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FIBERWISE GREEN FUNCTIONS OF SKEW PRODUCTS SEMICONJUGATE TO SOME POLYNOMIAL PRODUCTS ON $\mathbf{C}^2$

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Abstract

We consider the dynamics of polynomial skew products that are semiconjugate to some polynomial products on $\mathbf{C}^2$. We show that the fiberwise Green functions exist outside thin sets, whose upper semicontinuous regularizations are defined, continuous and plurisubharmonic on $\mathbf{C}^2$. This result is obtained from the existence of Green functions of polynomials outside thin sets.

1. Introduction

In [8] we considered the dynamics of a polynomial skew product on $\mathbf{C}^2$ of the form $f(z,w) = (p(z), q(z,w))$, where p and q are polynomials such that $p(z) = z^\delta + O(z^{\delta-1})$ and $q(z,w) = w^d + O_z(w^{d-1})$. Let $\delta \geq 2$ and $d \geq 2$. Then the dynamical degree λ_1 of f coincides with $\max\{\delta, d\}$. Let f^n be the n-th iterate of f. By definition, $f^n(z,w) = (p^n(z), Q_z^n(w))$, where $Q_z^n = q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$ and $q_z(w) = q(z,w)$. We investigated the existence of the Green function of f,

$$G_f(z,w) = \lim_{n\to\infty} \frac{1}{\lambda_1^n} \log^+|f^n(z,w)|,$$

and the *fiberwise Green function* of f,

$$G_z(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+|Q_z^n(w)|,$$

where $\log^+ = \max\{\log, 0\}$ and $|(z,w)| = \max\{|z|, |w|\}$. Besides giving an example of polynomial skew products whose Green and *fiberwise Green functions* are not defined on some curves in $\mathbf{C}^2$, we introduced the *weighted Green function* of f,

$$G_f^\alpha(z,w) = \lim_{n\to\infty} \frac{1}{\lambda_1^n} \log^+|f^n(z,w)|_\alpha,$$

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where $|(z,w)|_\alpha = \max\{|z|^\alpha, |w|\}$ and α is the rational number determined by the map f. Our main theorem in [8] shows that G_f^α is defined, continuous and plurisubharmonic on $\mathbf{C}^2$, which follows from the results on the existence and properties of G_z. Moreover, we showed that f extends to a rational map on the weighted projective space, which is holomorphic if and only if $\delta = d$, and that G_f^α determines the Fatou and Julia sets of the extension of f if $\delta \le d$.

However, the existence of the Green and *fiberwise Green functions* are still unclear.

In this study, we consider the dynamics of a polynomial skew product of the form $f(z,w) = (z^d, q(z,w))$, where $q(z,w) = w^d + O_z(w^{d-1})$, that is semiconjugate to a polynomial product $(z^d, h(w))$ by $\pi(z,w) = (z^r, z^s w)$ for some positive integers r and s. In this case, $h(w)$ is equal to $q(1,w) = w^d + O(w^{d-1})$. We investigate the existence of the *fiberwise Green function* G_z, which implies the existence of the Green function G_f. By the equality $|Q_z^n(w)| = |z^{\alpha d^n} h^n(z^{-\alpha}w)|$, where $\alpha = s/r$, we have the following theorem and corollary:

Theorem A. *The limit G_z is defined, continuous and plurisubharmonic on $\mathbf{C}^2 - E_f \cap (\{|z| > 1\} \times \mathbf{C})$, where*

$$E_f = \bigcup_{z \in \mathbf{C}} \{z\} \times z^\alpha E_h \quad \text{and} \quad E_h = \bigcap_{l \ge 0} \overline{\bigcup_{n \ge l} h^{-n}(0)}.$$

If $0 \notin E_h$, then G_z is defined, continuous and plurisubharmonic on $\mathbf{C}^2$, which coincides with G_f^α.

Corollary B. *It follows that the upper semicontinuous regularization*

$$\limsup_{w' \to w} \left(\limsup_{n \to \infty} \frac{1}{d^n} \log^+ |Q_z^n(w')| \right)$$

is defined, continuous and plurisubharmonic on $\mathbf{C}^2$. If $h(w) \ne w^d$, then it coincides with G_f^α.

Similar results hold for a *fiberwise Green function* $\tilde{G}_z(w) = \lim_{n\to\infty} d^{-n} \log|Q_z^n(w)|$. See Theorems 5.1 and 5.4 and Corollaries 5.2 and 5.5 for details.

These results on the existence of the *fiberwise Green functions* of f are obtained from an investigation of a Green function of the polynomial h,

$$\tilde{G}_h(w) = \lim_{n\to\infty} \frac{1}{d^n} \log|h^n(w)|.$$

Let A_h be the set of points whose orbits tend to infinity, and K_h be the set of points whose orbits are bounded.

Theorem C. *The limit $\tilde{G}_h$ is defined, continuous and subharmonic on $\mathbf{C} - E_h$. More precisely, if $h(w) = w^d$ then $\tilde{G}_h(w) = \log|w|$, and if $h(w) \ne w^d$ then $\tilde{G}_h = G_h$ on $\mathbf{C} - E_h$. If $0 \notin E_h$, then $\tilde{G}_h = G_h$ on $\mathbf{C}$.*

Corollary D. *It follows that the upper semicontinuous regularization*

$$\limsup_{w'\to w}\left(\limsup_{n\to\infty}\frac{1}{d^n}\log|h^n(w')|\right)$$

is defined, continuous and subharmonic on $\mathbf{C}$. *If* $h(w) \neq w^d$, *then it coincides with* G_h.

The organization of the paper is as follows. In Section 2 we recall the dynamics of polynomial skew products and some results in [8]. We begin the study of the dynamics of a skew product semiconjugate to a polynomial product of the form $(z^d, h(w))$ in Section 3, which contains necessary and sufficient conditions for a polynomial skew product of the form $(z^d, q(z,w))$ to be semiconjugate to a polynomial product. In Section 4 we analyze the existence of Green functions of polynomials, which induces the results on the existence of the Green and *fiberwise Green functions* of the polynomial skew product in Section 5.

2. Dynamics of polynomial skew products

In this section we briefly recall the dynamics of a polynomial skew product $f(z,w) = (p(z), q(z,w))$, where p and q are polynomials such that $p(z) = z^d + O(z^{d-1})$ and $q(z,w) = w^d + O_z(w^{d-1})$, and $d \geq 2$. Here we assume that $\deg p = \deg_w q$, where $\deg_w q$ denotes the degree of q with respect to w, although we did not impose this assumption in [8]. Roughly speaking, the dynamics of f consists of the dynamics on the base space and the fibers. The first component p defines the dynamics on the base space $\mathbf{C}$. Note that f preserves the set of vertical lines in $\mathbf{C}^2$. In this sense, we often use the notation $q_z(w)$ instead of $q(z,w)$. The restriction of f^n to a vertical line $\{z\} \times \mathbf{C}$ can be viewed as the composition of n polynomials on $\mathbf{C}$, $q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$.

A useful tool in the study of the dynamics of p on the base space is the Green function of p,

$$G_p(z) = \lim_{n\to\infty}\frac{1}{d^n}\log^+|p^n(z)|.$$

It is well known that G_p is defined, continuous and subharmonic on $\mathbf{C}$. More precisely, G_p is harmonic and positive on A_p and zero on K_p, where $A_p = \{z : p^n(z) \to \infty \text{ as } n \to \infty\}$ and $K_p = \{z : \{p^n(z)\}_{n\geq 1} \text{ bounded}\}$, and $G_p(z) = \log|z| + o(1)$ as $|z| \to \infty$. By definition, $G_p(p(z)) = dG_p(z)$. Note that $A_p \sqcup K_p = \mathbf{C}$ and G_p coincides with the Green function of K_p with a pole at infinity.

It is useful to consider the dynamics of the extension of p to a holomorphic map on the one-dimensional projective space $\mathbf{P}^1$. We define the Fatou set F_p of p as the maximal open set of $\mathbf{P}^1$ where the family of iterates of the extension of p is normal. A Fatou component of p means any connected component of the Fatou set of p. The Julia set J_p of p is defined as the complement of the Fatou set of p. It is well known that $F_p \cap \mathbf{C} = A_p \cup \operatorname{int} K_p$ and $J_p = \partial A_p = \partial K_p = \{z : G_p \text{ is not harmonic}\}$.

In a similar fashion, we consider the *fiberwise Green function* of f,

$$G_z(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+|Q_z^n(w)|,$$

where $Q_z^n = q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$. By definition, $G_{p(z)}(q_z(w)) = dG_z(w)$ if it exists. Roughly speaking, $G_f(z,w) = \max\{G_p(z), G_z(w)\}$. Since the limit G_p exists on $\mathbf{C}$, the existence of G_z implies that of G_f.

Known results about the existence of the limits G_f and G_z are as follows. If f is *regular* then G_f is defined, continuous and plurisubharmonic on $\mathbf{C}^2$. Several studies have been made on the dynamics of *regular* polynomial skew products (e.g. [3], [4], [5] and [1]). However, the existence of G_z is unclear even if f is *regular*. Conversely, the existence of G_z implies that of G_f. It is clear that G_z is well-behaved on $K_p \times \mathbf{C}$. Favre and Guedj [2] studied the existence and properties of G_z on $K_p \times \mathbf{C}$ without assuming that $\deg p = \deg_w q$ and the leading coefficient of q_z to be a constant. Using an argument in the proof of [2, Theorem 6.1], the existence of G_z on an open subset of $K_p^c \times \mathbf{C}$ is shown in [7, Lemma 2.3], which was improved in [8, Theorems 3.1 and 3.2].

In [8] we defined the rational number α of f as

$$\min\left\{ l \in \mathbf{Q} \,\middle|\, \begin{array}{l} ld \geq n_j + lm_j \text{ for any integers } n_j \text{ and } m_j \\ \text{s.t. } c_j z^{n_j} w^{m_j} \text{ is a term in } q \text{ for some } c_j \neq 0 \end{array} \right\}$$

if $\deg_z q > 0$ and as 0 if $\deg_z q = 0$. Since q has only finitely many terms, the minimum can be taken. Indeed, α is equal to

$$\max\left\{ \frac{n_j}{d - m_j} \,\middle|\, \begin{array}{l} c_j z^{n_j} w^{m_j} \text{ is a term in } q \\ \text{with } c_j \neq 0 \text{ and } m_j < d \end{array} \right\}.$$

This rational number α plays an important role in the study of the dynamics of f such as the existence of the limits G_z and G_f. Define $A_f = \bigcup_{n\geq 0} f^{-n}(W_R)$ and $W_R = \{|w| > R|z|^\alpha, |w| > R^{\alpha+1}\}$ for large $R > 0$.

Theorem 2.1 ([8, Theorem 3.1]). *The fiberwise Green function G_z is defined, continuous and pluriharmonic on A_f. Moreover, $G_z(w)$ tends to $\alpha G_p(z)$ as (z,w) in A_f tends to ∂A_f.*

Hence G_f is also defined, continuous and pluriharmonic on A_f. For an optimality of the minimum α and the region A_f, see [8, Remark 2] and [8, Examples 5.2 and 5.3]. Theorem 2.1 implies the existence of G_f^α.

Corollary 2.2 ([8, Theorem 4.1]). *The weighted Green function G_f^α is defined, continuous and plurisubharmonic on $\mathbf{C}^2$. More precisely,*

$$G_f^\alpha(z,w) = \begin{cases} G_z(w) & \text{on } A_f, \\ \alpha G_p(z) & \text{on } \mathbf{C}^2 - A_f. \end{cases}$$

In Theorem 5.4, we give a simple proof of the former statement of this corollary for skew products semiconjugate to some polynomial products.

It is useful to consider the dynamics of the extension of f to a holomorphic map on a weighted projective space. When we consider the extension of f, we assume that $\alpha \neq 0$; that is, f is not a polynomial product. Let r and s be the denominator and numerator of α respectively. The weighted projective space $\mathbf{P}(r,s,1)$ is a quotient space of $\mathbf{C}^3 - \{O\}$,

$$\mathbf{P}(r,s,1) = \mathbf{C}^3 - \{O\}/\sim,$$

where $(z,w,t) \sim (\lambda^r z, \lambda^s w, \lambda t)$ for any λ in $\mathbf{C} - \{0\}$. We denote a point in $\mathbf{P}(r,s,1)$ by weighted homogeneous coordinates $[z : w : t]$. It follows from the definition of α that f extends to a holomorphic map $\tilde{f}$ on $\mathbf{P}(r,s,1)$,

$$\tilde{f}[z : w : t] = \left[p\left(\frac{z}{t^r}\right)t^{dr} : q\left(\frac{z}{t^r}, \frac{w}{t^s}\right)t^{ds} : t^d\right].$$

We define the Fatou set of $\tilde{f}$ as the maximal open set of $\mathbf{P}(r,s,1)$ where the family of iterates $\{\tilde{f}^n\}_{n \geq 0}$ is normal. The Julia set of $\tilde{f}$ is defined as the complement of the Fatou set of $\tilde{f}$. We showed that the Julia set of $\tilde{f}$ coincides with the closure of the set where G_f^α is not pluriharmonic, where the closure is taken in $\mathbf{P}(r,s,1)$. In other words, the Julia set of $\tilde{f}$ coincides with the closure of

$$\bigcup_{|z|<1}\left(\{z\} \times \partial\left\{w : G_f^\alpha\left(\frac{w}{z^\alpha}\right) = 0\right\}\right) \cup \bigcup_{|z|=1}\left(\{z\} \times \left\{w : G_f^\alpha\left(\frac{w}{z^\alpha}\right) = 0\right\}\right)$$
$$\cup \bigcup_{|z|>1}\left(\{z\} \times \left\{w : G_f^\alpha\left(\frac{w}{z^\alpha}\right) = \alpha \log|z|\right\}\right),$$

where the closure is taken in $\mathbf{P}(r,s,1)$.

3. Skew products semiconjugate to some polynomial products

In this section we begin the study of the dynamics of a polynomial skew product of the form $f(z,w) = (z^d, q(z,w))$, where $q(z,w) = w^d + O_z(w^{d-1})$, that is semiconjugate to a polynomial product $(z^d, h(w))$ by $\pi(z,w) = (z^r, z^s w)$ for some positive integers r and s:

$$\begin{array}{ccc} \mathbf{C}^2 & \xrightarrow{(z^d,\,h(w))} & \mathbf{C}^2 \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi(z,w)=(z^r,z^s w)} \\ \mathbf{C}^2 & \xrightarrow{(z^d,\,q(z,w))} & \mathbf{C}^2. \end{array}$$

Note that $h(w) = q(1,w)$ and so the degree of h is also d; see Proposition 3.1 below. Since s is positive, f is not a polynomial product except (z^d, w^d), which occurs if and only if $h(w) = w^d$. The dynamics of a polynomial product $f(z,w) = (p(z), q(w))$ is relatively easy: $G_z(w) = G_q(w)$ coincides with G_f^α for

$\alpha = 0$, and f extends to a holomorphic map on the two-dimensional projective space $\mathbf{P}^2$.

For any polynomial $h(w)$ of degree d and positive integer s, there exists a polynomial skew product semiconjugate to $(z^d, h(w))$ by $\pi(z,w) = (z, z^s w)$. In fact, $(z^d, z^{sd}h(z^{-s}w))$ is the required map. On the other hand, we give necessary and sufficient conditions for a polynomial skew product $f(z,w) = (z^d, q(z,w))$ to be semiconjugate to a polynomial product.

Proposition 3.1. *Let $f(z,w) = (z^d, q(z,w))$ be a polynomial skew product, where $q(z,w) = w^d + O_z(w^{d-1})$. Assume that f is not a polynomial product. Then the following are equivalent for some mutually prime positive integers r and s:*

(1) *f is semiconjugate to a polynomial product $(z^d, q(1,w))$ by $\pi(z,w) = (z^r, z^s w)$,*
(2) *$q(z^r, z^s w) = z^{sd} q(1,w)$,*
(3) *$f\tau = \tau^d f$ for any $\tau(z,w) = (\lambda z, \kappa w)$ with $\lambda^s = \kappa^r$.*

Proof. Clearly, (1) and (2) are equivalent. Let us show the equivalence of (2) and (3). Suppose that (2) holds, and let $z^n w^m$ be a term of q with a nonzero coefficient for $m < d$. Then $rn + sm = sd$. Since (3) is equivalent to the equality $q(\lambda z, \kappa w) = \kappa^d q(z,w)$, it is enough to show that $\lambda^n \kappa^m = \kappa^d$. From the equality $\lambda^s = \kappa^r$ and the mutually primeness of r and s, it follows that

$$\lambda^n = \{(\kappa^r)^{1/s}\}^n = \{(\kappa^{1/s})^r\}^n = \kappa^{rn/s} = \kappa^{d-m}.$$

In particular, the equality of sets $(\kappa^r)^{1/s}$ and $(\kappa^{1/s})^r$ is guaranteed by the mutually primeness of r and s. The proof of the opposite direction from (3) to (2) is similar to above but relatively easy; from the equalities $\lambda^n \kappa^m = \kappa^d$ and $\lambda^s = \kappa^r$, it follows that $rn + sm = sd$. □

Any pair of multiple integers of r and s with the same positive multiplier satisfies (1) and (2) of Proposition 3.1. On the other hand, r and s in (3) of Proposition 3.1 should be mutually prime. We can restate the necessary and sufficient condition (3) for a polynomial skew product to be semiconjugate to a polynomial product as follows:

Proposition 3.2. *Let f be a polynomial skew product as in Proposition 3.1. If the equality $f\tau = \tau^d f$ holds for some $\tau(z,w) = (\lambda z, \kappa w)$ with $|\lambda| \neq 1$ and $\lambda\kappa \neq 0$, then f is semiconjugate to a polynomial product $(z^d, q(1,w))$ by $\pi(z,w) = (z^r, z^s w)$ for some positive integers r and s.*

Proof. Let $z^{n_j} w^{m_j}$ be a term of q with a nonzero coefficient for $m_j < d$. By assumption, λ and κ are related by $\lambda^{n_j}\kappa^{m_j} = \kappa^d$. The equalities $\lambda^{n_j} = \kappa^{d-m_j}$ and $\lambda^{n_i} = \kappa^{d-m_i}$ imply that $\lambda^{n_i(d-m_j)-n_j(d-m_i)} = 1$. Since $\lambda^n \neq 1$ for any nonzero integer n, we have $n_i(d-m_j) - n_j(d-m_i) = 0$. Hence the ratio of $d - m_j$ and n_j are independent of j. Therefore, (2) of Proposition 3.1 holds for any positive integers r and s whose ratio are equal to that of $d - m_j$ and n_j. □

Moreover, there are other necessary and sufficient conditions in terms of the *weighted homogeneous part* of the polynomial q, which is defined in [8], and the symmetries of the Julia set of f, which is described in [7, Proposition 3.9].

Now, we consider the existence of the *fiberwise Green function* G_z for $f(z,w) = (z^d, q(z,w))$ as above. Note that the ratio of r and s coincides with the rational number α of f defined in Section 2 unless $\alpha = 0$; thus let $\alpha = s/r$. If α is an integer, then we have the following equalities for any positive integer n:

$$q(z,w) = z^{\alpha d} h\left(\frac{w}{z^{\alpha}}\right) \quad \text{and} \quad Q_z^n(w) = z^{\alpha d^n} h^n\left(\frac{w}{z^{\alpha}}\right).$$

Even if α is not an integer, it follows that $|Q_z^n(w)| = |z^{\alpha d^n} h^n(z^{-\alpha} w)|$. Hence

$$\frac{1}{d^n} \log^+|Q_z^n(w)| = \frac{1}{d^n} \log^+\left|z^{\alpha d^n} h^n\left(\frac{w}{z^{\alpha}}\right)\right| = \max\left\{\alpha \log|z| + \frac{1}{d^n} \log\left|h^n\left(\frac{w}{z^{\alpha}}\right)\right|, 0\right\}.$$

Therefore, the existence of G_z follows from that of the limit of $d^{-n}\log|h^n|$, which is investigated in the next section.

4. Existence of Green functions of polynomials

Let $h(w) = w^d + O(w^{d-1})$ be a monic polynomial of degree $d \geq 2$. As we mentioned in Section 2, a useful tool of the study of the dynamics of h is the Green function of h,

$$G_h(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+|h^n(w)|.$$

Due to the term $\log^+ = \max\{\log, 0\}$, it follows that G_h is defined on $\mathbf{C}$. We investigate what happens if we replace $\log^+$ by log in this section. Define

$$\tilde{G}_h(w) = \lim_{n\to\infty} \frac{1}{d^n} \log|h^n(w)| \quad \text{and} \quad \overline{G}_h(w) = \limsup_{n\to\infty} \frac{1}{d^n} \log|h^n(w)|.$$

It follows that $\tilde{G}_h = G_h > 0$ on A_h and $\overline{G}_h$ is defined on $\mathbf{C}$. Note that $G_h = \max\{\overline{G}_h, 0\}$, or roughly $G_h = \max\{\tilde{G}_h, 0\}$. Here we define $\max\{\tilde{G}_h, 0\}$ as 0 when the limit $\tilde{G}_h(w)$ is not defined. Then $\max\{\tilde{G}_h, 0\}$ coincides with $\max\{\overline{G}_h, 0\}$ because $\overline{G}_h(w) \leq 0$ if $\tilde{G}_h(w)$ is not defined.

The point is that the function $\log|w|$ has singularity at $w = 0$, and so $\log|h^n(w)|$ has singularity on $h^{-n}(0)$. We show that $\tilde{G}_h$ has singularity only when the preimages of 0 has recurrence at 0 in Theorem 4.2 below; thus we define

$$E_h = \bigcap_{l\geq 0} \overline{\bigcup_{n\geq l} h^{-n}(0)}.$$

For example, if 0 is a fixed point, then $\tilde{G}_h$ is $-\infty$ on the preimages of 0 and discontinuous on E_h. If 0 is a periodic point, then $\tilde{G}_h$ is not defined on the preimages of 0, although $\overline{G}_h = 0$ on the preimages of 0. On the other hand, we show that the limit $\tilde{G}_h$ is well-behaved on $\mathbf{C} - E_h$ in Theorem 4.2 below, using the following fact by Sullivan and others such as Fatou and Julia.

Theorem 4.1 ([6, Theorems 16.1 and 16.4]). *Every Fatou component of a holomorphic map R on* $\mathbf{P}^1$ *is eventually periodic. If R maps the Fatou component U onto itself, then there are just four possibilities, as follows: Either U is the immediate basin of an attracting fixed point, or of a parabolic fixed point, or else U is a Siegel disk or Herman ring.*

A polynomial has no Herman rings. Because the dynamics of a holomorphic map on the periodic Fatou components is well understood, Theorem 4.1 induces the following key theorem, which includes Theorem C.

Theorem 4.2. *The limit* $\tilde{G}_h$ *is defined, continuous and subharmonic on* $\mathbf{C} - E_h$. *More precisely, if* $h(w) = w^d$ *then* $\tilde{G}_h(w) = \log|w|$, *and if* $h(w) \neq w^d$ *then* $\tilde{G}_h = G_h$ *on* $\mathbf{C} - E_h$. *If* $0 \notin E_h$, *then* $\tilde{G}_h = G_h$ *on* $\mathbf{C}$. *If* $0 \in E_h$, *then there are just three possibilities, as follows:*

(1) 0 *is an attracting periodic point,*
(2) 0 *is contained in a Siegel cycle,*
(3) 0 *is contained in the Julia set* J_h.

Proof. If $h(w) = w^d$, then clearly $\tilde{G}_h(w) = \log|w|$. If $0 \notin K_h$, then $\tilde{G}_h = G_h$. Hence we may assume that $h(w) \neq w^d$ and $0 \in K_h$. For the former statement, it is enough to show that $\tilde{G}_h = 0$ on $K_h - E_h$. The following proof of this equality also shows the latter statement.

First, we consider the case $0 \in F_h$; thus $0 \in \operatorname{int} K_h$. In this case, 0 is contained in the attracting basin of an attracting periodic point, or of a parabolic periodic point, or in the preimage of a Siegel cycle. Hereinafter we assume that the periodic point or the Siegel cycle is the fixed point or the Siegel disk for simplicity.

Let us assume that 0 is contained in the attracting basin. If 0 is not an attracting fixed point, then clearly $\tilde{G}_h = 0$ on K_h; thus $\tilde{G}_h = G_h$ on $\mathbf{C}$. If 0 is an attracting fixed point, then it is enough to show that $\tilde{G}_h = 0$ on $A_0 - E_h$, where A_0 denotes the attracting basin of 0. Let $\lambda = h'(0)$. If $0 < |\lambda| < 1$, then $h(w) = \lambda w + O(w^2)$. Hence there exist constants $c < |\lambda|$ and $r > 0$ such that $|h(w)| \geq c|w|$ for any $|w| < r$. Therefore, $|h^n(w)| \geq c^n|w|$ for any $n \geq 0$ and so $\tilde{G}_h = 0$ on $\{0 < |w| < r\}$ since $h^n(w)$ is bounded on K_h. Consequently, $\tilde{G}_h = 0$ on $A_0 - E_h$. If $\lambda = 0$, then $h(w) = aw^m + O(w^{m+1})$. Hence there exist constants $c < |a|$ and $r > 0$ such that $|h(w)| \geq c|w|^m$ for any $|w| < r$. Therefore, $|h^n(w)| \geq c^{1+m+\cdots+m^{n-1}}|w|^{m^n}$ for any $n \geq 0$ and so $\tilde{G}_h = 0$ on $\{0 < |w| < r\}$ since $h^n(w)$ is bounded on K_h. Consequently, $\tilde{G}_h = 0$ on $A_0 - E_h$.

If 0 is contained in the attracting basin of a parabolic fixed point that is not 0, then clearly $\tilde{G}_h = 0$ on K_h; thus $\tilde{G}_h = G_h$ on $\mathbf{C}$.

Let us assume that 0 is contained in the preimage of a Siegel disk. If 0 is not contained in the Siegel disk, then clearly $\tilde{G}_h = 0$ on K_h; thus $\tilde{G}_h = G_h$ on $\mathbf{C}$.

If 0 is contained in the Siegel disk D, then h is conjugate to $e^{\lambda}w$ on D. Hence $\tilde{G}_h = 0$ on $D - E_h$ and so $\tilde{G}_h = 0$ on $K_h - E_h$.

Next, we consider the case $0 \in J_h$; that is, $0 \in \partial K_h$. In this case $E_h = J_h$, and the proof depends on whether 0 is a parabolic point or not.

Let us assume that 0 is a parabolic fixed point and that $h(w) = w + aw^{m+1} + O(w^{m+2})$ for simplicity. Then $|h^n(w)| \sim (\sqrt[m]{m|a|n})^{-1}$ as $n \to \infty$ and so there exists a constant $c < 1$ such that $|h^n(w)| \geq c(\sqrt[m]{m|a|n})^{-1}$ on the attracting petals of 0. See [6] for details. Therefore, $\tilde{G}_h = 0$ on the parabolic basin of 0 since $h^n(w)$ is bounded on K_h. It follows from Theorem 4.1 that, except the parabolic basin of 0, there is no periodic Fatou component U such that 0 is contained in ∂U and attracts some points in U. Consequently, $\tilde{G}_h = 0$ on $K_h - E_h = K_h - J_h = \operatorname{int} K_h$.

For other cases of $0 \in J_h$, it follows that $\tilde{G}_h = 0$ on $K_h - E_h$ from the fact that there is no periodic Fatou component U such that 0 is contained in ∂U and attracts some points in U. □

The following corollary of Theorem 4.2 is identical with Corollary D.

Corollary 4.3. *It follows that the upper semicontinuous regularization*

$$\limsup_{w' \to w} \left(\limsup_{n \to \infty} \frac{1}{d^n} \log|h^n(w')| \right)$$

is defined, continuous and subharmonic on $\mathbf{C}$. *If* $h(w) \neq w^d$, *then it coincides with* G_h.

Proof. Clearly, $\overline{G}_h = G_h$ on A_h. We may assume that $h(w) \neq w^d$. It then follows from Theorem 4.2 that $\overline{G}_h = 0$ on $K_h - E_h$. We may assume that $0 \in K_h$; thus $h^n(w)$ is bounded on E_h. Hence it is enough to show that $\overline{G}_h(w)$ tends to 0 as w in $\mathbf{C} - E_h$ tends to E_h. If $0 \in F_h$, then this convergence holds because $\overline{G}_h = 0$ on $(K_h - E_h) \cup \partial K_h$ and because $K_h - E_h$ is dense in K_h. If $0 \in J_h$, then the convergence above holds because $\overline{G}_h(w)$ tends to 0 as w in A_h tends to ∂A_h and $\overline{G}_h = 0$ on $\operatorname{int} K_h$, and because $E_h = J_h = \partial A_h = \partial K_h$. □

Remark 4.4. We can replace $\limsup_{n\to\infty} d^{-n} \log|h^n|$ by $\liminf_{n\to\infty} d^{-n} \log|h^n|$ in Corollary 4.3 besides many places in the paper, because these functions are the same on $\mathbf{C} - E_h$.

5. Existence of fiberwise Green functions

In this section we investigate the existence of the *fiberwise Green function* G_z of a polynomial skew product $f(z,w) = (z^d, q(z,w))$, where $q(z,w) = w^d + O_z(w^{d-1})$, that is semiconjugate to a polynomial product $(z^d, h(w))$ by $\pi(z,w) = (z^r, z^s w)$ for some positive integers r and s. Results in this section are obtained from Theorem 4.2 and Corollary 4.3 in the previous section. Before describing the result on the existence of G_z, we consider that of $\tilde{G}_z(w) = \lim_{n\to\infty} d^{-n} \log|Q_z^n(w)|$. Define

$$E_f = \bigcup_{z\in\mathbf{C}} \{z\} \times z^\alpha E_h, \quad \text{where } \alpha = \frac{s}{r}.$$

Theorem 5.1. *The limit* $\tilde{G}_z$ *is defined, continuous and plurisubharmonic on* $\mathbf{C}^2 - E_f$. *More precisely, if* $h(w) \neq w^d$, *then it is equal to*

$$\alpha \log|z| + G_h\left(\frac{w}{z^\alpha}\right)$$

on $(\mathbf{C} - \{0\}) \times \mathbf{C} - E_f$ *and* $\log|w|$ *on* $\{0\} \times \mathbf{C}$. *If* $0 \notin E_h$, *then* $\tilde{G}_z$ *is defined and plurisubharmonic on* $\mathbf{C}^2$ *and continuous on* $\mathbf{C}^2 - \{O\}$.

Proof. If $z = 0$, then $\tilde{G}_0(w) = \log|w|$ since $f(0,w) = (0, w^d)$. For $z \neq 0$,

$$\frac{1}{d^n} \log|Q_z^n(w)| = \alpha \log|z| + \frac{1}{d^n} \log\left|h^n\left(\frac{w}{z^\alpha}\right)\right|.$$

Hence we have the following rough equality for $z \neq 0$:

$$\tilde{G}_z(w) = \alpha \log|z| + \tilde{G}_h\left(\frac{w}{z^\alpha}\right).$$

Therefore, applying Theorem 4.2 completes the proof. □

The following two corollaries follow from Theorem 5.1 and an argument similar to the proof of Corollary 4.3.

Corollary 5.2. *It follows that the upper semicontinuous regularization*

$$\limsup_{w'\to w}\left(\limsup_{n\to\infty} \frac{1}{d^n} \log|Q_z^n(w')|\right)$$

is defined and plurisubharmonic on $\mathbf{C}^2$ *and continuous on* $\mathbf{C}^2 - \{O\}$. *More precisely, if* $h(w) \neq w^d$, *then it is equal to*

$$\begin{cases} \alpha \log|z| + G_h\left(\dfrac{w}{z^\alpha}\right) & (z \neq 0), \\ \log|w| & (z = 0). \end{cases}$$

Corollary 5.3. *It follows that the upper semicontinuous regularization*

$$\limsup_{(z',w')\to(z,w)}\left(\limsup_{n\to\infty} \frac{1}{d^n} \log|f^n(z',w')|\right),$$

which is equal to

$$\limsup_{w'\to w}\left(\limsup_{n\to\infty}\frac{1}{d^n}\log|f^n(z,w')|\right),$$

is defined and plurisubharmonic on $\mathbf{C}^2$ *and continuous on* $\mathbf{C}^2-\{O\}$. *More precisely, if* $h(w)\neq w^d$, *then it is equal to*

$$\begin{cases}\max\left\{\log|z|,\alpha\log|z|+G_h\left(\dfrac{w}{z^\alpha}\right)\right\} & (z\neq 0),\\ \log|w| \quad (z=0).\end{cases}$$

We now describe the results on the existence of G_z, which includes Theorem A.

Theorem 5.4. *The limit* G_z *is defined, continuous and plurisubharmonic on* $\mathbf{C}^2-E_f\cap(\{|z|>1\}\times\mathbf{C})$. *More precisely, if* $h(w)\neq w^d$, *then it is equal to*

$$\max\left\{\alpha\log|z|+G_h\left(\frac{w}{z^\alpha}\right),0\right\}$$

on $(\mathbf{C}-\{0\})\times\mathbf{C}-E_f\cap(\{|z|>1\}\times\mathbf{C})$ *and* $\log^+|w|$ *on* $\{0\}\times\mathbf{C}$. *Moreover, if* $h(w)\neq w^d$, *then it follows that*

$$G_f^\alpha(z,w)=\begin{cases}\max\left\{\alpha\log|z|+G_h\left(\dfrac{w}{z^\alpha}\right),0\right\} & (z\neq 0),\\ \log^+|w| \quad (z=0).\end{cases}$$

In particular, the weighted Green function G_f^α *is defined, continuous and plurisubharmonic on* $\mathbf{C}^2$. *If* $0\notin E_h$, *then* $G_z=G_f^\alpha$ *on* $\mathbf{C}^2$.

Proof. The proof of the claims about G_z is similar to that of Theorem 5.1; we apply Theorem 4.2. Let us derive the form of G_f^α. If $z=0$, then $G_f^\alpha(0,w)=\log^+|w|$. If $z\neq 0$, then roughly

$$\begin{aligned}G_f^\alpha(z,w)&=\max\{\alpha\log|z|,\tilde{G}_z(w),0\}\\&=\max\left\{\alpha\log|z|,\alpha\log|z|+\tilde{G}_h\left(\frac{w}{z^\alpha}\right),0\right\}\\&=\max\left\{\alpha\log|z|+G_h\left(\frac{w}{z^\alpha}\right),0\right\}.\end{aligned}$$ □

The following two corollaries follow from Theorem 5.4 and an argument similar to the proof of Corollary 4.3. The former is identical with Corollary B.

Corollary 5.5. *It follows that the upper semicontinuous regularization*

$$\limsup_{w'\to w}\left(\limsup_{n\to\infty}\frac{1}{d^n}\log^+|Q_z^n(w')|\right)$$

is defined, continuous and plurisubharmonic on $\mathbf{C}^2$. *If* $h(w)\neq w^d$, *then it coincides with* G_f^α.

Corollary 5.6. *It follows that the upper semicontinuous regularization*

$$\limsup_{(z',w')\to(z,w)}\left(\limsup_{n\to\infty}\frac{1}{d^n}\log^+|f^n(z',w')|\right),$$

which is equal to

$$\limsup_{w'\to w}\left(\limsup_{n\to\infty}\frac{1}{d^n}\log^+|f^n(z,w')|\right),$$

is defined, continuous and plurisubharmonic on $\mathbf{C}^2$. *More precisely, if* $h(w)\neq w^d$, *then it is equal to*

$$\begin{cases}\max\left\{\log|z|,\alpha\log|z|+G_h\left(\dfrac{w}{z^\alpha}\right),0\right\} & (z\neq 0),\\ \log^+|w| \quad (z=0).\end{cases}$$

As shown in Section 2, the map f extends to a holomorphic map $\tilde{f}$ on $\mathbf{P}(r,s,1)$. By Theorem 5.4, the Julia set of $\tilde{f}$ can be written in terms of the dynamics of h; it coincides with the closure of

$$\bigcup_{|z|<1}\left(\{z\}\times\left\{w: G_h\left(\frac{w}{z^\alpha}\right)=-\alpha\log|z|\right\}\right)$$
$$\cup\bigcup_{|z|=1}(\{z\}\times z^\alpha K_h)\cup\bigcup_{|z|>1}(\{z\}\times z^\alpha J_h),$$

where the closure is taken in $\mathbf{P}(r,s,1)$.

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