q = \mathbf{0}$, i.e., $q_i \equiv 0$ for all $i = 1, \dots, I$.

In the simplest case $I = 1$, the above problem is reduced to a usual Sturm-Liouville problem on a finite interval $(0, a_1)$ with the Neumann boundary condition. In this case, in 1929, Ambarzumyan [1] showed that $\sigma(Q) = \sigma(0)$ implies $Q = 0$ almost everywhere. This seems to be the first example of an inverse problem where the potential function can be determined uniquely by the spectrum, without any additional data. Later, Chern and Shen [5] extended the result to vectorial Sturm-Liouville systems. In the case of the Dirichlet boundary condition, it was shown by Chern et al. [4] that the potential function must be identically equal to 0 if an additional condition

$$\int_0^{a_1} q_1(x) \cos\left(\frac{2\pi}{a_1}x\right) dx = 0$$

is imposed. Recently, the Ambarzumyan problem for periodic boundary conditions was studied by Yang et al [22]. They showed that for a vectorial Sturm-Liouville system of dimension d with the periodic boundary condition, if the eigenvalues are $(2n\pi)^2$ with multiplicities $2d$, then the vector potential must be 0. Thus the Ambarzumyan theorem, originally specified for the Neumann boundary condition, can be generalized to several Sturm-Liouville problems with different boundary conditions.

The aim of this paper is to study the Ambarzumyan problem for the Neumann Sturm-Liouville problems defined on trees. In this direction, Pivovarchik [16] showed that when Γ consists of three edges with equal length and one triple junction, an analogue of Ambarzumyan theorem is valid for the problem (1.1) with (A1)~(A3). Later, Carlson and Pivovarchik [3] extended the result to any trees such that $\{a_i/L\}$ are all rational numbers, where L is the total length of Γ given by

$$L := \sum_{i=1}^I a_i.$$

In this case, we can find infinitely many eigenvalues explicitly given by

$$(1.2) \quad \left(\frac{mm_0}{L}\pi\right)^2 \in \sigma(0), \quad m = 0, 1, 2, \dots$$

for some integer m_0 , which makes the analysis much easier than a more general case. As for the Dirichlet problem on star-shaped graphs, we refer to a recent paper by Hung et al [8].

In this paper we consider the Sturm-Liouville problem on trees such that $\{a_i/L\}$ are not necessarily rational. The following theorem is the main result of this paper, which gives a solution to the Ambarzumyan problem on compact metric trees.

THEOREM 1.1. *For the Neumann Sturm-Liouville operator defined on Γ , $\sigma(Q) = \sigma(\mathbf{0})$ implies $Q = \mathbf{0}$ almost everywhere.*

When we deal with compact metric trees, we encounter two difficulties. The first one is that we must handle trees with arbitrary number of edges. Moreover, even if the number of edges is given, the topologies might still be different. In order to handle this, we shall derive a recursive formula for characteristic functions whose zeros are the square roots of eigenvalues (see Lemma 2.1). The second difficulty is that the eigenvalues now are not explicit at all. Only when the edgelengths are in rational ratio, the eigenvalues would be as in (1.2). To overcome this difficulty, we approximate $\{a_i/L\}$ precisely by rational numbers at the same time by applying a (generalized) pigeon hole argument.

We believe that the recursive formula as given in Lemma 2.1 will be useful in future studies about quantum trees (see also Remark 2.2).

In Section 2, we shall study direct problems and derive recursive formulas for characteristic functions. In Section 3, we compute the expansion of characteristic functions. In Section 4, we present a key lemma for the approximation of eigenvalues. Finally Section 5 is devoted to the proof of Theorem 1.1.

2. Recursive formula for characteristic functions

Let $\rho > 0$, and let $y = C_i(x; \rho)$ and $y = S_i(x; \rho)$ be (linearly independent) solutions of

$$(2.1) \quad -y'' + q_i(x)y = \rho^2 y, \quad 0 < x < a_i$$

with the initial conditions

$$y(0) = 1, \quad y'(0) = 0,$$

and

$$y(0) = 0, \quad y'(0) = 1,$$

respectively. We sometimes write $C_i(x; \rho)$ as $C_i(x)$ and $S_i(x; \rho)$ as $S_i(x)$ just for simplicity. For each edge γ_i , we write y_i as

$$y_i = A_i C_i(x) + B_i S_i(x).$$

Then from (A1)~(A3), we have a system of linear equations for the unknowns $\{A_i\}$ and $\{B_i\}$. Thus we may express the coefficient matrix of the system of linear equations as

$$(2.2) \quad \Phi_N(\rho) = \begin{bmatrix} C'_1(0) & S'_1(0) & 0 & 0 & 0 & \dots & 0 \\ C_1(a_1) & S_1(a_1) & -C_2(0) & -S_2(0) & 0 & \dots & 0 \\ C'_1(a_1) & S'_1(a_1) & -C'_2(0) & -S'_2(0) & * & \dots & * \\ 0 & 0 & * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & * & * & \dots & * \end{bmatrix},$$

where the first row corresponds to the Neumann boundary condition at $v_1 \in \Gamma$, and the second and third rows correspond to the continuity condition and the Kirchhoff condition at v_2 . Note that in the first row, we include the term $S'_1(0)$ although the corresponding coefficient $B_1 = 0$. We do this in order to have a systematic form of Φ_N during reductions, as we shall see later. We define a characteristic function by

$$\varphi_N(\rho) := \det \Phi_N(\rho),$$

so that $\lambda = \rho^2 \in \sigma(Q)$ if and only if $\varphi_N(\rho) = 0$. We note that the characteristic function depends on the orientation of edges and how to express the coefficient matrix, but the set of zeros of $\varphi_N(\rho)$ does not depend on them.

Next we introduce another eigenvalue problem by replacing (A1) with the following condition:

- (A4) Let v_1 be a vertex of a boundary edge γ_1 in a subtree, with local coordinate 0 (or a_1). The solution of (1.1) satisfies the zero Dirichlet boundary condition at v_1 :

$$y_1(0) = 0 \quad (\text{or } y_1(a_1) = 0),$$

while at all other pendant vertices, the solution satisfies the homogeneous Neumann boundary condition as in (A1).

Hereafter, we call (1.1) with (A2)~(A4) the Dirichlet-Neumann problem. Suppose that we impose zero Dirichlet condition at this pendant vertex $v_1 \in \partial\Gamma$, then $A_1 = 0$. Thus we may express the corresponding coefficient matrix by

$$\Phi_D(\rho) = \begin{bmatrix} C_1(0) & S_1(0) & 0 & 0 & 0 & \cdots & 0 \\ C_1(a_1) & S_1(a_1) & -C_2(0) & -S_2(0) & 0 & \cdots & 0 \\ C_1'(a_1) & S_1'(a_1) & -C_2'(0) & -S_2'(0) & * & \cdots & * \\ 0 & 0 & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & * & * & * & \cdots & * \end{bmatrix}.$$

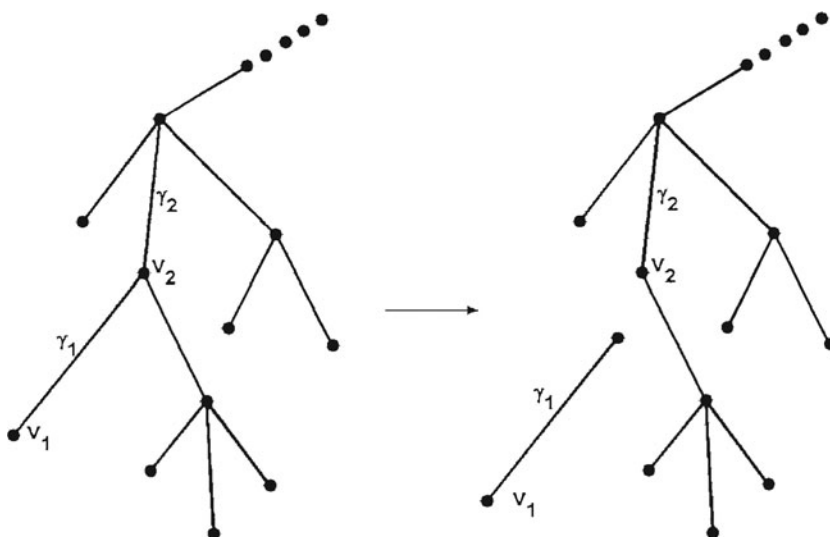


FIGURE 1

Then we define a characteristic function for the Dirichlet-Neumann problem by

$$\varphi_D(\rho) := \det \Phi_D(\rho).$$

Again, the set of zeros of $\varphi_D(\rho)$ does not depend on the orientation of edges and how to express the coefficient matrix. Our interest will be only in zeros of the characteristic functions, and hence the non-uniqueness of characteristic functions will not affect the following argument.

For a compact metric tree, it is not easy to express explicitly the characteristic functions φ_N and φ_D . Instead, we may compute these functions recursively as follows. Let \mathbf{v}_1 be a pendant vertex which is an endpoint of the edge γ_1 , while \mathbf{v}_2 is the other vertex of γ_1 . We also let γ_2 be incident to \mathbf{v}_2 , and is closer to the root \mathbf{v}_0 than γ_1 (cf. fig. 1). Let $\tilde{\Gamma}$ be a subtree of Γ obtained by removing γ_1 . We denote by $\tilde{\varphi}_N$ the corresponding characteristic function of the problem with Neumann condition at \mathbf{v}_2 in case $I(\mathbf{v}_2) = \{1, 2\}$. However if the degree of \mathbf{v}_2 is greater than 2, then we take the continuity and Kirchhoff conditions at \mathbf{v}_2 instead. Thus we say $\tilde{\varphi}_N$ the characteristic function for a Neumann/Kirchhoff problem on $\tilde{\Gamma}$.

If we remove γ_1 and replace the matching conditions (A2) and (A3) by zero Dirichlet condition at the vertex \mathbf{v}_2 , then $\tilde{\Gamma}$ is broken into several subtrees, all of which have \mathbf{v}_2 as a pendant vertex. For each subtree, a characteristic function of the Dirichlet-Neumann problem is defined as above. We denote by $\tilde{\varphi}_D(\rho)$ the product of these characteristic functions.

For trees with two or more edges, we have the following recursive formulas.

LEMMA 2.1. *Assume that $I \geq 2$. Then the characteristic functions have the following properties:*

- (a) $\varphi_N(\rho) = C_1(a_1)\tilde{\varphi}_N(\rho) - C'_1(a_1)\tilde{\varphi}_D(\rho)$.
- (b) $\varphi_D(\rho) = -S_1(a_1)\tilde{\varphi}_N(\rho) + S'_1(a_1)\tilde{\varphi}_D(\rho)$.

Proof. Since $C'_1(0) = 0$ and $S'_1(0) = 1$, we have

$$\begin{aligned} \det \Phi_N(\rho) &= -\det \begin{bmatrix} C_1(a_1) & -C_2(0) & -S_2(0) & 0 & \cdots & 0 \\ C'_1(a_1) & -C'_2(0) & -S'_2(0) & * & \cdots & * \\ 0 & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & * & \cdots & * \end{bmatrix} \\ &= C_1(a_1) \det \tilde{\Phi}_N(\rho) - C'_1(a_1) \det \tilde{\Phi}_D(\rho), \end{aligned}$$

after an expansion with respect to the first row, where $\tilde{\Phi}_N$ and $\tilde{\Phi}_D$ are $(2I - 2) \times (2I - 2)$ matrices given by

$$\tilde{\Phi}_N(\rho) = \begin{bmatrix} C'_2(0) & S'_2(0) & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{bmatrix}$$

and

$$\tilde{\Phi}_D(\rho) = \begin{bmatrix} C_2(0) & S_2(0) & 0 & \cdots & 0 \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{bmatrix},$$

respectively. Noting that $\tilde{\Phi}_N$ describes conditions on the Neumann/Kirchhoff problem for $\tilde{\Gamma}$, we have

$$\det \tilde{\Phi}_N(\rho) = \tilde{\varphi}_N(\rho).$$

Similarly, since $\tilde{\Phi}_D$ describes conditions on the Dirichlet-Neumann problems for subtrees of $\tilde{\Gamma}$, with Dirichlet conditions at \mathbf{v}_2 , we have

$$\det \tilde{\Phi}_D(\rho) = \tilde{\varphi}_D(\rho).$$

Thus the proof of (a) is completed.

Next, let us consider the Dirichlet-Neumann problem. In this case, we have the expansion, since $C_1(0) = 1$ and $S_1(0) = 0$,

$$\begin{aligned} \det \Phi_D(\rho) &= \det \begin{bmatrix} S_1(a_1) & -C_2(0) & -S_2(0) & 0 & \cdots & 0 \\ S'_1(a_1) & -C'_2(0) & -S'_2(0) & * & \cdots & * \\ 0 & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & * & \cdots & * \end{bmatrix} \\ &= -S_1(a_1) \det \tilde{\Phi}_N(\rho) + S'_1(a_1) \det \tilde{\Phi}_D(\rho). \end{aligned}$$

This proves (b). □

Remark 2.2. More general recursive formulas were obtained in a recent paper by Law and Pivovarchik [15]. Interested readers might like to read a spectral determinant approach to the same formulas [20].

Now we compute the characteristic function of a compact metric tree as follows. Given a tree with two or more edges, we remove one of the edges of Γ and use the recursive formulas. Repeating this procedure, we will reach to problems on single edges. For a single edge γ_i , we may define its characteristic functions by

$$(2.3) \quad \tilde{\varphi}_N(\rho) := -C'_i(a_i), \quad \tilde{\varphi}_D := S'_i(a_i).$$

Thus we can express $\varphi_N(\rho)$ and $\varphi_D(\rho)$ as polynomials of $\{C_i(a_i)\}$ and $\{S_i(a_i)\}$.

Next, we consider the zero potential $Q = \mathbf{0}$, and denote by $\psi_N(\rho)$ and $\psi_D(\rho)$ the corresponding characteristic functions of the Neumann/Kirchhoff problem and the Dirichlet-Neumann problem, respectively. Similarly, we denote by $\tilde{\psi}_N$ and $\tilde{\psi}_D$ be characteristic functions for the Neumann and Dirichlet-Neumann

problems for Γ with γ_1 removed. For a tree with a single edge γ_i , to be consistent with (2.3), we define its characteristic functions by

$$(2.4) \quad \tilde{\psi}_N(\rho) := \sin(\rho a_i), \quad \tilde{\psi}_D(\rho) := \cos(\rho a_i).$$

For trees with two or more edges, the characteristic functions can be computed by using the following recursive formulas repeatedly and (2.4).

LEMMA 2.3. *Assume $I \geq 2$. Then the characteristic functions for $Q = \mathbf{0}$ have the following properties:*

- (a) $\psi_N(\rho) = \cos(\rho a_1)\tilde{\psi}_N(\rho) + \sin(\rho a_1)\tilde{\psi}_D(\rho)$.
- (b) $\psi_D(\rho) = -\sin(\rho a_1)\tilde{\psi}_N(\rho) + \cos(\rho a_1)\tilde{\psi}_D(\rho)$.

Proof. For $q_i(x) \equiv 0$, the solutions of (2.1) are given by $C_i(x; \rho) = \cos(\rho x)$ and $S_i(x; \rho) = \sin(\rho x)$. Then the above recursive formulas can be obtained in the same way as Lemma 2.1. □

3. Expansion of characteristic functions

In this section we show the following asymptotic formulas for the characteristic functions $\varphi_N(\rho)$ and $\varphi_D(\rho)$, as $\rho \rightarrow \infty$. First, the following lemma is not new and can be easily derived from the integral equations:

$$C_i(x) = \cos(\rho x) + \frac{1}{\rho} \int_0^x \sin(\rho(x-t))q_i(t)C_i(t) dt,$$

$$S_i(x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho^2} \int_0^x \sin(\rho(x-t))q_i(t)S_i(t) dt.$$

LEMMA 3.1. *As $\rho \rightarrow \infty$, one has*

$$C_i(a_i; \rho) = \cos(\rho a_i) + \rho^{-1}K_i \sin(\rho a_i) + o(\rho^{-1}),$$

$$C'_i(a_i; \rho) = -\rho \sin(\rho a_i) + K_i \cos(\rho a_i) + o(1),$$

$$S_i(a_i; \rho) = \rho^{-1} \sin(\rho a_i) - \rho^{-2}K_i \cos(\rho a_i) + o(\rho^{-2}),$$

$$S'_i(a_i; \rho) = \cos(\rho a_i) + \rho^{-1}K_i \sin(\rho a_i) + o(\rho^{-1}),$$

where $K_i := \frac{1}{2} \int_0^{a_i} q_i(x) dx$.

LEMMA 3.2. *The characteristic functions φ_N and φ_D have the following properties:*

- (a) $\varphi_N(\rho) = \rho\psi_N(\rho) - \left(\sum_{i=1}^I K_i\right)\psi_D(\rho) + o(1)$ as $\rho \rightarrow \infty$.
- (b) $\varphi_D(\rho) = \psi_D(\rho) + \rho^{-1}\left(\sum_{i=1}^I K_i\right)\psi_N(\rho) + o(\rho^{-1})$ as $\rho \rightarrow \infty$.

Proof. If Γ consists of a single edge, then by Lemma 3.1, we have

$$\begin{aligned}\varphi_N(\rho) &= -C'_1(a_1) = \rho \sin(\rho a_1) - K_1 \cos(\rho a_1) + o(1), \\ \varphi_D(\rho) &= S'_1(a_1) = \cos(\rho a_1) + \rho^{-1} K_1 \sin(\rho a_1) + o(\rho^{-1}),\end{aligned}$$

so that

$$\begin{aligned}\varphi_N(\rho) &= \rho \psi_N(\rho) - K_1 \psi_D(\rho) + o(1), \\ \varphi_D(\rho) &= \psi_D(\rho) + \rho^{-1} K_1 \psi_N(\rho) + o(\rho^{-1}).\end{aligned}$$

Hence (a) and (b) hold in this case.

Suppose now that (a) and (b) hold for any subtree of Γ . Then by Lemmas 2.1 and 2.3, we have

$$\begin{aligned}\varphi_N(\rho) &= C_1(a_1) \tilde{\varphi}_N(\rho) - C'_1(a_1) \tilde{\varphi}_D(\rho) \\ &= \{\cos(\rho a_1) + \rho^{-1} K_1 \sin(\rho a_1) + o(1)\} \left\{ \rho \tilde{\psi}_N(\rho) - \left(\sum_{i=2}^I K_i \right) \tilde{\psi}_D(\rho) + o(1) \right\} \\ &\quad - \{-\rho \sin(\rho a_1) + K_1 \cos(\rho a_1) + o(1)\} \\ &\quad \times \left\{ \tilde{\psi}_D(\rho) + \rho^{-1} \left(\sum_{i=2}^I K_i \right) \tilde{\psi}_N(\rho) + o(\rho^{-1}) \right\} \\ &= \rho \cos(\rho a_1) \tilde{\psi}_N(\rho) + K_1 \sin(\rho a_1) \tilde{\psi}_N(\rho) - \cos(\rho a_1) \left(\sum_{i=2}^I K_i \right) \tilde{\psi}_D(\rho) \\ &\quad + \rho \sin(\rho a_1) \tilde{\psi}_D(\rho) - K_1 \cos(\rho a_1) \tilde{\psi}_D(\rho) + \sin(\rho a_1) \left(\sum_{i=2}^I K_i \right) \tilde{\psi}_N(\rho) + o(1) \\ &= \rho \psi_N(\rho) - K_1 \psi_D(\rho) - \left(\sum_{i=2}^I K_i \right) \psi_D(\rho) + o(1), \\ &= \rho \psi_N(\rho) - \left(\sum_{i=1}^I K_i \right) \psi_D(\rho) + o(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\varphi_D(\rho) &= -S_1(a_1) \tilde{\varphi}_N(\rho) + S'_1(a_1) \tilde{\varphi}_D(\rho) \\ &= \{-\rho^{-1} \sin(\rho a_1) - \rho^{-2} K_1 \cos(\rho a_1) + o(\rho^{-2})\} \\ &\quad \times \left\{ \rho \tilde{\psi}_N(\rho) - \left(\sum_{i=2}^I K_i \right) \tilde{\psi}_D(\rho) + o(1) \right\} \\ &\quad + \{\cos(\rho a_1) + \rho^{-1} K_1 \sin(\rho a_1) + o(1)\}\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \tilde{\psi}_D(\rho) + \rho^{-1} \left(\sum_{i=2}^I K_i \right) \tilde{\psi}_N(\rho) + o(\rho^{-1}) \right\} \\
& = -\sin(\rho a_1) \tilde{\psi}_N(\rho) \\
& \quad + \rho^{-1} \left\{ K_1 \cos(\rho a_1) \tilde{\psi}_N(\rho) + \sin(\rho a_1) \left(\sum_{i=2}^I K_i \right) \tilde{\psi}_D(\rho) \right\} + o(\rho^{-1}) \\
& \quad + \cos(\rho a_1) \tilde{\psi}_D(\rho) \\
& \quad + \rho^{-1} \left\{ K_1 \sin(\rho a_1) \tilde{\psi}_D(\rho) + \cos(\rho a_1) \left(\sum_{i=2}^I K_i \right) \tilde{\psi}_N(\rho) \right\} + o(1) \\
& = \psi_D(\rho) + \rho^{-1} \left\{ K_1 \psi_N(\rho) + \left(\sum_{i=2}^I K_i \right) \psi_N(\rho) \right\} + o(\rho^{-1}) \\
& = \psi_D(\rho) + \rho^{-1} \left(\sum_{i=1}^I K_i \right) \psi_N(\rho) + o(\rho^{-1}).
\end{aligned}$$

Hence the assertion holds for the whole tree. Thus by induction, the proof is complete. \square

4. Approximation of eigenvalues

In this section we compute approximate eigenvalues for Γ with $Q = \mathbf{0}$. We begin with the following lemma, which is a slight generalization of the Dirichlet approximation theorem (see e.g. [19]).

LEMMA 4.1. *There exist infinite sequences of natural numbers $\{m_n\}$ and $\{k_{i,n}\}$ ($i = 1, \dots, I$) such that $m_n \rightarrow \infty$ as $n \rightarrow \infty$ and*

$$\left| \frac{a_i}{L} - \frac{k_{i,n}}{m_n} \right| < m_n^{-1-1/I}$$

for all $i = 1, 2, \dots, I$ and $n = 1, 2, \dots$

Proof. If a_i/L ($i = 1, \dots, I$) are all rational numbers, we can find natural numbers p and q_i such that

$$\frac{a_i}{L} = \frac{q_i}{p}, \quad i = 1, 2, \dots, I.$$

Then we may take

$$k_{i,n} = nq_i, \quad m_n = np.$$

Assume that not all of a_i/L ($i = 1, \dots, I$) are rational. We consider the I -dimensional unit cube

$$\{x = (x_1, x_2, \dots, x_I) \in \mathbf{R}^I : 0 \leq x_i \leq 1 \text{ for } i = 1, 2, \dots, I\},$$

and divide it into n^I smaller cubes with edge length $1/n$. We shall approximate the vector $(a_1/L, \dots, a_I/L)$ by a rational point in the unit cube as follows. For each $p = 0, 1, 2, \dots, n^I$, there exists $q_i \in \mathbf{N} \cup \{0\}$ such that

$$0 \leq p \frac{a_i}{L} - q_i < 1.$$

Then by the pigeonhole principle, there exist two distinct vectors U_1 and U_2 of the form

$$U_j = \left(p_j \frac{a_1}{L} - q_{1,j}, \dots, p_j \frac{a_I}{L} - q_{I,j} \right), \quad j = 1, 2,$$

that fall into the same small cube. Let $p_2 > p_1$. Then for $m_n := p_2 - p_1 \leq n^I$ and $k_{i,n} := q_{i,2} - q_{i,1}$, we have

$$(4.1) \quad \left| m_n \frac{a_i}{L} - k_{i,n} \right| < \frac{1}{n},$$

so that

$$\left| \frac{a_i}{L} - \frac{k_{i,n}}{m_n} \right| < \frac{1}{m_n n} \leq \frac{1}{m_n^{1+1/I}}.$$

The proof is complete since from (4.1), $m_n \rightarrow \infty$ as $n \rightarrow \infty$. □

Remark 4.2.

(a) Since

$$\left| m_n - \sum_{i=1}^I k_{i,n} \right| \leq m_n \sum_{i=1}^I \left| \frac{a_i}{L} - \frac{k_{i,n}}{m_n} \right| < \frac{I}{n},$$

the equality

$$(4.2) \quad m_n = \sum_{i=1}^I k_{i,n}$$

holds for all $n \geq I$.

(b) Using $\sum_{i=1}^I (a_i/L) = 1$, we may apply the pigeonhole principle on the $(I-1)$ -dimensional hyperplane

$$\{x = (x_1, x_2, \dots, x_I) \in \mathbf{R}^I : x_1 + x_2 + \dots + x_I = 1\}.$$

Then we can improve the result in Lemma 4.1 to

$$\left| \frac{a_i}{L} - \frac{k_{i,n}}{m_n} \right| = O(m_n^{-1-1/(I-1)}).$$

Let $\{m_n\}$ be the sequence given in Lemma 4.1. We define a sequence $\{\mu_n\}$

$$\mu_n := \frac{2m_n\pi}{L}, \quad n = 1, 2, \dots,$$

and compute the values of ψ_N, ψ_D , and their derivatives at $\rho = \mu_n$ as follows.

LEMMA 4.3. *The characteristic functions ψ_N and ψ_D have the following properties:*

- (a) $\psi_N(\mu_n) = O(\mu_n^{-1/I})$ and $\frac{d\psi_N}{d\rho}(\mu_n) = L + O(\mu_n^{-1/I})$ as $n \rightarrow \infty$.
- (b) $\psi_D(\mu_n) = 1 + O(\mu_n^{-1/I})$ and $\frac{d\psi_D}{d\rho}(\mu_n) = O(\mu_n^{-1/I})$ as $n \rightarrow \infty$.

Proof. By Lemma 2.3 and

$$\mu_n a_i = \frac{m_n\pi}{L} a_i = 2k_{i,n}\pi + O(\mu_n^{-1/I}),$$

we have

$$\psi_N(\mu_n) = \cos(\mu_n a_1) \tilde{\psi}_N(\mu_n) + \sin(\mu_n a_1) \tilde{\psi}_D(\mu_n) = \tilde{\psi}_N(\mu_n) + O(\mu_n^{-1/I}) \tilde{\psi}_D(\mu_n),$$

and

$$\psi_D(\mu_n) = -\sin(\mu_n a_1) \tilde{\psi}_N(\mu_n) + \cos(\mu_n a_1) \tilde{\psi}_D(\mu_n) = \tilde{\psi}_D(\mu_n) + O(\mu_n^{-1/I}) \tilde{\psi}_N(\mu_n).$$

Using these equalities repeatedly on (2.4), we obtain

$$\psi_N(\mu_n) = O(\mu_n^{-1/I}), \quad \tilde{\psi}_N(\mu_n) = O(\mu_n^{-1/I}),$$

and

$$\psi_D(\mu_n) = 1 + O(\mu_n^{-1/I}), \quad \tilde{\psi}_D(\mu_n) = 1 + O(\mu_n^{-1/I}).$$

Also, we have

$$\begin{aligned} \frac{d\psi_N}{d\rho}(\mu_n) &= \frac{d}{d\rho} \{ \cos(\rho a_1) \tilde{\psi}_N + \sin(\rho a_1) \tilde{\psi}_D \}(\mu_n) \\ &= -a_1 \sin(\mu_n a_1) \tilde{\psi}_N(\mu_n) + \cos(\mu_n a_1) \frac{d\tilde{\psi}_N}{d\rho}(\mu_n) \\ &\quad + a_1 \cos(\mu_n a_1) \tilde{\psi}_D(\mu_n) + \sin(\mu_n a_1) \frac{d\tilde{\psi}_D}{d\rho}(\mu_n) \\ &= \frac{d\tilde{\psi}_N}{d\rho}(\mu_n) + a_1 + O(\mu_n^{-1/I}) \end{aligned}$$

and

$$\begin{aligned}
\frac{d\psi_D}{d\rho}(\mu_n) &= \frac{d}{d\rho} \{-\sin(\rho a_1)\tilde{\psi}_N + \cos(\rho a_1)\tilde{\psi}_D\}(\mu_n) \\
&= -a_1 \cos(\mu_n a_1)\tilde{\psi}_N(\mu_n) - \sin(\mu_n a_1)\frac{d\tilde{\psi}_N}{d\rho}(\mu_n) \\
&\quad - a_1 \sin(\mu_n a_1)\tilde{\psi}_D(\mu_n) + \cos(\mu_n a_1)\frac{d\tilde{\psi}_D}{d\rho}(\mu_n) \\
&= \frac{d\tilde{\psi}_D}{d\rho}(\mu_n) + O(\mu_n^{-1/I}).
\end{aligned}$$

Using these equalities repeatedly on (2.4), we obtain

$$\frac{d\psi_N}{d\rho}(\mu_n) = \sum_{i=1}^I a_i + O(\mu_n^{-1/I})$$

and

$$\frac{d\psi_D}{d\rho}(\mu_n) = O(\mu_n^{-1/I}).$$

The proof is complete. □

As an immediate consequence of Lemma 4.3, we have the following result concerning the location of zeros of ψ_N .

LEMMA 4.4. *There exists a sequence of positive numbers $\{\rho_n\}$ such that $\psi_N(\rho_n) = 0$ for $n = 1, 2, \dots$ and $\rho_n = \mu_n + O(\mu_n^{-1/I})$ as $n \rightarrow \infty$.*

Proof. By Lemma 4.3, there exists $M > 0$ independent of n such that $|\psi_N(\mu_n)| \leq M\mu_n^{-1/I}$ for all large n . On the other hand, by (2.4) and Lemma 2.3, $\psi_N(\rho)$ and $\psi_D(\rho)$ must be some polynomials of $\sin(\rho a_i)$ and $\cos(\rho a_i)$ ($i = 1, 2, \dots, I$) with constant coefficients. This implies that $|(d^2/d\rho^2)\psi_N|$ is uniformly bounded in $\rho > 0$. Hence by Lemma 4.3, there exists $\delta > 0$ independent of n such that

$$\frac{d}{d\rho}\psi_N(\rho) > \frac{L}{2} \quad \text{for } \rho \in (\mu_n - \delta, \mu_n + \delta).$$

Therefore $\psi_N(\rho)$ must vanish at some ρ_n such that

$$|\rho_n - \mu_n| \leq (2M/L)\mu_n^{-1/I}.$$

This completes the proof. □

5. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. By Lemma 3.2, φ_N satisfies

$$\varphi_N(\rho) = \rho\psi_N(\rho) - \left(\sum_{i=1}^I K_i\right)\psi_D(\rho) + o(1) \quad \text{as } \rho \rightarrow \infty.$$

Here, setting $\rho = \rho_n$, we have $\varphi_N(\rho_n) = 0 = \psi_N(\rho_n)$ by $\sigma(Q) = \sigma(0)$. Also, by Lemmas 4.3 (b) and 4.4, we have

$$\psi_D(\rho_n) = 1 + O(\rho_n^{-1/I}) \quad \text{as } n \rightarrow \infty.$$

Thus we obtain

$$\left(\sum_{i=1}^I K_i\right)\{1 + O(\rho_n^{-1/I})\} + o(1) = 0.$$

Letting $n \rightarrow \infty$, we conclude

$$(5.1) \quad \sum_{i=1}^I K_i = \frac{1}{2} \sum_{i=1}^I \int_0^{a_i} q_i(x) \, dx = 0.$$

On the other hand, since $\lambda = 0$ is the first eigenvalue, it follows from the variational principle that the inequality

$$\frac{\sum_{i=1}^I \int_0^{a_i} \{(y'_i)^2 + q_i(x)(y_i)^2\} \, dx}{\sum_{i=1}^I \int_0^{a_i} (y_i)^2 \, dx} \geq 0$$

holds for any test function in $H^1(\Gamma)$. By (5.1), the infimum is attained by $y_i(x) = 1$ for $i = 1, \dots, I$. This implies that the constant vector $(1, \dots, 1)$ is an eigenfunction associated with $\lambda = 0$. Then by simple substitution, we conclude that $q_i = 0$ a.e. on $[0, a_i]$, $i = 1, 2, \dots, I$. Thus the proof is complete. \square

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