T. S. PHẠM
KODAI MATH. J.
35 (2012), 311–319

AN EXPLICIT BOUND FOR THE ŁOJASIEWICZ EXPONENT OF REAL POLYNOMIALS

Tiến Sơn Phạm

Dedicated to Professor Mutsuo Oka on the occasion of his 65th birthday

Abstract

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a polynomial function of degree d with $f(0) = 0$. The classical Łojasiewiz inequality states that there exist $c > 0$ and $\alpha > 0$ such that $|f(x)| \geq cd(x, f^{-1}(0))^{\alpha}$ in a neighbourhod of the origin $0 \in \mathbf{R}^n$, where $d(x, f^{-1}(0))$ denotes the distance from x to the set $f^{-1}(0)$. We prove that the smallest such exponent α is not greater than $R(n,d)$ with $R(n,d) := \max\{d(3d-4)^{n-1}, 2d(3d-3)^{n-2}\}$.

1. Introduction

Let $f : U \to \mathbf{R}$ be an analytic function defined in a neighborhood U of the origin $0 \in \mathbf{R}^n$, $f(0) = 0$, and let $Z := \{x \in U \mid f(x) = 0\}$. Then the classical Łojasiewicz inequality ([29]) asserts that there exist constants $r > 0$, $c > 0$ and $\alpha > 0$ such that

$$|f(x)| \geq cd(x, Z)^{\alpha}, \quad \text{for all } \|x\| \leq r,$$

where $d(x, Z) := \inf\{\|x - y\| \mid y \in Z\}$, and $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbf{R}^n$.

The Łojasiewicz exponent of f at the origin $0 \in \mathbf{R}^n$, denoted by α_f, is the infimum of the exponents α satisfying the above Łojasiewicz's inequality. Bocknak and Risler [7] (see also [37]) proved that α_f is a rational number. Moreover, the Łojasiewicz's inequality holds with exponent α_f and some constant $c > 0$.

The computation or estimation of the Łojasiewicz exponent is a quite interesting problem. For instance, if f is a real polynomial of degree d in n variables, one would like to have an explicit bound for α_f in terms of d and n. The complex analytic variant of this question has been settled in the papers [1], [2], [5], [6], [9], [10], [16], [19], [20], [22], [23], [24], [31], [33], [34], [36], [37].

1991 *Mathematics Subject Classification*. Primary 32B20, 32S05; Secondary 14B05, 14P10.

Key words and phrases. Łojasiewicz inequalities, real polynomials.

This work was supported by the National Foundation for Science and Technology Development, Vietnam.

Received July 19, 2011; revised October 21, 2011.

In the case $n = 2$, a formula for computing the Łojasiewicz exponent α_f was given by Kuo in [25]. A similar formula for the Łojasiewicz exponent at infinity in the real plane is given in the paper [40] (see also [41]). However, it seems more difficult to obtain effective estimates in the general case.

We now assume that f is a real polynomial of degree d in n variables. It is known that α_f can be bounded by some rational number depending only on n and d (see, for example, [23], [38]). If f has an isolated zero at the origin (that is, f has a strict local extremum at 0), then Gwoździewicz [13] (see also [24], [17]) established the following nice estimate:

$$\alpha_f \le (d-1)^n + 1.$$

In this paper we consider the general case, that is, the case where f may have a non-isolated zero at the origin. Precisely, for any integer $d \ge 2$ and for any polynomial f in n variables with $\deg f = d$ and $f(0) = 0$ we have the following explicit estimate:

$$\alpha_f \le \max\{d(3d-4)^{n-1}, 2d(3d-3)^{n-2}\}.$$

The proof of this inequality is based on an explicit bound for the Łojasiewicz exponent in the gradient inequality for real polynomials [11] and the Ekeland's variational principle [14]. Note that this principle is also used recently by Tiep, Vui and Thao [39] in order to study the (global) Łojasiewicz inequality for polynomial functions.

The paper is organized as follows: The results are given in Section 2 and the proofs are given in Section 3.

2. Results

Let $f : U \to \mathbf{R}$ be an analytic function defined in a neighborhood U of the origin $0 \in \mathbf{R}^n$ and let $Z := \{x \in U \mid f(x) = 0\}$. We can write

$$f = f_m + f_{m+1} + \cdots,$$

where f_i is a homogeneous form of degree i, and $f_m \not\equiv 0$. We denote by $m_f := m$, the *multiplicity* of f. Note that $m_f \ge 1$ with the equality if and only if $\nabla f(0) \ne 0$.

THEOREM 2.1. *Let $f : U \to \mathbf{R}$ be an analytic function defined in a neighborhood U of the origin $0 \in \mathbf{R}^n$, $f(0) = 0$. We have*

(i) $\alpha_f \ge m_f$.

(ii) $\alpha_f = 1$ *if and only if* $m_f = 1$.

Remark 2.1. In the complex case, Risler and Trotman proved in [35] that $\alpha_f = m_f$.

The main result of this paper is as follows.

Theorem 2.2. *Let* $f : \mathbf{R}^n \to \mathbf{R}$ *be a real polynomial of degree* $d \geq 2$. *Assume that* $f(0) = 0$ *and* $\nabla f(0) = 0$. *Then the Łojasiewicz exponent* α_f *satisfies*

$$\alpha_f \leq R(n, d),$$

where $R(n, d) := \max\{d(3d-4)^{n-1}, 2d(3d-3)^{n-2}\}$.

3. Proofs

3.1. Proof of Theorem 2.1. Let $f : U \to \mathbf{R}$ be an analytic function defined in a neighborhood U of the origin $0 \in \mathbf{R}^n$, $f(0) = 0$, and let $Z := \{x \in U \mid f(x) = 0\}$. The *directional set* $D(Z)$ *of* Z *at* $0 \in \mathbf{R}^n$ is defined by

$$D(Z) := \left\{ v \in \mathbf{S}^{n-1} \mid \exists \{x_k\} \subset Z \backslash \{0\}, x_k \to 0 \in \mathbf{R}^n \text{ s.t. } \frac{x_k}{\|x_k\|} \to v, k \to \infty \right\}.$$

Here $\mathbf{S}^{n-1}$ denotes the unit sphere centred at $0 \in \mathbf{R}^n$. We refer the reader to [21] for the basic properties of the directional set $D(Z)$. We note that the set $D(Z)$ is simply the intersection of the usual tangent cone of Z at $0 \in \mathbf{R}^n$ (i.e. the Painlevé-Kuratowski upper limit: $\limsup_{t \to 0+} \frac{1}{t} Z$) with the sphere $\mathbf{S}^{n-1}$. Therefore, it is straightforward that $D(Z)$ is a closed subanalytic subset of $\mathbf{S}^{n-1}$ (since it is described by a first-order formula and since Z is an analytic set). Moreover, we have

Lemma 3.1. *The directional set* $D(Z)$ *is a subanalytic set of dimension* $\leq n-2$.

Proof. See, for example, [27, Proposition 1], [21, Proposition 2.2], [28]. □

Proof of Theorem 2.1. (i) By Lemma 3.1, there is $v \in \mathbf{S}^{n-1} \backslash D(Z)$ such that $f_{m_f}(v) \neq 0$. We have, for all $0 < t \ll 1$,

$$f(tv) = f_{m_f}(v) t^{m_f} + \text{terms of higher in } t.$$

Therefore

$$f(tv) \simeq t^{m_f} \quad \text{for } 0 \leq t \ll 1. \tag{1}$$

On the other hand, by the monotonicity lemma (e.g. [12, Theorem 4.1], [8, Theorem 2.1]), the function $t \mapsto d(tv, Z)$ is analytic for $0 \leq t \ll 1$. We will prove that there exists a constant $c > 0$ such that

$$d(tv, Z) \geq ct \quad \text{for } 0 \leq t \ll 1. \tag{2}$$

By contrary, assume that

$$\lim_{t \to 0+} \frac{d(tv, Z)}{t} = 0.$$

Let $x(t)$, $0 \le t \ll 1$, be a curve in Z such that $d(tv, Z) = \|tv - x(t)\|$. Clearly, $x(t) \neq 0$ for $0 < t \ll 1$. Moreover, we have, for all $0 < t \ll 1$,

$$\frac{d(tv, Z)}{t} = \frac{\|tv - x(t)\|}{t} = \left\| v - \frac{x(t)}{t} \right\| \ge \left| \|v\| - \left\| \frac{x(t)}{t} \right\| \right| = \left| 1 - \left\| \frac{x(t)}{t} \right\| \right|.$$

Consequently, $\lim_{t\to 0+} \frac{x(t)}{t} = v$ and $\lim_{t\to 0+} \frac{\|x(t)\|}{t} = 1$. Therefore

$$\lim_{t\to 0+} \frac{x(t)}{\|x(t)\|} = \lim_{t\to 0+} \frac{x(t)}{t} \frac{t}{\|x(t)\|} = v,$$

which contradicts to the fact that $v \notin D(Z)$.

Now it follows immediately from (1), (2) and the definition of the exponent α_f that $\alpha_f \ge m_f$.

(ii) By the statement (i), if $\alpha_f = 1$ then $m_f = 1$.

We now assume that $m_f = 1$, which means that $\nabla f(0) \neq 0$. Then there exist positive constants r and c such that

$$\|\nabla f(x)\| \ge c \quad \text{for all } x \in \mathbf{B}^n(2r).$$

Here and in the following $\mathbf{B}^n(r) := \{x \in \mathbf{R}^n \mid \|x\| \le r\}$ denotes the closed ball centered at the origin with radius r.

Without loss of generality, we may assume that the function f is of class C^1 on $\mathbf{R}^n$.

Take any $x \in \mathbf{B}^n(r)$. By [15, Corollary 16], there exists $x' \in \mathbf{R}^n$ such that

$$\|x - x'\| \le d(x, Z),$$
$$\|\nabla f(x')\| d(x, Z) \le |f(x)|.$$

The first inequality implies that

$$\|x'\| \le \|x' - x\| + \|x\| \le d(x, Z) + \|x\| \le 2\|x\| \le 2r.$$

Thus

$$|f(x)| \ge \|\nabla f(x')\| d(x, Z) \ge c d(x, Z). \tag{3}$$

On the other hand, since the function f is of class C^1, f is Lipschitz on the closed ball $\mathbf{B}^n(2r)$. That is there exists $L > 0$ such that

$$|f(b) - f(a)| \le L\|b - a\| \quad \text{for all } a, b \in \mathbf{B}^n(2r).$$

Let $a \in Z$ be such that $\|x - a\| = d(x, Z)$. Observe that $a \in \mathbf{B}^n(2r)$. Hence

$$|f(x)| = |f(x) - f(a)| \le L\|x - a\| = L d(x, Z). \tag{4}$$

The desired result now follows immediately from (3), (4) and the definition of the Łojasiewicz exponent α_f. □

3.2. Proof of Theorem 2.2. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a real polynomial of degree $d \geq 2$. Assume that $f(0) = 0$, $\nabla f(0) = 0$, and $Z := \{x \in \mathbf{R}^n \mid f(x) = 0\}$. We have

LEMMA 3.2. *Let $r > 0$, $c > 0$ and $l > 0$ be constants such that*

$$\|\nabla f(x)\| \geq c d(x, Z)^l \quad \textit{for all } x \in \mathbf{B}^n(2r).$$

Then

$$|f(x)| \geq c' d(x, Z)^{l+1} \quad \textit{for all } x \in \mathbf{B}^n(r),$$

where $c' := c \dfrac{l^l}{(l+1)^{l+1}}$.

Proof. This proof follows that of [42].

Let $\phi(s) := cs^l$ and $\psi(s) := \max_{0 \leq \lambda \leq s} \lambda \phi(s - \lambda)$. Then it is easy to see that
(a) the function ϕ is nondecreasing on $[0, r]$;
(b) $\psi(s) = c' s^{l+1}$; and
(c) for each $s > 0$ there exists $\lambda \in (0, s)$ such that

$$\frac{1}{\lambda} \psi(s) \leq \phi(s - \lambda).$$

By the assumption, $\|\nabla f(x)\| \geq \phi(d(x, Z))$ for all $x \in \mathbf{B}^n(2r)$. We will prove the following inequality

$$|f(x)| \geq \psi(d(x, Z)), \quad \text{for all } x \in \mathbf{B}^n(r).$$

By contrary, assume that there exists $x_0 \in \mathbf{B}^n(r)$ such that

$$|f(x_0)| < \psi(d(x_0, Z)).$$

Then $x_0 \notin Z$ and $\psi(d(x_0, Z)) > 0$. Moreover, there exists $c_0 \in (0, 1)$ such that

$$|f(x_0)| < c_0 \psi(d(x_0, Z)).$$

Let $\varepsilon := c_0 \psi(d(x_0, Z)) > 0$ and $s := d(x_0, Z) > 0$.

In view of Item (c) above, it is clear that there exists $\lambda \in \mathbf{R}$ such that the following inequalities hold

$$0 < \lambda < s = d(x_0, Z),$$

$$\frac{1}{\lambda} \psi(s) \leq \phi(s - \lambda).$$

By the Ekeland's variational principle ([14, Theorem 1.1]), there exists $x' \in \mathbf{R}^n$ such that

$$\|x' - x_0\| \leq \lambda,$$

$$|f(x')| \leq |f(x_0)|,$$

$$|f(z)| + \frac{\varepsilon}{\lambda} \|z - x'\| \geq |f(x')| \quad \text{for all } z \in \mathbf{R}^n.$$

Consequently, we have

$$\|x'\| \le \|x' - x_0\| + \|x_0\| \le \lambda + \|x_0\| < d(x_0, Z) + \|x_0\| \le 2\|x_0\| \le 2r,$$

and

$$d(x', Z) \ge d(x_0, Z) - \|x' - x_0\| \ge d(x_0, Z) - \lambda > 0.$$

This implies that $x' \notin Z$; i.e., $f(x') \neq 0$. We may assume that $f(x') = |f(x')| > 0$ (otherwise, we replace f by $-f$). Then $f(z) = |f(z)| \ge 0$ for $\|z - x'\| \ll 1$. Therefore

$$f(z) + \frac{\varepsilon}{\lambda}\|z - x'\| \ge f(x') \quad \text{for } \|z - x'\| \ll 1.$$

Take any $u \in \mathbf{R}^n$, and set $z = x' + tu$ in the preceding inequality, with $0 < t \ll 1$. This yields

$$\frac{1}{t}[f(x' + tu) - f(x')] \ge -\frac{\varepsilon}{\lambda}\|u\|.$$

Letting $t \to 0+$, we get

$$\langle \nabla f(x'), u \rangle \ge -\frac{\varepsilon}{\lambda}\|u\|.$$

Taking the infimum of both sides over all $u \in \mathbf{R}^n$ with $\|u\| = 1$, we get

$$-\|\nabla f(x')\| \ge -\frac{\varepsilon}{\lambda},$$

which means that

$$\|\nabla f(x')\| \le \frac{\varepsilon}{\lambda} = \frac{c_0 \psi(d(x_0, Z))}{\lambda}.$$

And thus we obtain the following contradiction

$$\begin{aligned} \|\nabla f(x')\| &\ge \phi(d(x', Z)) \ge \phi(d(x_0, Z) - \lambda) \\ &\ge \frac{1}{\lambda}\psi(d(x_0, Z)) > \frac{c_0}{\lambda}\psi(d(x_0, Z)) \ge \|\nabla f(x')\|. \end{aligned}$$

(The first inequality follows from the assumption and the fact that $x' \in \mathbf{B}^n(2r)$.) □

Remark 3.1. In Lemma 3.2, we assume that f was polynomial function. However, it is enough to assume that f is C^1-function.

Proof of Theorem 2.2. The well known Łojasiewicz's gradient inequality ([29] or [30]) states that there exist $r > 0$, $c_1 > 0$, $\theta > 0$ such that for any $x \in \mathbf{B}^n(2r)$ we have

$$\|\nabla f(x)\| \ge c_1 |f(x)|^{\theta}. \tag{5}$$

Let θ_f be the infimum of the exponents θ satisfying the Łojasiewicz's gradient inequality. It is known (see [29], [7], [3]) that $\theta_f \in (0,1)$ and the inequality (5) holds with the exponent θ_f and some constant $c_1 > 0$. Moreover, D'Acunto and Kurdyka proved [11] that

$$\theta_f \le 1 - \frac{1}{R(n,d)}. \tag{6}$$

We observe from (5) that

$$\{x \in \mathbf{B}^n(2r) \mid \nabla f(x) = 0\} \subset \{x \in \mathbf{B}^n(2r) \mid f(x) = 0\}.$$

By Łojasiewicz's inequality ([7], [3]), there exist constants $c_2 > 0$, $\beta > 0$ such that we have for any $x \in \mathbf{B}^n(2r)$,

$$\|\nabla f(x)\| \ge c_2 d(x,Z)^{\beta}. \tag{7}$$

Let β_f be the infimum of the exponents β satisfying the inequality (7). It is known (see, for example, [7]) that the inequality (7) holds with the exponent β_f and some constant $c_2 > 0$, i.e.,

$$\|\nabla f(x)\| \ge c_2 d(x,Z)^{\beta_f} \quad \text{for all } x \in \mathbf{B}^n(2r).$$

It follows from Lemma 3.2 that

$$|f(x)| \ge c_2' d(x,Z)^{\beta_f+1} \quad \text{for all } x \in \mathbf{B}^n(r),$$

where $c_2' := c_2 \dfrac{\beta_f^{\beta_f}}{(\beta_f+1)^{\beta_f+1}}$. By the definition of the Łojasiewicz exponent α_f, then

$$\beta_f + 1 \ge \alpha_f. \tag{8}$$

On the other hand, we have for all $\|x\| \le r$,

$$|f(x)| \ge c d(x,Z)^{\alpha_f},$$

after perhaps reducing r. This yields

$$\|\nabla f(x)\| \ge c_1 |f(x)|^{\theta_f} \ge c_1 c^{\theta_f} d(x,Z)^{\alpha_f \theta_f} \quad \text{for all } \|x\| \le r.$$

By the definition of β_f, then

$$\alpha_f \theta_f \ge \beta_f.$$

This, together with the inequality (8), implies that

$$\alpha_f \le \frac{1}{1-\theta_f}.$$

The desired result follows immediately from the inequality (6). □

Remark 3.2. After the submission of this paper for publication we have learnt that Theorem 2.2 was also proved by a different argument by Kurdyka and Spodzieja [26] (see also [4, Theorem 2.8]).

Acknowledgments. The author is grateful to Professor Hà Huy Vui for pointing out the results of Ekeland [14], [15].

REFERENCES

[1] O. M. ABDERRAHMANE, On the Łojasiewicz exponent and Newton polyhedron, Kodai Math. Journal **28** (2005), 106–110.

[2] W. D. BROWNAWELL, Bounds for the degrees in the Nullstellensatz, Ann. of Math. **126** (1987), 577–591.

[3] E. BIERSTONE AND P. D. MILMAN, Semianalytic and subanalytic sets, Publ. Math. I.H.E.S. **67** (1988), 5–42.

[4] E. BIERSTONE AND P. D. MILMAN, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. **128** (1997), 207–302.

[5] C. BIVIÀ-AUSINA, Łojasiewicz exponents, the integral closure of ideals and Newton polyhedra, J. Math. Soc. Japan **55** (2003), 655–668.

[6] C. BIVIÀ-AUSINA, Jacobian ideals and the Newton non-degeneracy condition, Proc. Edinb. Math. Soc. **48** (2005), 21–36.

[7] J. BOCHNAK AND J. J. RISLER, Sur les exposants de Łojasiewicz, Comment. Math. Helv. **50** (1975), 493–507.

[8] M. COSTE, An introduction to o-minimal geometry, RAAG Notes, Institut de Recherche Mathématiques de Rennes, November, 1999.

[9] E. CYGAN, T. KRASIŃSKI AND P. TWORZEWSKI, Separation of algebraic sets and the Łojasiewicz exponent of polynomial mappings, Invent. Math. **136** (1999), 75–87.

[10] E. CYGAN, A note on separation of algebraic sets and the Łojasiewicz exponent for polynomial mappings, Bull. Sci. Math. **129** (2005), 139–147.

[11] D. D'ACUNTO AND K. KURDYKA, Explicit bounds for the Łojasiewicz exponent in the gradient inequality for polynomials, Ann. Pol. Math. **87** (2005), 51–61.

[12] L. VAN DEN DRIES AND C. MILLER, Geometric categories and o-minimal structures, Duke Math. J. **84** (1996), 497–540.

[13] J. GWOŹDZIEWICZ, The Łojasiewicz exponent of an analytic function at an isolated zero, Comment. Math. Helv. **74** (1999), 364–375.

[14] I. EKELAND, On the variational principle, J. Math. Anal. Appl. **47** (1974), 324–357.

[15] I. EKELAND, Nonconvex minimization problems, Bull. AMS. **1** (1979), 443–474.

[16] T. FUKUI, Łojasiewicz type inequalities and Newton diagrams, Proc. Amer. Math. **112** (1991), 1169–1183.

[17] A. HARAUX AND T. S. PHẠM, On the gradient of quasi-homogeneous polynomials, Univ. Iagel. Acta Math. **49** (2011).

[18] H. HIRONAKA, Sub-analytic sets; number theory, algebraic geometry and commutative algebra, in honor of Y. Akisuki, Tokyo, 1973, 435–493.

[19] Z. JELONEK, On the effective Nullstellensatz, Invent. Math. **162** (2005), 1–17.

[20] Z. JELONEK, On the Łojasiewicz exponent, Hokkaido Math. J. **35** (2006), 471–485.

[21] S. KOIKE AND L. PAUNESCU, The directional dimension of subanalytic sets is invariant under bi-Lipschitz homeomorphisms, Ann. Inst. Fourier (Grenoble) **59** (2009), 2445–2467.

[22] J. KOLLÁR, Sharp effective Nullstellensatz, J. Amer. Math. Soc. **1** (1988), 963–975.

[23] S. JI, J. KOLLÁR AND B. SHIFFMAN, A global Łojasiewicz inequality for algebraic varieties, Trans. Amer. Math. Soc. **329** (1992), 813–818.

[24] J. KOLLÁR, An effective Łojasiewicz inequality for real polynomials, Period. Math. Hungar. **38** (1999), 213–221.

[25] T. C. Kuo, Computation of Łojasiewicz exponent of $f(x, y)$, Comment. Math. Helv. **49** (1974), 201–213.
[26] K. Kurdyka and S. Spodzieja, Separation of real algebraic sets and the Łojasiewicz exponent, preprint, UŁ, 2011/7.
[27] T. L. Loi, Density of Morse functions on sets definable in o-minimal structures, Ann. Polon. Math. **89** (2006), 289–299.
[28] T. L. Loi, An observation on definable bi-Lipschitz homeomorphism, Vietnam J. Math. **38** (2010), 281–286.
[29] S. Łojasiewicz, Ensembles semi-analytiques, Publ. Math. I.H.E.S. (1964), preprint.
[30] S. Łojasiewicz, Sur les trajectoires du gradient d'une fonction analytique, Semin. Geom. Univ. Studi Bologna 1982/1983 (1984), 115–117.
[31] B. Lichtin, Estimation of Łojasiewicz exponents and Newton polygons, Invent. Math. **64** (1981), 417–429.
[32] J. Milnor, Singular points of complex hypersurfaces, Annals of Mathematics Studies **61**, Princeton University Press, 1968.
[33] A. Płoski, Sur l'exposant d'une application analytique, I, II, Bull. Polish Acad. Sci. Math. **32** (1984), 669–673, **33** (1985), 123–127.
[34] T. S. Phạm, On the effective computation of Łojasiewicz exponents via Newton polyhedra, Period. Math. Hungar. **54** (2007), 201–213.
[35] J.-J. Risler and D. Trotman, Bi-Lipschitz invariance of the multiplicity, Bull. London. Math. Soc. **29** (1997), 200–204.
[36] T. Rodak and S. Spodzieja, Effective formulas for the local Łojasiewicz exponent, Math. Z. **268** (2011), 37–44,
[37] S. Spodzieja, The Łojasiewicz exponent of subanalytic sets, Ann. Polon. Math. **87** (2005), 247–263.
[38] P. Solernó, Effective Łojasiewicz inequalities in semialgebraic geometry, Appl. Algebra Engrg. Comm. Comput. **2** (1991), 2–14.
[39] D. S. Tiep, H. H. Vui and N. T. Thao, Łojasiewicz inequality for polynomial functions on non compact domains, Internat. J. Math. **23**, No. 4 (2012), 1250033.
[40] H. H. Vui and N. H. Duc, A formula for the Łojasiewicz exponent at infinity in the real plane via real approximations, Hokkaido Math. J. **38** (2009), 417–425.
[41] H. H. Vui and N. H. Duc, Łojasiewicz inequality at infinity for polynomials in two real variables, Math. Z. **266** (2010), 243–264.
[42] R. Zhang, Weakly upper Lipschitz multifunctions and applications in parametric optimization, Math. Program., Ser. A. **102** (2005), 153–166.

Tiến Sơn Phạm
Department of Mathematics
University of Dalat
1, Phu Dong Thien Vuong
Dalat
Vietnam
E-mail: sonpt@dlu.edu.vn