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THE TANAKA-WEBSTER CONNECTION AND REAL HYPERSURFACES IN A COMPLEX SPACE FORM

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Abstract

We classify parallel real hypersurfaces in a complex space form for the generalized Tanaka-Webster connection.

1. Introduction

Tanaka-Webster connection ([16], [18]) is defined as a canonical affine connection on a non-degenerate CR-manifold. A real hypersurface in a Kählerian manifold has an (integrable) CR-structure (η, J) which is associated with an almost contact metric structure (η, ϕ, ξ, g), but the Levi form is not guaranteed to be non-degenerate, in general. In this context, the first author [5], [6] defined the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$, $k \neq 0$ for real hypersurfaces in a Kählerian manifold. In particular, if the shape operator A of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then its associated CR-structure is strongly pseudo-convex, and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 2 in section 2).

On the other hand, U-H. Ki [9] proved that there are no real hypersurfaces with parallel Ricci tensor (for Levi-Civita connection) in a non-flat complex space form $\tilde{M}_n(c)$, $(c \neq 0)$ when $n \geq 3$. This is also true when $n = 2$ ([10]). These results imply, in particular, that there do not exist locally symmetric $(\nabla R = 0)$ real hypersurfaces in a non-flat complex space form. As the CR-geometric counterpart of local symmetry, we introduce *g.-Tanaka-Webster parallellity in a real hypersurfaces of a Kähler manifold*, whose g.-Tanaka-Webster torsion tensor $\hat{T}$ and g.-Tanaka-Webster curvature tensor $\hat{R}$ are parallel with respect to $\hat{\nabla}^{(k)}$:

$$\hat{\nabla}^{(k)}\hat{T} = 0, \quad \hat{\nabla}^{(k)}\hat{R} = 0.$$

In section 3, we classify such spaces in a non-flat complex space form. Namely, we prove

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Main Theorem. *Let M be a real hypersurface of a non-flat complex space form $\tilde{M}_n(c)$, $n \geq 3$, $c \neq 0$. Then M is g.-Tanaka-Webster parallel if and only if M is locally congruent to one of the following*:

(I) *in case that $\tilde{M}_n(c) = P_n\mathbf{C}$ with the Fubini-Study metric of $c = 4$,*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*

(A_2) *a tube of radius r over a totally geodesic $P_k\mathbf{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*

(B) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$;*

(II) *in case that $\tilde{M}_n(c) = H_n\mathbf{C}$ with the Bergman metric of $c = -4$,*

(A_0) *a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbf{C}$,*

(A_2) *a tube over a totally geodesic $H_k\mathbf{C}$ $(1 \leq k \leq n-2)$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbf{R}$.*

In [8], J. T. Cho and M. Kimura gave a classification of real hypersurfaces in a non-flat complex space form such that the holomorphic sectional curvature for $\hat{\nabla}^{(k)}$ is constant. Then we can find that among above examples in Main Theorem the holomorphic sectional curvature is constant only for type (A_0) in $H_n\mathbf{C}$ and (A_1) in $P_n\mathbf{C}$ or $H_n\mathbf{C}$.

2. Preliminaries

In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

First, we give a brief review of several fundamental notions and formulas which we will need later on.

– Almost contact metric structures and the associated CR-structures

An odd-dimensional differentiable manifold M has an *almost contact structure* if it admits a (1,1)-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Then one can find always a compatible Riemannian metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields on M. We call (η, ϕ, ξ, g) an *almost contact metric structure* of M and $M = (M; \eta, \phi, \xi, g)$ an *almost contact metric manifold.* From (1) and (2) we easily get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{3}$$

For an almost contact metric manifold M, we define its fundamental 2-form Φ by $\Phi(X, Y) = g(\phi X, Y)$. If M satisfies in addition

$$\Phi = d\eta, \tag{4}$$

M is called a *contact metric manifold.* For more details about the general theory of almost contact metric manifolds, we refer to [3].

For an almost contact metric manifold $M = (M; \eta, \phi, \xi, g)$, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. The restriction $J = \phi \mid D$ of ϕ to D defines an almost complex structure in D. As soon as the following conditions are further satisfied:

$$[JX, JY] - [X, Y] \in D \quad (\text{or } [X, JY] + [JX, Y] \in D) \tag{5}$$

and

$$[J, J](X, Y) = 0 \tag{6}$$

for all $X, Y \perp \xi$, where $[J, J]$ is the Nijenhuis torsion of J, then the pair (η, J) is called an (integrable) CR-structure associated with the almost contact metric structure (η, ϕ, ξ, g). If its Levi form L defined by $L(X, Y) = d\eta(X, JY)$, $X, Y \perp \xi$, is non-degenerate (positive or negative definite, resp.), then (η, J) is called a non-degenerate (strongly pseudo-convex, resp.) CR-structure. In particular, for a contact metric manifold its associated Levi-form is hermitian and positive definite, but its associated almost complex structure is not in general integrable. For further details about CR-structures, we refer for example to [1], [17].

– The generalized Tanaka-Webster connection for real hypersurfaces

Let M be an (oriented) real hypersurface of a Kählerian manifold $\tilde{M} = (\tilde{M}; \tilde{J}, \tilde{g})$ and N a global unit normal vector on M. By $\tilde{\nabla}$, A we denote the Levi-Civita connection in $\tilde{M}$ and the shape operator with respect to N, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric of M induced from $\tilde{g}$. An eigenvector(resp. eigenvalue) of the shape operator A is called a principal curvature vector(resp. principal curvature). For any vector field X tangent to M, we put

$$\tilde{J}X = \phi X + \eta(X)N, \quad \tilde{J}N = -\xi. \tag{7}$$

We easily see that the structure (η, ϕ, ξ, g) is an almost contact metric structure on M i.e. satisfies (1) and (2). From the condition $\tilde{\nabla}\tilde{J} = 0$, the relations (7) and by making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{8}$$

$$\nabla_X \xi = \phi AX. \tag{9}$$

By using (8) and (9), we see that a real hypersurface in a Kählerian manifold always satisfies (5) and (6), the CR-integrability condition. From (4) and (9) we have

PROPOSITION 1. *Let $M = (M; \eta, \phi, \xi, g)$ be a real hypersurface of a Kählerian manifold. The almost contact metric structure of M is contact metric if and only if $\phi A + A\phi = 2\phi$.*

Let $\tilde{M} = \tilde{M}_n(c)$ be a complex space form of constant holomorphic sectional curvature $4c$ and M a real hypersurface of $\tilde{M}$. Then we have the following Gauss and Codazzi equations:

$$
\begin{aligned}
(10)\qquad R(X, Y)Z = {} & c\{g(Y,Z)X - g(X,Z)Y \\
& + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\
& + g(AY,Z)AX - g(AX,Z)AY,
\end{aligned}
$$

$$
(11)\qquad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}
$$

for any tangent vector fields X, Y, Z on M.

The Tanaka-Webster connection ([16], [18]) is the canonical affine connection defined on a non-degenerate CR-manifold. S. Tanno [17] defined the generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. We define the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) for real hypersurfaces in Kählerian manifolds by the naturally extended one of Tanno's generalized Tanaka-Webster connection. Now, we recall the generalized Tanaka-Webster connection $\hat{\nabla}$ for contact metric manifolds:

$$
\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y
$$

for all vector fields X and Y.

Making use of (9), we define the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for real hypersurfaces of Kählerian manifolds by

$$
(12)\qquad \hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y
$$

for a non-zero real number k. We put

$$
(13)\qquad F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y.
$$

Then the torsion tensor $\hat{T}$ is given by:

$$
\begin{aligned}
(14)\qquad \hat{T}(X, Y) &= F_X Y - F_Y X \\
&= g((\phi A + A\phi)X, Y)\xi - \eta(Y)\phi AX + \eta(X)\phi AY \\
&\quad - k(\eta(X)\phi Y - \eta(Y)\phi X)
\end{aligned}
$$

Furthermore, by using (2), (3), (8), (9) and (12) we can see that

$$\hat{\nabla}^{(k)}\eta = 0, \quad \hat{\nabla}^{(k)}\xi = 0, \quad \hat{\nabla}^{(k)}g = 0, \quad \hat{\nabla}^{(k)}\phi = 0, \tag{15}$$

and

$$\hat{T}(X, Y) = 2\, d\eta(X, Y)\xi, \quad X, Y \in D.$$

We note that the associated Levi form is $L(X, Y) = \frac{1}{2}g((J\bar{A} + \bar{A}J)X, JY)$, where we denote by $\bar{A}$ the restriction A to D. If M satisfies $\phi A + A\phi = 2k\phi$, then we see that the associated CR-structure is strongly pseudo-convex and further satisfies $\hat{T}(\xi, \phi Y) = -\phi\hat{T}(\xi, Y)$. Hence, the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see [5], [6]). Namely, we have

Proposition 2. *Let $M = (M; \eta, \phi, \xi, g)$ be a real hypersurface of a Kählerian manifold. If M satisfies $\phi A + A\phi = 2k\phi$, then the associated CR-structure is strongly pseudo-convex and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.*

Remark 1. From Propositions 1 and 2, we can find examples M in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ whose almost contact metric structures are not contact metric but their associated CR-structures are strongly pseudo-convex and moreover, the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection. In reality, a real hypersurface M in $P_n\mathbf{C}$ or $H_n\mathbf{C}$ satisfies $\phi A + A\phi = 2k\phi$ if and only if M is locally congruent to one of real hypersurfaces of type (A_0) in $H_n\mathbf{C}$, (A_1) or (B) in $P_n\mathbf{C}$, $H_n\mathbf{C}$ (cf. [12] and [14]). But, with the help of the tables in [2] and [15], we see that $k = 1$ only for a geodesic hypersphere of radius $\frac{\pi}{4}$ in $P_n\mathbf{C}$ and for a horosphere in $H_n\mathbf{C}$.

3. g.-Tanaka-Webster parallel spaces

We define the g.-Tanaka-Webster curvature tensor of $\hat{R}$ (with respect to $\hat{\nabla}^{(k)}$) by

$$\hat{R}(X, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X, Y]}Z$$

for all vector fields X, Y, Z in M. From the definition of $\hat{R}$, together with (12) and (13), we have

$$\hat{R}(X, Y)Z = R(X, Y)Z + (\nabla_X F)_Y Z + F_X F_Y Z - (\nabla_Y F)_X Z - F_Y F_X Z$$

for all vector fields X, Y, Z tangent to M. We put

$$E(X, Y)Z = (\nabla_X F)_Y Z + F_X F_Y Z - (\nabla_Y F)_X Z - F_Y F_X Z.$$

Use (9) to get

(16) $$\begin{aligned} E(X,Y)Z &= (\nabla_X F)_Y Z - (\nabla_Y F)_X Z + F_X F_Y Z - F_Y F_X Z \\ &= g(\phi((\nabla_X A)Y - (\nabla_Y A)X), Z)\xi + 2g(\phi AY, Z)\phi AX \\ &\quad - 2g(\phi AX, Z)\phi AY + g((\nabla_X \phi)AY - (\nabla_Y \phi)AX, Z)\xi \\ &\quad - \eta(Z)(\phi((\nabla_X A)Y - (\nabla_Y A)X) + (\nabla_X \phi)AY - (\nabla_Y \phi)AX) \\ &\quad - k(g((\phi A + A\phi)X, Y)\phi Z + \eta(Y)(\nabla_X \phi)Z - \eta(X)(\nabla_Y \phi)Z) \\ &\quad + g(\phi AX, F_Y Z)\xi - \eta(F_Y Z)\phi AX - k\eta(X)\phi F_Y Z \\ &\quad - g(\phi AY, F_X Z)\xi + \eta(F_X Z)\phi AY + k\eta(Y)\phi F_X Z. \end{aligned}$$

Then E is a tensor field of type $(1,3)$, and

(17) $$\hat{R}(X,Y)Z = R(X,Y)Z + E(X,Y)Z$$

for all vector fields X, Y, Z in M.

We proved the following result in [7].

Proposition 3. *Let M be a Hopf hypersurface of a non-flat complex space form $\tilde{M}_n(c)$, $c \neq 0$. Then M admits a flat g.-Tanaka-Webster structure, namely, $\hat{R} = 0$ if and only if M is locally congruent to a horosphere in $H_n\mathbf{C}$, or dim $M = 3$ and a homogeneous tube over a complex quadric Q^{n-1} and $P_n\mathbf{R}$ (resp. $H_n\mathbf{R}$) in $P_n\mathbf{C}$ (resp. $H_n\mathbf{C}$).*

Very recently, the second author [13] proved that for real hypersurfaces of a complex projective space $P_n\mathbf{C}$, $n \geq 3$, the g.-Tanaka-Webster Ricci tensor $\hat{S}$ vanishes if and only if it is locally congruent to a geodesic sphere with $k^2 \geq 4n(n-1)$.

As an analogue of local symmetry in Riemannian geometry, we now introduce a g.-Tanaka-Webster parallel spaces.

Definition 1. A real hypersurface in a Kähler manifold is a *g.-Tanaka-Webster parallel space* (g.-T.-W. parallel space, for short) if its g.-Tanaka-Webster torsion tensor $\hat{T}$ and its curvature tensor $\hat{R}$ satisfy

$$\hat{\nabla}^{(k)}\hat{T} = 0, \quad \hat{\nabla}^{(k)}\hat{R} = 0.$$

For contact strictly pseudo-convex pseudo-Hermitian manifolds, we defined a g.-Tanaka-Webster parallel space and studied in [4].

In [11], S. Kobayashi and K. Nomizu call a connection *invariant by parallelism* if for any points p and q in M and for any curve γ from p to q, there exists a (unique) local affine isomorphism f such that $f(p) = q$ and such that the differential of f at p coincides with the parallel displacement $\tau_\gamma : T_pM \to T_qM$ along γ. By [11, Corollary 7.6], this is equivalent to the connection having parallel torsion and curvature tensor. In other words, a g.-T.-W. parallel space is one for which the generalized Tanaka-Webster connection is an invariant connection by parallelism.

In a former paper, the first author proved

Theorem 4 ([5]). *Let M be a real hypersurface of a non-flat complex space form $\tilde{M}_n(c)$, $c \neq 0$. Then the shape operator is parallel for the g.-Tanaka-Webster connection if and only if M is locally congruent to one of real hypersurfaces of type (A) or (B).*

Now, we prove

Lemma 5. *If a real hypersurface in a Kählerian manifold satisfies $\hat{\nabla}^{(k)}\hat{T} = 0$, then*

$$(\hat{\nabla}^{(k)}_X A)Y = 0$$

for any tangent vector X of M and any tangent vector Y orthogonal to ξ.

Proof. Since $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$ and $\hat{\nabla}^{(k)}\eta = 0$, it follows from (14) that

$$g(\phi(\hat{\nabla}^{(k)}_Z A)X + (\hat{\nabla}^{(k)}_Z A)\phi X, Y)\xi - \eta(Y)\phi(\hat{\nabla}^{(k)}_Z A)X + \eta(X)\phi(\hat{\nabla}^{(k)}_Z A)Y = 0. \tag{18}$$

The scalar product with ξ in (18) yields

$$g(\phi(\hat{\nabla}^{(k)}_Z A)X + (\hat{\nabla}^{(k)}_Z A)\phi X, Y) = 0. \tag{19}$$

Thus we have $\phi(\hat{\nabla}^{(k)}_Z A) = -(\hat{\nabla}^{(k)}_Z A)\phi$. Using (19), (18) reduces again to

$$\eta(Y)\phi(\hat{\nabla}^{(k)}_Z A)X - \eta(X)\phi(\hat{\nabla}^{(k)}_Z A)Y = 0. \tag{20}$$

Suppose $g(X,\xi) = 0$ and $Y = \xi$, we have $\phi(\hat{\nabla}^{(k)}_Z A)X = -(\hat{\nabla}^{(k)}_Z A)\phi X = 0$. This proves our lemma. ■

Lemma 6. *If a real hypersurface in a non-flat complex space form $\tilde{M}_n(c)$ $(c \neq 0)$, $n \geq 3$, satisfies $\hat{\nabla}^{(k)}\hat{T} = 0$, then it is a Hopf hypersurface.*

Proof. By the definition of g.-Tanaka-Webster connection, we have

$$\begin{aligned}(\hat{\nabla}^{(k)}_X A)Y = {} & (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX \\ & - k\eta(X)\phi AY - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y.\end{aligned} \tag{21}$$

Using the equation of Codazzi, we obtain

$$\begin{aligned}(\hat{\nabla}^{(k)}_X A)Y - (\hat{\nabla}^{(k)}_Y A)X \\ = {} & c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi) \\ & + 2g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY - g(\phi AX, Y)A\xi \\ & + \eta(Y)A\phi AX + k\eta(X)A\phi Y + \eta(AX)\phi AY + k\eta(Y)\phi AX \\ & + g(\phi AY, X)A\xi - \eta(X)A\phi AY - k\eta(Y)A\phi X.\end{aligned} \tag{22}$$

We can choose an orthonormal frame $\{e_1, \dots, e_{2n-2}, \xi\}$ of $T_x(M)$ such that the shape operator A is represented by a matrix form

$$A = \left(\begin{array}{ccc|c} a_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{2n-2} & h_{2n-2} \\ \hline h_1 & \cdots & h_{2n-2} & \alpha \end{array}\right),$$

where we have put $h_i = g(Ae_i, \xi)$, $i = 1, \dots, 2n-2$ and $\alpha = g(A\xi, \xi)$. By the direct computation using Lemma 5, we have

$$g((\hat{\nabla}^{(k)}_{e_i} A)e_j - (\hat{\nabla}^{(k)}_{e_j} A)e_i, \xi) = (-2c + 2a_i a_j - a_i\alpha - a_j\alpha)g(\phi e_i, e_j) = 0, \tag{23}$$

$$g((\hat{\nabla}^{(k)}_{e_i} A)e_j - (\hat{\nabla}^{(k)}_{e_j} A)e_i, e_i) = -h_i(a_i + 2a_j)g(\phi e_i, e_j) = 0, \tag{24}$$

$$\begin{aligned} &g((\hat{\nabla}^{(k)}_{e_i} A)e_j - (\hat{\nabla}^{(k)}_{e_j} A)e_i, \phi e_i) \\ &\quad = -h_j a_i - a_i g(\phi e_i, e_j)g(A\xi, \phi e_i) - a_j g(\phi e_i, e_j)g(A\xi, \phi e_i) = 0, \end{aligned} \tag{25}$$

$$g((\hat{\nabla}^{(k)}_{\xi} A)e_i - (\hat{\nabla}^{(k)}_{e_i} A)\xi, e_i) = 2h_i g(A\xi, \phi e_i) = 0, \tag{26}$$

$$\begin{aligned} &g((\hat{\nabla}^{(k)}_{\xi} A)e_i - (\hat{\nabla}^{(k)}_{e_i} A)\xi, e_j) \\ &\quad = (c - a_i k + a_j k + a_i\alpha - a_i a_j)g(\phi e_i, e_j) + h_i g(A\xi, \phi e_j) + h_j g(A\xi, \phi e_i) \\ &\quad = 0, \end{aligned} \tag{27}$$

where $i \neq j$. From (26), we have $h_i = 0$ or $g(A\xi, \phi e_i) = 0$ for $i = 1, \dots, 2n-2$. By the suitable permutation of the orthonormal basis, we can represent A as

$$A = \left(\begin{array}{cccccc|c} a_1 & & & & & & h_1 \\ & \ddots & & & & & \vdots \\ & & a_q & & & & h_q \\ & & & a_{q+1} & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & a_{2n-2} & 0 \\ \hline h_1 & \cdots & h_q & 0 & \cdots & 0 & \alpha \end{array}\right),$$

where $h_1, \dots, h_q \neq 0$. Let H_1 and H_2 be subspaces of the tangent space $T_x(M)$ spanned by $\{e_1, \dots, e_q\}$ and $\{e_{q+1}, \dots, e_{2n-2}\}$, respectively. We use indices $s, t, u, \dots$ for $H_1 = \{e_s\}$ and $x, y, z, \dots$ for $H_2 = \{e_x\}$. We notice that $g(A\xi, \phi e_s) = 0$ for all $e_s \in H_1$.

When $\dim H_1 = 0$, M is a Hopf hypersurface. In the following we consider the case that $\dim H_1 \neq 0$. When $\dim H_1 = 1$, then the shape operator A can be represented as follows:

(i)

$$A = \left(\begin{array}{cccc|c} a_1 & & & & h_1 \\ & a_2 & & & 0 \\ & & \ddots & & \vdots \\ & & & a_{2n-2} & 0 \\ \hline h_1 & 0 & \cdots & 0 & \alpha \end{array}\right).$$

Next, when $\dim H_1 \geq 2$, substituting $e_i = e_s$, $e_j = e_t \in H_1$ in (25), we see that $h_t a_s = 0$ for any $s \neq t$. If there exists a non-zero a_s, then $h_t = 0$ for any $s \neq t$. This is a contradiction. So we have $a_1 = \cdots = a_q = 0$. Hence the shape operator A can be represented as follows:

(ii)

$$A = \left(\begin{array}{cccccc|c} 0 & & & & & & h_1 \\ & \ddots & & & & & \vdots \\ & & 0 & & & & h_q \\ & & & a_{q+1} & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & a_{2n-2} & 0 \\ \hline h_1 & \cdots & h_q & 0 & \cdots & 0 & \alpha \end{array}\right).$$

Case (i). Since $\dim H_1 = 1$, there exists $e_x \in H_2$ such that $g(\phi e_1, e_x) \neq 0$. By (24),

$$h_1(a_1 + 2a_x)g(\phi e_1, e_x) = 0,$$

from which we obtain $a_1 = -2a_x$. On the other hand, putting $e_i = e_1$ and $e_j = e_x$ in (23),

$$2c - a_x\alpha + 4a_x^2 = 0.$$

Thus we have $a_x \neq 0$. Substituting $e_i = e_x$ and $e_j = e_1$ in (25),

$$\begin{aligned} 0 &= -h_1 a_x - a_x g(\phi e_x, e_1) g(A\xi, \phi e_x) - a_1 g(\phi e_x, e_1) g(A\xi, \phi e_x) \\ &= -h_1 a_x (1 - g(\phi e_x, e_1)^2), \end{aligned}$$

here we used $A\xi = h_1 e_1 + \alpha\xi$. Since $h_1 a_x \neq 0$, we obtain $\phi e_1 = \pm e_x$. We only have to consider the case that $\phi e_1 = e_x$. Since $\dim H_1 = 1$ and $n \geq 3$, we have $\dim H_2 \geq 3$. Taking $e_y \neq e_x$, we have $g(\phi e_y, e_1) = 0$. Thus (25) implies $h_1 a_y = 0$, from which we obtain $a_y = 0$ for $e_y \neq e_x$. So there exist i, j such that $\phi e_i = e_j$, $i, j \neq 1, x$ and $a_i = a_j = 0$. Using (27), we have $g(\phi e_i, e_j) = 0$. This is a contradiction.

Case (ii). From (23), we have $g(e_s, \phi e_t) = 0$ for any $e_s, e_t \in H_1$. So we see that $\dim H_2 \neq 0$ and $\phi H_1 \subseteq H_2$. Thus, for any $e_s \in H_1$, there exists $e_x \in H_2$ such that $g(\phi e_s, e_x) \neq 0$. Substituting $e_i = e_s \in H_1$, $e_j = e_x \in H_2$ in (23) and (24), we have

$$-2c - a_x \alpha = 0, \quad 2h_s a_x = 0$$

for any $e_s \in H_1$. From these equations, we get $h_s = 0$ for any s. This is a contradiction.

Therefore, $\dim H_1 = 0$ and M is a Hopf hypersurface. ■

LEMMA 7. *Let M be a real hypersurface of a complex space form $\tilde{M}_n(c)$, $c \neq 0$, $n \geq 3$. Then $\hat{\nabla}^{(k)} \hat{T} = 0$ if and only if the shape operator A is parallel with respect to the g.-Tanaka-Webster connection.*

Proof. First we suppose $\hat{\nabla}^{(k)} \hat{T} = 0$. From Lemma 6, M is a Hopf hypersurface and the shape operator A satisfies $A\xi = \alpha\xi$ for some constant α. Using $\hat{\nabla}^{(k)} \xi = 0$ and (12), we have

$$(\hat{\nabla}^{(k)}_X A)\xi = \hat{\nabla}^{(k)}_X A\xi - A\hat{\nabla}^{(k)}_X \xi = \hat{\nabla}^{(k)}_X (\alpha\xi) = 0.$$

Together with Lemma 5, we have $\hat{\nabla}^{(k)} A = 0$.

Conversely, if $\hat{\nabla}^{(k)} A = 0$, then (14) implies $\hat{\nabla}^{(k)} \hat{T} = 0$. Thus we have our result. ■

By Theorem 4 and Lemma 7, we have

THEOREM 8. *Let M be a real hypersurface of a complex space form $\tilde{M}_n(c)$, $c \neq 0$, $n \geq 3$. Then $\hat{\nabla}^{(k)} \hat{T} = 0$ if and only if M is locally congruent to one of real hypersurfaces of type (A) or (B).*

If a real hypersurface in a complex space form $\tilde{M}_n(c)$, $c \neq 0$, $n \geq 3$ satisfies $\hat{\nabla}^{(k)} \hat{T} = 0$, then $\hat{\nabla}^{(k)} F = 0$ by Lemma 7 and (15). Moreover, use (8), (11) and (15) in (16) to find $\hat{\nabla}^{(k)} E = 0$. Hence, from (17) we obtain

$$(\hat{\nabla}^{(k)}_W \hat{R})(X, Y)Z = (\hat{\nabla}^{(k)}_W R)(X, Y)Z.$$

Using the Gauss equation and Lemma 7 again, the righthand side of the equation vanishes. So we have $\hat{\nabla}^{(k)} \hat{R} = 0$.

After all, by Theorem 8 and the mentioned above we have completed our main Theorem.

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