N. YAGITA
KODAI MATH. J.
34 (2011), 446–463

NOTE ON CHOW RINGS OF NONTRIVIAL G-TORSORS OVER A FIELD

Nobuaki Yagita

Abstract

Let G_k be a split reductive group over a field k corresponding to a compact Lie group G. Let $\mathbf{G}_k$ be a nontrivial G_k-torsor over a field k. In this paper we study the Chow ring of $\mathbf{G}_k$. For example when $(G, p) = (G_2, 2)$, we have the isomorphism $CH^*(\mathbf{G}_k)_{(2)} \cong \mathbf{Z}_{(2)}$.

1. Introduction

Let k be a subfield of $\mathbf{C}$ which contains primitive p-th root of the unity. Let G be a compact connected Lie group. Let us denote by G_k the split reductive group over k which corresponds G. By definition, a G_k-torsor $\mathbf{G}_k$ over k is a variety over k with a free G_k-action such that the quotient variety is $Spec(k)$. A G_k-torsor over k is called trivial, if it is isomorphic to G_k or equivalently it has a k-rational point. Let p be a prime number. In this paper, we always assume that $\mathbf{G}_k$ is nontrivial over any finite extension K/k of degree coprime to p. (We simply say that $\mathbf{G}_k$ is a *nontrivial* torsor over k at p.)

Let H be a subgroup of G. Given a torsor $\mathbf{G}_k$ over k, we can form the twisted form of G/H by

$$(\mathbf{G}_k \times G_k/H_k)/G_k \cong \mathbf{G}_k/H_k.$$

We mainly study the cases that G are exceptional Lie groups and the (p component) torsion index $t(G)_{(p)} = p$. Let T be a maximal torus and B be the Borel subgroup $T \subset B$. In particular, when $(G, p) = (G_2, 2)$, we compute $CH^*(\mathbf{G}_k/T_k) \cong CH^*(\mathbf{G}_k/B_k)$ explicitly. Moreover we show $CH^*(\mathbf{G}_k)_{(2)} \cong \mathbf{Z}_{(2)}$. We also study the case $(G, p) = (SO_{2^{n+1}-1}, 2)$, $n \geq 3$. This case $CH^*(\mathbf{G}_k)_{(2)} \subset CH^*(G_k)_{(2)}$ but it is not isomorphic to $\mathbf{Z}_{(2)}$ nor $CH^*(G_k)_{(2)}$. We also have a partial result for the case $(G, p) = (F_4, 3)$. These are the first examples that Chow rings are computed for nontrivial torsors.

For these groups, Petrov, Semenov and Zainoulline [Pe-Se-Za] showed that the Chow motive of $\mathbf{G}_k/B_k$ is isomorphic to a direct sum of the generalized Rost

2000 *Mathematics Subject Classification*. Primary 14C15; Secondary 14L30, 55R35.
Key words and phrases. motivic cobordism, Rost motive, compact Lie groups.
Received November 11, 2010; revised March 4, 2011.

motives ([Vo4], [Ro2], [Su-Jo], [Vi-Za]). The algebraic cobordism $MGL^{2*,*}(-)$ of the Rost motives are given in [Vi-Ya], [Ya4]. From this, we show the multiplicative structure of $CH^*(\mathbf{G}_k/T_k)$. The algebraic cobordism $MGL^{2*,*}(G_k)$ is studied in [Ya1]. By using arguments in [Ya1], we can compute $CH^*(\mathbf{G}_k)_{(p)}$.

The author thanks Burt Totaro, Michishige Tezuka, Masaki Kameko and Kirill Zainoulline who teach him theories of torsors and algebraic groups. He also thanks the referee who corrected errors in the first version of this paper.

2. Rost motive

Let k be a field of $ch(k) = 0$ and X a smooth variety over k. We consider the Chow ring $CH^*(X)$ generated by cycles modulo rational equivalence. For a non zero symbol $a = \{a_0, \dots, a_n\}$ in the mod 2 Milnor K-theory $K^M_{n+1}(k)/2$, let $\phi_a = \langle\langle a_0, \dots, a_n \rangle\rangle$ be the $(n+1)$-fold Pfister form. Let X_{ϕ_a} be the projective quadric of dimension $2^{n+1} - 2$ defined by ϕ_a. The Rost motive $M_a (= M_{\phi_a})$ is a direct summand of the motive $M(X_{\phi_a})$ representing X_{ϕ_a} so that $M(X_{\phi_a}) \cong M_a \otimes M(\mathbf{P}^{2^n - 1})$.

Moreover for an odd prime p and nonzero symbol $0 \neq a \in K^M_{n+1}(k)/p$, we can define ([Ro], [Vo], [Su-Jo], [Vi-Za]) the generalized Rost motive M_a, which is irreducible and is split over K/k if and only if $a|_K = 0$ (as the case $p = 2$).

The Chow ring of the Rost motive is well known. Let $\bar{k}$ be an algebraic closure of k, $X|_{\bar{k}} = X \otimes_k \bar{k}$, and $i_{\bar{k}} : CH^*(X) \to CH^*(X|_{\bar{k}})$ the restriction map.

Lemma 2.1 (Rost [Ro1,2], [Vo4], [Vi-Ya], [Ya3,4]). *The Chow ring $CH^*(M_a)$ is only dependent on n. There are isomorphisms*

$$CH^*(M_a) \cong \mathbf{Z}\{1\} \oplus (\mathbf{Z}\{c_0\} \oplus \mathbf{Z}/p\{c_1, \dots, c_{n-1}\})[y]/(c_i y^{p-1})$$
$$\text{and} \quad CH^*(M_a|_{\bar{k}}) \cong \mathbf{Z}[y]/(y^p)$$

where $|y| = 2(p^{n-1} + \cdots + p + 1)$ and $|c_i| = |y| + 2 - 2p^i$. Moreover the restriction map is given by $i_{\bar{k}}(c_0 y^{j-1}) = p y^j$ and $i_{\bar{k}}(c_i y^{j-1}) = 0$ for $i, j > 0$.

Remark. The element y does not exist in $CH^*(M_a)$ while $c_i y$ exists. Usually $CH^*(M_a)$ is defined only additively, however when $CH^*(M_a)$ has the natural ring structure (e.g., $p = 2$), the multiplications are given by $c_i \cdot c_j = 0$ for all $0 \leq i, j \leq n-1$.

Remark. In this paper the degree $|x|$ of an element $x \in CH^*(X)$ means the $2-$*times* of the usual degree of the Chow ring so that it is compatible with the degree of the (topological) cohomology $H^*(X(\mathbf{C}))$.

Let us use notation $\Omega^{2*}(X)$ for the motivic cobordism $MGL^{2*,*}(X)_{(p)}$ defined by Voevodsky. (Hence it is the algebraic cobordism defined by Levine and Morel [Le-Mo1,2], [Le].) It is known that

$$\Omega^{2*} = \Omega^{2*}(pt.) \cong MU^{2*}(pt.)_{(p)} \cong \mathbf{Z}_{(p)}[x_1, x_2, \dots]$$

where $MU^{2*}(pt.)$ is the complex cobordism ring and $|x_i| = -2i$. It is known that there is a relation ([Le-Mo1,2], [Le], [Ya2])

$$(2.1) \qquad \Omega^*(X) \otimes_{\Omega^*} \mathbf{Z}_{(p)} \cong CH^*(X)_{(p)}.$$

We can take for x_{p^i-1} the cobordism class of a $2(p^i-1)$-dimensional manifold whose characteristic numbers are divisible by p but the additive characteristic number s_{p^i-1} is not divisible by p^2. Let us denote x_{p^i-1} as v_i. Let I_n be the ideal in Ω^* generated by $v_0, \dots, v_{n-1}$, i.e.,

$$(2.2) \qquad I_n = (p = v_0, v_1, \dots, v_{n-1}) \subset \Omega^*.$$

Then it is well known that I_n and I_∞ are the only prime ideals stable under the Landweber-Novikov cohomology operations ([Ra]) in Ω^*.

The category of cobordism motives is defined and studied in [Vi-Ya]. In particular, we can define the algebraic cobordism of motives. The following fact is the main result in [Vi-Ya] (in [Ya4] for odd primes).

Lemma 2.2 ([Vi-Ya], [Ya4]). *The restriction map*

$$i_{\bar{k}} : \Omega^*(M_a) \to \Omega^*(M_a|_{\bar{k}}) \cong \Omega^*[y]/(y^p)$$

is injective and there is an Ω^*-module isomorphism

$$\Omega^*(M_a) \cong \Omega^*\{1\} \oplus I_n\{y, \dots, y^{p-1}\} \subset \Omega^*[y]/(y^p)$$

such that $v_i y = c_i$ in $\Omega^*(M_a) \otimes_{\Omega^*} \mathbf{Z}_{(p)} \cong CH^*(M_a)_{(p)}$ in (2.1).

Remark. Let $BP\langle n\rangle^* = \mathbf{Z}_{(p)}[v_1, \dots, v_n]$. Recall ([Ya2]) that

$$ABP\langle n\rangle^{2*,*}(X) \cong \Omega^{2*}(X) \otimes_{\Omega^*} BP\langle n\rangle^*$$

for smooth X. Then we also see that

$$i_{\bar{k}} : ABP\langle n-1\rangle^{2*,*}(M_a) \to ABP\langle n-1\rangle^{2*,*}(M_a|_{\bar{k}})$$

is injective. In particular, when $n = 1$, $ABP\langle 0\rangle^{2*,*}(-) = CH^{2*}(-)_{(p)}$ and hence

$$CH^*(M_a)_{(p)} \cong \mathbf{Z}_{(p)}\{1\} \oplus \mathbf{Z}_{(p)}[y]/(y^{p-1})\{py\} \subset \mathbf{Z}_{(p)}[y]/(y^p) \cong CH^*(M_a|_{\bar{k}})_{(p)}.$$

3. Compact Lie group G

Let G be a compact connected Lie group. By the Borel theorem, we have the ring isomorphism for p odd

$$(3.1) \qquad H^*(G; \mathbf{Z}/p) \cong P(y)/(p) \otimes \Lambda(x_1, \dots, x_l) \quad \text{with } P(y) = \bigotimes_{i=1}^{k} \mathbf{Z}[y_i]/(y_i^{p^{r_i}})$$

where $|y_i| = even$ and $|x_j| = odd$. When $p = 2$, for each y_i, there is x_j with $x_j^2 = y_i$. Hence we have $gr\, H^*(G; \mathbf{Z}/2) \cong P(y)/(2) \otimes \Lambda(x_1, \dots, x_l)$.

Let T be the maximal torus of G and BT the classifying space of T. We consider the fibering

$$G \xrightarrow{\pi} G/T \xrightarrow{i} BT \tag{3.2}$$

and the induced spectral sequence

$$E_2^{*,*} = H^*(BT; H^*(G; \mathbf{Z}/p)) \Rightarrow H^*(G/T; \mathbf{Z}/p).$$

The cohomology of the classifying space of the torus is given by

$$H^*(BT) \cong \mathbf{Z}[t_1, \dots, t_\ell] \quad \text{with } |t_i| = 2.$$

where ℓ is also the number of the odd degree generators x_i in $H^*(G; \mathbf{Z}/p)$. It is known that y_i are permanent cycles and that there is a regular sequence ([Tod], [Mi-Ni]) $(\bar{b}_1, \dots, \bar{b}_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i) = \bar{b}_i$. Thus we get

$$E_\infty^{*,*'} \cong P(y) \otimes \mathbf{Z}/p[t_1, \dots, t_\ell]/(\bar{b}_1, \dots, \bar{b}_\ell).$$

Moreover we know that G/T is a manifold (flag manifold) with torsion free cohomology, and we get

$$H^*(G/T)_{(p)} \cong \mathbf{Z}_{(p)}[y_1, .., y_k, t_1, \dots t_\ell]/(f_1, \dots, f_k, b_1, \dots, b_\ell) \tag{3.3}$$

where $b_i = \bar{b}_i \ mod(p)$ and $f_i = y_i^{p^{r_i}} \ mod(t_1, \dots, t_\ell)$. We also know

$$MU^*(G/T)_{(p)} \cong \Omega^*[y_1, \dots, y_k, t_1, \dots t_\ell]/(\tilde{f}_1, \dots, \tilde{f}_k, \tilde{b}_1, \dots, \tilde{b}_\ell) \tag{3.4}$$

where $\tilde{b}_i = b_i \ mod(MU^{<0})$ and $\tilde{f}_i = f_i \ mod(MU^{<0})$.

Let G_k be the split reductive algebraic group corresponding G and T_k the split maximal torus. Since G_k/B_k is cellular, we have

$$CH^*(G_k/T_k) \cong CH^*(G_{\mathbf{C}}/T_{\mathbf{C}}) \cong H^*(G/T),$$
$$\text{and} \quad \Omega^*(G_k/T_k) \cong \Omega^*(G_{\mathbf{C}}/T_{\mathbf{C}}) \cong MU^*(G/T).$$

Next we consider the relation between $CH^*(\mathbf{G}_k)$ and $CH^*(\mathbf{G}_k/T_k)$ (or $\Omega^*(\mathbf{G}_k)$ and $\Omega^*(\mathbf{G}_k/T_k)$).

THEOREM 3.1 (Grothendieck [Gr], [Ya1]). *Let $\mathbf{G}_k$ be a G_k-torsor over k. (Here we do not assume the nontriviality of $\mathbf{G}_k$). Let $h^*(X) = CH^*(X)$ or $\Omega^*(X)$. Then*

$$h^*(\mathbf{G}_k) \cong h^*(\mathbf{G}_k/T_k)/(i^*h^*(BT_k)) \cong h^*(\mathbf{G}_k/T_k)/(t_1, \dots, t_\ell).$$

Proof. Let $L_i \to \mathbf{G}_k/T_k$ be the line bundle corresponding the element $t_i \in CH^2(\mathbf{G}_k/T_k)$. Then we can embed the T_k-bundle $\mathbf{G}_k \to \mathbf{G}_k/T_k$ into the associated vector bundle $\bigoplus_i L_i \to \mathbf{G}_k/T_k$ such that $\mathbf{G}_k$ is an open subscheme of $\bigoplus_i L_i$. Consider the localization exact sequence

$$\bigoplus_i h^*\left(\bigoplus_{j \neq i} L_j\right) \xrightarrow{\oplus s_{i*}} h^*\left(\bigoplus_i L_i\right) \longrightarrow h^*(\mathbf{G}_k) \longrightarrow 0$$

where $s_i : \mathbf{G}_k/T_k \to L_i$ is a zero section. Since L_i are vector bundles

$$h^*(\mathbf{G}_k/T_k) \cong h^*\left(\bigoplus_{i \neq j} L_j\right) \cong h^*\left(\bigoplus_i L_i\right).$$

By the definition of the first Chern class, we know $t_i = c_1(L_i) = s_i^* s_{i*}(1)$. Thus we get the desired result $h^*(\mathbf{G}_k) \cong h^*(\mathbf{G}_k/T_k)/(t_1, \dots, t_\ell)$. □

Note that $CH^*(G_k) \cong CH^*(G_{\mathbf{C}})$ from $CH^*(G_k/T_k) \cong CH^*(G_{\mathbf{C}}/T_{\mathbf{C}})$.

Corollary 3.2 ([Ya1], [Ka]). $CH^*(G_k)_{(p)} \cong P(y)_{(p)}/(py_i \mid 1 \le i \le k)$.

The following theorem for $\Omega^*(G_{\mathbf{C}})$ is one of the main result in [Ya1]. Let Q_i be the Milnor primitive operation in $H^*(X; \mathbf{Z}/p)$ inductively defined by $Q_i = [Q_{i-1}, P^{p^{i-1}}]$ and $Q_0 = \beta$ where β is the Bockstein operation and $P^{p^{p-1}}$ is the p^{p-1}-th reduced power operation. It is known that we can take generators such that $Q_i(x_j) \in P(y)/(p)$ for all $i \ge 0$, $1 \le j \le \ell$ ([Mi-Ni]).

Theorem 3.3 ([Ya1]). *Take generators so that $Q_i(x_j) \in P(y)/(p)$ for all $i \ge 0$, $1 \le j \le \ell$. Then there is an Ω^*-module isomorphism*

$$\Omega^*(G_k)/I_\infty^2 \cong \Omega^* \otimes P(y) \Big/ \left(I_\infty^2, \sum_i v_i Q_i(x_j) \mid 1 \le j \le \ell\right).$$

Let P be a parabolic subgroup. Then the inclusion $T \subset P$ induces the fibering

$$P/T \to G/T \xrightarrow{p} G/P \tag{3.5}$$

and the spectral sequence (see [Tod])

$$E(G/T)_2^{*,*'} \cong H^*(G/P) \otimes H^{*'}(P/T) \Rightarrow H^*(G/T).$$

Since these cohomology have no torsion and are even dimensionally generated, this spectral sequence collapses,

$$gr\, H^*(G/T) \cong H^*(G/P) \otimes H^*(P/T). \tag{3.6}$$

Hence $H^*(G/P)$ can be computed from $H^*(G/T)$ (while some cases $H^*(G/P)$ are more easy). The cohomology $H^*(P/T)$ can be computed by the fibering $P/T \to BT \xrightarrow{i} BP$. Indeed, if $i^* \mid H^*(BP)$ is injective, then $H^*(P/T) \cong H^*(BT)/(i^*\tilde{H}^*(BP))$. Note here when $P = B$ the Borel subgroup, we know $H^*(G/T) \cong H^*(G/B)$ (similar isomorphisms hold for $CH^*(-)$ and $\Omega^*(-)$).

4. Exceptional groups of type (I)

Let G be a simply connected compact Lie group with the flag manifold G/T of dimension $2d$. The torsion index is defined by

$$t(G) = |H^{2d}(G/T; \mathbf{Z})/i^* H^{2d}(BT; \mathbf{Z})|.$$

By Grothendieck, it is known that any G_k-torsor $\mathbf{G}_k$ splits over some fields L_i over k with $gcd[L_i:k]$ dividing $t(G)$. By Totaro all $t(G)$ are recently known [To2,3]. Let us write by $t(G)_{(p)}$ the p-component of $t(G)$. In this section, we restrict the cases $t(G)_{(p)} = p$ (for ease of arguments) and G are simply connected exceptional Lie groups. We call such (G,p) is of type (I), that is

$$(G_2,2),\ (F_4,2),\ (E_6,2)$$
$$(F_4,3),\ (E_6,3),\ (E_7,3),\quad \text{and}\quad (E_8,5).$$

Throughout this section, we assume (G,p) are type of (I). For these cases, the ordinary $mod(p)$ cohomology is well known

$$gr\, H^*(G;\mathbf{Z}/p) \cong \mathbf{Z}/p[y]/(y^p) \otimes \Lambda(x_1,\dots,x_\ell)$$

where $\ell = rank(G) \geq 2$, $|y| = 2p+2$, $|x_1| = 3$, $|x_2| = 2p+1$. Moreover

$$Q_1(x_1) = y, \quad Q_0(x_2) = y.$$

From Corollary 3.2, we see

Corollary 4.1. $CH^*(G_k)_{(p)} \cong \mathbf{Z}_{(p)}[y]/(y^p, py)$.

From Theorem 3.3 and the Q_i-actions, we see

$$\Omega^*(G_k)/I_\infty^2 \cong \Omega^*[y]/(py, v_1 y, y^p, I_\infty^2),$$

while we have more strong result (Theorem 5.1 in [Ya1]).

Corollary 4.2. $\Omega^*(G_k) \cong \Omega^*[y]/(py, v_1 y, y^p)$.

Remark. In the Atiyah-Hirzebruch spectral sequence ([Ya2])

$$E_2^{*,*',*''} \cong H^{*,*'}(G_k; MU^{*''}) \Rightarrow MGL^{*,*'}(G_k)$$

we know that

$$d_{2p-1}(x_1) = v_1 \otimes Q_1(x_1) = v_1 y.$$

Thus we get also $E_\infty^{2*,*,*''} \cong MU^*[y]/(py, v_1 y, y^p)$.

For general G, recall that the polynomial parts $P(y)$ of $H^*(G;\mathbf{Z}/p)$ is written as $\bigotimes_i^k \mathbf{Z}/p[y_i]/(y_i^{p^{r_i}})$. In [Pe-Se-Za], Petrov, Semenov and Zainoulline defined the J-invariant $J_p(\mathbf{G}_k) = (i_1,\dots,i_k)$ of $\mathbf{G}_k$ (roughly speaking) as the smallest number i_s such that

$$y_s^{p^{i_s}} \in Im(CH^*(\mathbf{G}_k/T_k) \xrightarrow{i_{\bar{k}}} CH^*(G_k/T_k) \xrightarrow{\pi^*} CH^*(G_k))$$

with some changes for generators. (More accurate definition, see 4.6 in [Pe-Se-Za].) In particular, $J_p(\mathbf{G}_k) = (0,\dots,0)$ if and only if $\mathbf{G}_k$ splits by a finite extension K/k of degree coprime to p (4.7, Corollary 6.7 in [Pa-Se-Za]). Hence if G is a group of type (I) and $\mathbf{G}_k$ is nontrivial at p, then $J(\mathbf{G}_k) = (1)$.

Theorem 4.3 (Theorem 5.13 in [Pe-Se-Za]). *Let $J_p(\mathbf{G}_k) = (1)$. Then there is a mod(p) indecomposable motive $R_p(G)$ such that*

(1) $$CH^*(R_p(G)|_{\bar{k}}))/p \cong \mathbf{Z}/p[y]/(y^p)$$

(2) $$M(\mathbf{G}_k/T_k; \mathbf{Z}/p) \cong \bigoplus_s R_p(G) \otimes \mathbf{T}^{\otimes j_s} \cong R_p(G) \otimes H^*(G/T; \mathbf{Z}/p)/(y)$$

where we identify $H^*(G/T; \mathbf{Z})/(y)$ as the sum of mod p Tate motives $\bigoplus T^{\otimes j_s}$.

We say that L is splitting field of a variety X if the motive $M(X|_L)$ of $X|_L$ is isomorphic to a direct sum of twisted Tate motives $\mathbf{T}^{\otimes i}$. A smooth scheme X is said to be generically split over k if its function field $L = k(X)$ is a splitting field. The complete flag variety $\mathbf{G}_k/B_k$ is always generically split.

Theorem 4.4 (Theorem 3.7 in [Pe-Se-Za]). *Let $Q_k \subset P_k$ be parabolic subgroups of G_k which are generically split over k. There is a decomposition of motive $M(\mathbf{G}_k/Q_k)_{(p)} \cong M(\mathbf{G}_k/P_k)_{(p)} \otimes H^*(P/Q)$.*

For $p = 2, 3$, from Proposition 5.21 (for $m = p$) and §7 in [Pe-Se-Za], we have the *integral* motivic decomposition which deduces the *mod*(p) decomposition in Theorem 4.3. Moreover when $(G, p) = (G_2, 2)$ or $(F_4, 3)$ from Bonnet, Semenov and Zainoulline (see Corollary 6 in [Vi-Za], and also [Se], [Bo], [Ni-Se-Za]), we know that the integral motive corresponding $R_p(G)$ is really generalized Rost motive M_2.

Corollary 4.5. *Let $(G, p) = (G_2, 2)$ or $(F_4, 3)$, and assume that $\mathbf{G}_k$ is nontrivial at p. Then for each parabolic subgroup P_k, $\mathbf{G}_k/P_k$ is generically split and*

$$CH^*(G_k/P_k)_{(p)} \cong \mathbf{Z}[y]/(y^p) \otimes A \quad \textit{and} \quad M(\mathbf{G}_k/P_k)_{(p)} \cong M_2 \otimes A$$

where A is a sum of twisted Tate motives and $M_2 = M_a$ is the generalized Rost motive for some $0 \neq a \in K_3^M(k)/p$.

The following theorem implies $CH^*(\mathbf{G}_k)_{(2)} \cong \mathbf{Z}_{(2)}$ when $(G, p) = (G_2, 2)$.

Theorem 4.6. *Let G be type (I), and assume that*

$$M(\mathbf{G}_k/B_k)_{(p)} \cong M_2 \otimes H^*(G/T)_{(p)}/(y).$$

Then the Chow ring $CH^*(\mathbf{G}_k/T_k)_{(p)}$ is multiplicatively generated by $t_1, \ldots, t_\ell$ when $p = 2$ (for $* < 2p + 6$ when $p = odd$). Hence $CH^*(\mathbf{G}_k)_{(p)} \cong \mathbf{Z}_{(p)}$ when $p = 2$ (for $* < 2p + 6$ when $p = odd$).

Proof. We consider the restriction map

$$i_{\bar{k}} : \Omega^*(\mathbf{G}_k/T_k) \to \Omega^*(\mathbf{G}_k/T_k|_{\bar{k}}) \cong MU^*(G/T)_{(p)}.$$

Since $i_{\bar{k}} \,|\, \Omega^*(M_2)$ is injective, so is $i_{\bar{k}}$ above. Let us write

$$Im(i_{\bar{k}}) = i_{\bar{k}}(\Omega^*(\mathbf{G}_k/T_k)) \subset \Omega^*(G_k/T_k) = MU^*(G/T)_{(p)}.$$

Of course $py^i, v_1y^i \in Im(i_{\bar{k}})$ for $i > 0$ since so in $\Omega^*(M_2|_{\bar{k}})$. Note that $t_1,..,t_\ell \in Im(i_{\bar{k}})$ because they exist in $CH^*(\mathbf{G}_k/T_k)$ since so in $CH^*(BT_k)$.

Recall that each element $x \in \Omega^*(\mathbf{G}_k/T_k|_{\bar{k}}) \cong \Omega^*(G_k/T_k)$ is represented as

$$(*) \qquad x = \sum_{i=0}^{p-1}\sum_s v(s,i)t(s,i)y^i, \quad v(s,i) \in \Omega^*,\ t(s,i) \in \mathbf{Z}_{(p)}[t_1,\dots,t_\ell]$$

while if $x \in Im(i_{\bar{k}})$, then $v(s,i) \in Ideal(p, v_1)$ for $i > 0$.

From Corollary 4.2, we see $py = v_1y = 0$ in $\Omega^*(G_k)$. From Theorem 3.1, this means

$$(**) \qquad py, v_1y \ \in (t_1,\dots,t_\ell)\Omega^*(G_k/T_k).$$

(But note that this does not mean $py, v_1y \in (t_1,\dots,t_\ell)\ Im(i_{\bar{k}})$ while we will see it.) Let us write $v_1y = \sum v(s,i)t(s,i)y^i$ as $(*)$. The above fact $(**)$ implies $|t(s,i)| > 0$ for $i > 0$, and hence $|v(s,i)| < 0$.

Now we consider $\Omega\langle 1\rangle^*(-)$-theory. Let us write

$$\Omega\langle 1\rangle^*(X) = \Omega^*(X) \otimes_{\Omega^*} \mathbf{Z}_{(p)}[v_1] = ABP\langle 1\rangle^{2*,*}(X).$$

In $\Omega\langle 1\rangle^*(G_k/T_k)$, the fact $|v(s,i)| < 0$ means

$$v(s,i) \in (v_1) = \mathbf{Z}_{(p)}[v_1]^{<0} = \Omega\langle 1\rangle^{<0}.$$

Hence $v_1y \in (t_1,\dots,t_\ell)\ Im(i_{\bar{k}})$ in $\Omega\langle 1\rangle^*(-)$ theory.

Thus we can write

$$v_1y = \sum_{i>0}^{p-1}\sum_s v(s,i)'t(s,i)'v_1y^i + \sum_s v(s,0)'t(s,0)' \quad \text{in } \Omega\langle 1\rangle^*(\mathbf{G}_k/T_k).$$

If $v(s,i)' \neq 0$ for $i > 0$, then apply the same equation to the right hand side v_1y in the above equation. Since $t(s,i) = 0$ when $|t(s,i)| > dim(G/T)$, we can write

$$v_1y = \sum_s v(s,0)''t(s,0)''.$$

We have the similar result for py. Hence $i_{\bar{k}}(\Omega\langle 1\rangle^*(\mathbf{G}_k/T_k))$ is generated as an $\Omega\langle 1\rangle^*$-algebra by $t_1,\dots,t_\ell$ when $p = 2$ (for $* < |v_1y^2| = 2p+6$ when $p = odd$).

Since we know the isomorphisms

$$CH^*(\mathbf{G}_k/T_k)_{(p)} \cong \Omega^*(\mathbf{G}_k/T_k) \otimes_{\Omega^*} \mathbf{Z}_{(p)} \cong \Omega\langle 1\rangle^*(\mathbf{G}_k/T_k) \otimes_{\Omega\langle 1\rangle^*} \mathbf{Z}_{(p)},$$

we get the desired results. □

5. Exceptional Lie group G_2

In this section we study $CH^*(\mathbf{G}_k/T_k)$ for the case $(G,p)=(G_2,2)$. We recall the cohomology from Toda-Watanabe [To-Wa]

$$H^*(G/T;\mathbf{Z}) \cong \mathbf{Z}[t_1,t_2,y]/(t_1^2+t_1t_2+t_2^2, t_2^3-2y, y^2)$$

with $|t_i|=2$ and $|y|=6$. Let $P(=P_1)$ be the maximal parabolic such that G/P is isomorphic to a quadric. Then from (3.6) and $H^*(P/T)\cong \mathbf{Z}\{1,t_1\}$, we have

$$H^*(G/P;\mathbf{Z}) \cong \mathbf{Z}[t_2,y]/(t_2^3-2y,y^2) \cong \mathbf{Z}\{1,y\}\otimes\{1,t_2,t_2^2\}.$$

By Bonnet, we have the decomposition

Theorem 5.1 ([Bo], §7 in [Pe-Se-Za]).

$$M(\mathbf{G}_k/P_k) \cong M_2 \oplus M_2(1) \oplus M_2(2).$$

Theorem 5.2. *There is a ring isomorphism*

$$CH^*(\mathbf{G}_k/P_k)_{(2)} \cong \mathbf{Z}_{(2)}[t_2,u]/(t_2^6, 2u, t_2^3u, u^2)$$
$$\cong \mathbf{Z}_{(2)}[t_2]/(t_2^6) \oplus \mathbf{Z}/2[t_2]/(t_2^3)\{u\}$$

with $|t_2|=2$, $|u|=4$.

Proof. From Lemma 2.2, we know

$$\Omega^*(M_2) \cong \Omega^*\{1,2y,v_1y\} \subset \Omega^*\{1,y\}.$$

From the preceding theorem, we have the Ω^*-module isomorphism

$$\Omega^*(\mathbf{G}_k/P_k) \cong \Omega^*\{1,v_1y,2y\}\otimes\{1,t_2,t_2^2\} \subset \Omega^*(G_k/P_k).$$

Since $CH^*(X)_{(p)} \cong \Omega^*(X)\otimes_{\Omega^*}\mathbf{Z}_{(p)}$, we have the isomorphism

$$CH^*(\mathbf{G}_k/P_k)_{(2)} \cong \mathbf{Z}_{(2)}\{1,2y\}\{1,t_2,t_2^2\} \oplus \mathbf{Z}/2\{v_1y\}\{1,t_2,t_2^2\}.$$

(Note $2v_1y = v_1(2y) \in \Omega^{<0}\Omega^*(\mathbf{G}_k/P_k)$.)

Here the multiplications are given as follows. Since $2y = t_2^3 \ mod(\Omega^{<0})$ in $\Omega^*(G_k/T_k)$, we can take $2y = t_2^3 \in CH^*(\mathbf{G}/P_k)_{(2)}$ so that

$$\mathbf{Z}_{(2)}\{1,2y\}\{1,t_2,t_2^2\} = \mathbf{Z}_{(2)}[t_2]/(t_2^6) \subset CH^*(\mathbf{G}/P_k)_{(2)}.$$

Let us write $u=v_1y$ in $CH^*(\mathbf{G}_k/T_k)_{(2)}$. Then $t_2^3u = 2yv_1y = 0$ and $u^2 = v_1^2y^2 = 0$ in $\Omega^*(\mathbf{G}_k/T_k)\otimes_{\Omega^*}\mathbf{Z}_{(2)}$. Hence we have the isomorphism in the theorem. □

Remark. The space $\mathbf{G}_k/P_k$ is isomorphic to the quadric defined by the maximal neighbor of the 3-Pfister form. Hence its Chow ring is computed in [Ya3]. (See also Lemma 7.2 and 7.4 below.)

Next consider $CH^*(\mathbf{G}_k/T_k)_{(2)}$.

THEOREM 5.3. *There is a ring isomorphism*

$$CH^*(\mathbf{G}_k/T_k)_{(2)} \cong \mathbf{Z}_{(2)}[t_1, t_2]/(t_2^6, 2u, t_2^3u, u^2)$$

where $u = t_1^2 + t_1t_2 + t_2^2$.

Proof. The Chow ring is isomorphic to

$$(*) \qquad CH^*(\mathbf{G}_k/T_k)_{(2)} \cong CH^*(\mathbf{G}_k/P_k)\{1, t_1\}$$
$$\cong (\mathbf{Z}_{(2)}\{1, 2y\} \oplus \mathbf{Z}/2\{v_1y\})\{1, t_2, t_2^2\}\{1, t_1\}.$$

Here $2y = t_2^3$. Since $v_1y \in (t_1, t_2)$ and $v_1y = 0 \in CH^*(G_k/T_k)$, we see

$$v_1y = \lambda(t_1^2 + t_1t_2 + t_2^2) \quad mod((t_1, t_2)\Omega^{<0}\Omega^*(G_k/T_k))$$

for $\lambda \in \mathbf{Z}_{(2)}$. We can take $\lambda = 1$ $mod(2)$. Otherwise $v_1y = 0 \in \Omega^*(G_k/T_k)/2$, which is $\Omega^*/2$-free, and this is a contradiction. Hence we can take $t_1^2 + t_1t_2 + t_2^2$ as v_1y. (This is also proved by Lemma 4.3 in [Ya1], since $Q_1(x_1) = y$ and $d_3(x_1) = t_1^2 + t_1t_2 + t_2^2$.) Hence in $CH^*(\mathbf{G}_k/T_k)$ we have the relation

$$(t_2^3)^2 = 0, \quad (t_2^3)u = 0, \quad u^2 = 0, \quad 2u = 0.$$

We consider the mod 2 Poincare polynomial

$$\sum_i rank_{\mathbf{Z}/2}(CH^{2i}(\mathbf{G}_k/T_k)/2)t^i = (1 + t^2 + t^3)(1 + t + t^2)(1 + t)$$

$$= 1 + 2t + 3t^2 + 4t^3 + 4t^5 + 3t^5 + t^6 = \frac{(1 - t^6)(1 - t^4)}{(1 - t)(1 - t)} - t^5(1 + t)^2$$

which is the $(mod(2))$ Poincare series of the right hand side ring of the theorem. (Note (t_2^6, u^2) is a regular sequence in $\mathbf{Z}/2[t_1, t_2]$ but $(t_2^6, u^2, (t_2^3)u)$ is not.) □

The author learned the following remarks by Kirill Zainoulline.

Remark. It is well known that there is a bijection between $H^1(k; G_2)$ and the class of Cayley algebras C from the fact $G_2 = Aut(C|_{\bar{k}})$. Hence each torsor $\mathbf{G}_k$ over k corresponds a Cayley algebra. Moreover $\mathbf{G}_k/B_k$ and $\mathbf{G}_k/P_k$ correspond the following varieties [Ca-Pe-Se-Za]. By an i-space $(i = 1, 2)$, we mean i-dimensional subspace V_i of C such that $u \cdot v = 0$ for every $u, v \in V_i$. The flag variety corresponding $\mathbf{G}_k/B_k$ is the full flag variety

$$X(1, 2) = \{V_1 \subset V_2 | V_i; i - subspaces \subset C\}$$

and the flag variety corresponding $\mathbf{G}_k/P_k$ is

$$X(2) = \{V_2 | V_2; 2 - subspaces \subset C\}.$$

Let g be the map

$$g : H^1(k; G_2) \to H^3(k; \mathbf{Z}/2) \cong K_3^M(k)/2$$

induced from the Rost cohomological invariant. The symbol of the Rost motive in Theorem 5.1 is $g(\mathbf{G}_k)$ i.e., $M_2 = M_{g(\mathbf{G}_k)}$.

Remark. Similar facts hold for $(G, p) = (F_4, 3)$. This case, the corresponding algebras are exceptional Jordan algebras of dimension 27 over k, and the symbol for the generalized motive is the image of also the Rost cohomological invariant.

6. Exceptional group F_4 for $p = 3$

Let $(G, p) = (F_4, 3)$ throughout this section. Let $\mathbf{G}_k$ be a nontrivial G_k-torsor at 3. Let P be a maximal parabolic subgroup of G given by the the first three vertexes of the Dynkin diagram.

$$\overset{1}{\circ} -- \overset{2}{\circ}\Rightarrow\!=\!\overset{3}{\circ} -- \overset{4}{\circ}.$$

We also note $G/P \cong F_4/B_3 \cdot S^1$.

Theorem 6.1 (Corollary 6 in [Vi-Za], [Se]). *Let M_2 be the generalized Rost motive. Then there is an isomorphism $M(\mathbf{G}_k/P_k) \cong \bigoplus_{i=0}^{7} M_2(i)$.*

We first recall the ordinary cohomology of G/P ([Is-To], Theorem 2 in [Du-Za]).

$$H^*(G/P) \cong \mathbf{Z}[t, y]/(r_8, r_{12}), \quad |t| = 2,\ |y| = 8$$

where $r_8 = 3y^2 - t^8$ and $r_{12} = 26y^3 - 5t^{12}$. Hence we can rewrite

$$H^*(G/P)_{(3)} \cong \mathbf{Z}_{(3)}\{1, t, \dots, t^7\} \otimes \{1, y, y^2\}.$$

Recall the Chow rings of the Rost motive

$$CH^*(M_2|_{\bar{k}}) \cong \mathbf{Z}[y]/(y^3),$$

$$CH^*(M_2) \cong \mathbf{Z}\{1\} \oplus \mathbf{Z}\{3y, 3y^2\} \oplus \mathbf{Z}/3\{v_1 y, v_1 y^2\}.$$

Of course, the above $y \in CH^*(M_a)$ can be identified with the same named element in $H^*(G_k/P_k)_{(3)}$ by the restriction map $CH^*(M_a) \to CH^*(M_a|_{\bar{k}}) \subset CH^*(G_k/P_k)_{(3)}$. From the above theorem, we have the decomposition

$$(*) \quad CH^*(\mathbf{G}_k/P_k)_{(3)} \cong \mathbf{Z}_{(3)}\{1, t, \dots, t^7\} \otimes (\mathbf{Z}_{(3)}\{1, 3y, 3y^2\} \oplus \mathbf{Z}/3\{v_1 y, v_1 y^2\}).$$

The ring structure is given as follows.

Theorem 6.2.

$$CH^*(\mathbf{G}_k/P_k)_{(3)} \cong \mathbf{Z}_{(3)}[t, b, a_1, a_2]/(t^{16}, t^8 b, b^2 = 3t^8, ba_i, 3a_i, t^8 a_i, a_1 a_2)$$

$$\cong \mathbf{Z}_{(3)}\{1, t, \dots, t^7\} \otimes (\mathbf{Z}_{(3)}\{1, b = \sqrt{3}t^4, t^8\} \oplus \mathbf{Z}/3\{a_1, a_2\})$$

where $|b| = 8$ and $|a_1| = 4$, $|a_2| = 12$.

Proof. From the relation r_8 in $CH^*(G/P)$, we have

$$3y^2 = t^8 + vx \in \Omega^*(G/P) \quad \text{for } v \in \Omega^{<0}.$$

Hence we can take t^8 instead of $3y^2$ in $(*)$. Of course

$$(3y)^2 = 3t^8 + 3vx \in \Omega^*(G/P).$$

Hence we write by $b = \sqrt{3t^4}$ the element $3y$. Write by a_1, a_2 the elements $v_1 y$, $v_1 y^2$ respectively. Elements in $I_\infty \Omega^{<0} \subset \Omega(G_k/P_k)$ reduces to zero in $CH^*(\mathbf{G}_k/T_k)$. Therefore we have the desired multiplicative results. □

The cohomology $H^*(G/T)$ is given by Toda-Watanabe [To-Wa]

$$H^*(G/T)_{(3)} \cong \mathbf{Z}_{(3)}[t_1, t_2, t_3, t_4, y]/(\rho_2, \rho_4, \rho_6, \rho_8, \rho_{12}).$$

Here relations ρ_i are written by the elementary symmetric functions $c_i = \sigma_i(t_1, t_2, t_3, t_4)$, that is,

$$\rho_2 = c_2 - (1/2)c_1^2, \quad \rho_4 = c_4 - c_3 c_1 + (1/2)^3 c_1^4 - 3y, \quad \rho_6 = -c_4 c_1^2 + c_3^2,$$
$$\rho_8 = 3c_4 c_1^4 - (1/2)^4 c_1^8 + 3y(2^4 y + 2^3 c_3 c_1), \quad \rho_{12} = y^3.$$

By the arguments similar to the proof of Theorem 5.3 (or Lemma 4.3 in [Ya1]), we can prove

Theorem 6.3. *Let* $\pi : \mathbf{G}_k/T_k \to \mathbf{G}_k/P_k$. *Then*

$$\pi^*(t) = c_1, \quad \pi^*(a_1) = \rho_2, \quad \pi^*(b) = c_4 - c_3 c_1 - (2)^{-3} c_1^4.$$

Hence there is an epimorphism

$$\mathbf{Z}_{(3)}[t_1, t_2, t_3, t_4]/(c_1^{16}, c_1^8 \pi^*(b), \pi^*(b)^2 - 3c_1^8, \pi^*(b)\rho_j, 3\rho_j, c_1^8 \rho_j, \rho_2 \rho_6)$$
$$\to CH^*(\mathbf{G}_k/T_k)_{(3)}/(\pi^*(a_2) - \rho_6),$$

where $j = 2, 6$.

Proof. We consider the composition of maps

$$CH^*(\mathbf{G}_k/P_k) \xrightarrow{\pi^*} CH^*(\mathbf{G}_k/T_k) \xrightarrow{i_{\bar{k}}} CH^*(G_k/T_k).$$

It is known $\pi_*(t) = c_1$ in $CH^*(G_k/T_k)$. By dimensional reason, so is in $CH^*(\mathbf{G}_k/T_k)$. Note $i_{\bar{k}}\pi_*(a_i) = i_{\bar{k}}\pi_*(v_1 y^i) = 0 \in CH^*(G_k/T_k)$ and hence $\pi_*(a_i) \in Ideal(\rho_2, \dots, \rho_{12})$. By dimensional reason, we see $\pi_*(a_1) = \rho_2$ and $\pi_*(a_2) - \rho_6 \in Ker(i_{\bar{k}})$. The element b is defined from $3y \in \Omega^*(G_k/T_k)$. So we have the result for $\pi_*(b)$ from the relation ρ_4. □

If we can take a_2 with $\pi^*(a_2) = \rho_6$, then we get $CH^*(\mathbf{G}_k)_{(3)} \cong \mathbf{Z}_{(3)}$. Otherwise we see $CH^{12}(\mathbf{G}_k)_{(3)} \neq 0$.

7. The orthogonal group $SO(m)$ and $p = 2$

We consider the orthogonal groups $G = SO(m)$ and $p = 2$. The mod 2-cohomology is written as (see for example [Ni])

$$gr\, H^*(SO(m); \mathbf{Z}/2) \cong \Lambda(x_1, x_2, \dots, x_{m-1})$$

where the multiplications are given by $x_s^2 = x_{2s}$. We write $y_{2(odd)} = x_{2(odd)} = x_{odd}^2$. Hence we can write

$$H^*(SO(m); \mathbf{Z}/2) \cong \mathbf{Z}/2[y_{4i+2} \,|\, 2 \le 4i+2 \le m-1]/(y_{4i+2}^{s(i)}) \otimes \Lambda(x_1, x_3, \dots x_{\bar{m}})$$

where $s(i)$ is the smallest number such that $2^{s(i)}(4i+2) \ge m$ and $\bar{m} = m-1$ (resp. $\bar{m} = m-2$) if m is even (resp. odd).

The Q_i-operations are given by Nishimoto [Ni]

$$Q_n x_{odd} = x_{odd+|Q_n|}, \quad Q_n x_{even} = Q_n y_{even} = 0.$$

Relations in $\Omega^*(SO(m))$ are given by

$$\sum_n v_n Q_n(x_{odd}) = \sum_n v_n x_{odd+|Q_n|} = 0 \quad mod(I_\infty^2).$$

For example, the relation in $\Omega^*(SO(m))/I_\infty^2$ starting with $2y_6$ are written as

$$\begin{aligned} &2Q_0(x_5) + v_1 Q_1(x_5) + v_2 Q_2(x_5) + v_3 Q_3(x_5) + \cdots \\ &\quad = 2x_6 + v_1 x_8 + v_2 x_{12} + v_2 x_{20} + \cdots \\ &\quad = 2y_6 + v_1 y_2^4 + v_2 y_6^2 + v_3 y_{10}^2 + \cdots = 0 \quad mod(I_\infty^2). \end{aligned}$$

Theorem 7.1 ([Ya1]). *There is an $\Omega_{(2)}^*$-algebra isomorphism*

$$\Omega^*(SO(m))/I_\infty^2 \cong \Omega^*[y_{4i+2} \,|\, 2 \le 4i+2 \le m-1]/(R, I_\infty^2)$$

where $R = \{$relations starting with $y_{4i+2}^{2^{s(i)}}, 2y_{4i+2}, v_1 y_{4i'+2}, i' \neq 0\}$.

For ease of arguments, we only consider the case $G = SO(odd)$. Let $G = SO(2m'+1)$ and $P = SO(2m'-1) \times SO(2)$. Then it is well known [To-Wa]

Lemma 7.2. *$H^*(G/P) \cong \mathbf{Z}[t, y]/(t^{m'} - 2y, y^2)$ $|y| = 2m'$.*

By Toda-Watanabe [To-Wa], we also know

Theorem 7.3 ([To-Wa]).

$$H^*(G/T) \cong \mathbf{Z}[t_i, y_{2i}, t_{m'}, y]/(c_i - 2y_{2i}, J_{2i}, t_{m'}^{m'} - 2y, y^2)$$

where $1 \le i \le m'-1$, $c_i = \sigma(t_1, \dots, t_{m'})$ and

$$J_{2i} = 1/4\left(\sum_{j=0}^{2i} (-1)^j c_j c_{2i-j}\right) = y_{4i} - \sum_{0<j<2i} (-1)^j y_{2j} y_{4i-2j}.$$

Hence we can write

$$gr\,H^*(G/T) \cong H^*(G/P) \otimes A, \quad A = \mathbf{Z}[t_i, y_i]/(c_i' - 2y_i, J_{2i} \mid 1 \le i \le m' - 1)$$

where $c_i' = \sigma(t_1, \dots, t_{m'-1})$. More precisely, we can write

$$gr\,A = P(y)' \otimes P(t)'$$

where $P(y)' = \bigotimes_{i<2^{n-1}-1} \mathbf{Z}[y_{4i+2}]/(y^{2^{s_i}})$ so that $P(y) = P(y)' \otimes \mathbf{Z}[y]/(y^2)$ and where

$$P(t)' = H^*(BT_{m'-1})/(H^*(BU(m'-1))) \cong Z[t_1, \dots, t_{m'-1}]/(c_1', \dots, c_{m'-1}')$$

Indeed, it is also known that

$$gr\,H^*(G/(U(m'-1) \times SO(2))) \cong P(y)' \otimes H^*(G/P).$$

Now we recall arguments for quadrics. Let $m = 2m' + 1$. and let us write the quadratic form $q(x)$ defined by

$$q(x_1, \dots, x_m) = x_1x_2 + \cdots + x_{m-2}x_{m-1} + x_m^2$$

and the projective quadric X_q defined by the quadratic form q. Then it is well known that (in fact $SO(m)$ acts on the affine quadric in $\mathbf{A}^m - 0$)

$$X_q \cong SO(m)/(SO(m-2) \times SO(2)).$$

Hereafter we assume that $G = SO(m)$ and $P = SO(m-2) \times SO(2)$ and $\mathbf{G}_k$ is nontrivial (at $p = 2$). Moreover we consider the case $m = 2^{n+1} - 1$.

The quadric q is always split over k and we know $CH^*(G_k/P_k) \cong CH^*(X_q)$. Define the quadratic form q' by

$$q'(x_1, \dots, x_m) = x_1^2 + \cdots + x_m^2.$$

Then this q' is a subform of

$$\langle\langle -1, \dots, -1 \rangle\rangle = \phi_{\rho^{n+1}}$$

the $(n+1)$-th Pfister form associated to ρ^{n+1}, where $\rho = (-1) \in K_1^M(k) \cong k^*/(k^*)^2$. (That is, q' is the maximal neighbor of the $(n+1)$-th Pfister form.) Of course $q|_{\bar{k}} = q'|_{\bar{k}}$ and we can identify $\mathbf{G}_k/P_k \cong X_{q'}$. From Lemma 7.2 (or Rost's result), we know

$$CH^*(X_{q'}|_{\bar{k}}) \cong \mathbf{Z}[t, y]/(t^{2^n-1} - 2y, y^2).$$

(Here note that from the existence of nontrivial $\mathbf{G}_k$, we know $0 \ne \rho^{n+1} \in K_{n+1}^M(k)/2$.) As stated in §2, there is a decomposition of motives

$$M(X_{q'}) \cong M_n \otimes \mathbf{Z}/2[t]/(t^{2^n-1}).$$

Hence we have the additive isomorphism

$$CH^*(X_{q'}) \cong \mathbf{Z}[t]/(t^{2^n-1}) \otimes (\mathbf{Z}\{1, c_{n,0}\} \oplus \mathbf{Z}/2\{c_{n,1}, \dots, c_{n,n-1}\}).$$

With identification $t^{2^n-1}=2y=c_{n,0}$, and $u_i=c_{n,i}$ for $i>0$, we also get the ring isomorphism

LEMMA 7.4 (§6 or Lemma 2.2 in [Ya3]). *There is a ring isomorphism*

$$CH^*(\mathbf{G}_k/P_k)\cong \mathbf{Z}[t]/(t^{2^{n+1}-2})\oplus \mathbf{Z}/2[t]/(t^{2^n-1})\{u_1,\dots,u_{n-1}\}$$

where $u_i=v_iy\in\Omega^*(\mathbf{G}_k/P_k)\otimes_{\Omega^*}Z_{(2)}$ *so* $u_iu_j=0$.

By the projection $\mathbf{G}_k/T_k\to\mathbf{G}_k/P_k$, Petrov, Semenov and Zainoulline also show that the J-invariant $J_2(\mathbf{G}_k)=(0,\dots,0,1)$ (7.5 in [Pe-Se-Za]). So we have

THEOREM 7.5. *The restriction map* $i_{\bar{k}}:\Omega^*(\mathbf{G}_k/B_k)\to\Omega^*(\mathbf{G}_k/B_k|_{\bar{k}})=\Omega^*(G_k/B_k)$ *is injective and*

$$gr\ CH^*(\mathbf{G}_k/B_k)\cong gr\ CH^*(\mathbf{G}_k/P_k)\otimes A,$$
$$gr\ \Omega^*(\mathbf{G}_k/B_k)\cong gr\ \Omega^*(\mathbf{G}_k/P_k)\otimes A$$

where $A=\mathbf{Z}[t_i,y_{2i}]/(c_i'-2y_i,J_{2i}\,|\,1\le i\le m'-1)$.

As a corollary, we see that t_i, y_{2i} are all in $CH^*(\mathbf{G}_k/T_k)$ (but y is not). Hence $CH^*(\mathbf{G}_k/T_k)$ is multiplicatively generated by t_i, y_i, t and $u_1,\dots,u_{n-1}$.

THEOREM 7.6. *We have an isomorphism*

$$CH^*(\mathbf{G}_k)_{(2)}\cong P(y)'/(2)\subset P(y)'\otimes\mathbf{Z}/2[y]/(y^2)\cong CH^*(G_k)_{(2)}.$$

Proof. The proof is quite similar to that of Theorem 4.6. Let us write

$$\Omega\langle n-1\rangle^*(X)=\Omega^*(X)\otimes_{\Omega^*}\mathbf{Z}_{(2)}[v_1,\dots,v_{n-1}]\cong ABP\langle n-1\rangle^{2*,*}(X).$$

By Theorem 3.1, we want to prove

$$(1)\qquad u_1,\dots,u_{n-1}\in(t_1,\dots,t_{m'})CH^*(\mathbf{G}_k/T_k).$$

This means

$$u_1,\dots,u_{n-1}\in((t_1,\dots,t_{m'})+\Omega^{<0})\Omega\langle n-1\rangle^*(\mathbf{G}_k/T_k).$$

Let us write

$$Im(i_{\bar{k}})=i_{\bar{k}}^*(\Omega\langle n-1\rangle^*(\mathbf{G}_k/T_k))\subset\Omega\langle n-1\rangle^*(G_k/T_k),$$
$$I(t,\Omega^{<0})=((t_1,\dots,t_{m'})+\Omega^{<0}))\ Im(i_{\bar{k}}).$$

(Note $I_\infty^2\subset\Omega^{<0}\ Im(i_{\bar{k}})$.) Thus it is sufficient for (1) to prove

$$(2)\qquad 2y,\dots,v_{n-1}y\in I(t,\Omega^{<0}).$$

At first we will show $v_{n-1}y\in I(t,\Omega^{<0})$. Recall $y=y_{2^{n+1}-2}=x_{2^{n+1}-2}$. From Theorem 7.1 and Nishimoto's result, we see

$$(3)\quad \begin{aligned} x &= 2Q_0(x_{2^n-1}) + v_1Q_1(x_{2^n-1}) + \cdots + v_{n-2}Q_{n-2}(x_{2^n-1}) + v_{n-1}Q_{n-1}(x_{2^n-1}) \\ &= 2x_{2^n} + v_1x_{2^n+2} + \cdots + v_{n-2}x_{2^n+2^{n-1}-2} + v_{n-1}x_{2^{n+1}-2} \\ &= 0 \quad \text{in } \Omega\langle n-1\rangle^*(G_k)/(I_\infty^2). \end{aligned}$$

So $x \in ((t_1,..,t_{m'}) + I_\infty^2)\Omega\langle n-1\rangle^*(G_k/T_k)$ from Theorem 3.1.

Each element $z \in \Omega\langle n-1\rangle^*(G_k/T_k)$ is written (not uniquely) by

$$(4)\qquad z = \sum v_I t_J y_K + \sum v_{I'} t_{J'} y_{K'} y$$

with $v_I, v_{I'} \in \Omega\langle n-1\rangle^*$, $t_J, t_{J'} \in \mathbf{Z}_{(2)}[t_1,\dots,t_{m'}]$ and $y_K, y_{K'} \in P(y)'$. Note that if $z \in (t_1,\dots,t_{m'})\Omega\langle n-1\rangle^*(G_k/T_k)$, then we can take $|t_J| > 0$ and $|t_{J'}| > 0$.

Consider the case $z = x$ in (3). Since $y_K \in Im(i_{\bar{k}})$, we see

$$v_I t_J y_K \in (t_1,\dots,t_{m'})\ Im(i_{\bar{k}}).$$

Since $|y| < |t_{J'} y_{K'} y|$, we know $|v_{I'}| < 0$, i.e., $v_{I'} y \in Im(i_{\bar{k}})$ because $v_{I'} \in \Omega\langle n-1\rangle^- = \mathbf{Z}_{(2)}[v_1,\dots,v_{n-1}]$. Thus we know $v_{I'} t_{J'} y_{K'} y \in (t_1,\dots,t_{m'})\ Im(i_{\bar{k}})$. Therefore we see

$$(5)\qquad x \in I(t, \Omega^{<0}).$$

In (3), $x_{2^n+2} = y_{2^n+2},\dots,x_{2^n+2^{n-1}-2}$ are in $Im(i_{\bar{k}})$. So we get

$$v_1 x_{2^n+2} + \cdots + v_{n-2} x_{2^n+2^{n-1}-2} \in \Omega^{<0}\ Im(i_{\bar{k}}).$$

Hence we obtain

$$(6)\qquad 2x_{2^n} + v_{n-1} y \in I(t, \Omega^{<0}).$$

Similarly, we have $2x_{2^{n+1}-2^{i+1}} + v_i y \in I(t, \Omega^{<0})$, for $0 < i < n-1$.

Next we will see

$$(7)\qquad 2y_2,\dots,2y_{2^n-2} \in I(t, \Omega^{<0}).$$

Then in particular, $2x_{2^n} = 2(x_2)^{2^{n-1}} = 2x_2 x_2^{2^{n-1}-1} \in I(t, \Omega^{<0})$ implies $v_{n-1} y \in I(t, \Omega^{<0})$ from (6). Similarly we can prove $v_{n-2} y,\dots,2y \in I(t, \Omega^{<0})$ by using the arguments (3)–(7). Thus we see (2) and so (1).

We prove (7) for $2y_2$ and the other cases are similar. By also using Nishimoto's result and Theorem 3.3, we have the relation

$$x' = 2x_2 + v_1 x_4 + \cdots v_{n-1} x_{2^n} = 0 \in \Omega\langle n-1\rangle^*(G_k)/I_\infty^2.$$

By using arguments similar to (3)–(5), we have $x' \in I(t, \Omega^{<0})$. Of course $v_1 x_4 + \cdots v_{n-1} x_{2^n} \in \Omega^{<0}\ Im(i_{\bar{k}})$. Thus we see $2y_2 \in I(t, \Omega^{<0})$. □

References

[Bo] J. Bonnet, Un isomorphisme motivique entre deux variétés homogénes projective sous l'action d'un groupe de type G_2, Doc. Math. **8** (2003), 247–277.

[Ca-Pe-Se-Za] B. Calmès, V. Petrov, N. Semenov and K. Zainoulline, Chow motives of twisted flag varieties, Compositio Math. **142** (2006), 1063–1080.

[Du-Za] H. Duan and X. Zhao, The Chow ring of generalized Grassmanianns, Found. Comput. Math. **10** (2010), 245–274.

[Gr] A. Grothendieck, Torsion homologique et sections rationnelles, Sem. C. Chevalley, ENS 1958, expose 5, Secreatariat Math., IHP, Paris, 1958.

[Is-To] K. Ishitoya and H. Toda, On the cohomology of irreducible symmetric spaces of exceptional type, J. Math. Kyoto Univ. **17** (1977), 225–243.

[Ka] V. G. Kac, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups, Invent. Math. **80** (1985), 69–79.

[Le] M. Levine, Comparison of cobordism theories, J. Algebra **322** (2009), 3291–3317.

[L-M 1] M. Levine and F. Morel, Cobordisme algébrique I, C. R. Acad. Sci. Paris **332** (2001), 723–728.

[L-M 2] M. Levine and F. Morel, Cobordisme algébrique II, C. R. Acad. Sci. Paris **332** (2001), 815–820.

[Mi-Ni] M. Mimura and T. Nishimoto, Hopf algebra structure of Morava K-theory of exceptional Lie groups, Contemp. Math. **293** (2002), 195–231.

[Ni-Se-Za] S. Nikolenko, N. Semenov and K. Zainoulline, Motivic decomposition of anisotropic varieties of type F_4 into generalized Rost motives, J. K-theory **3** (2009), 85–102.

[Ni] T. Nishimoto, Higher torsion in Morava K-theory of $SO(m)$ and $Spin(m)$, J. Math. Soc. Japan. **52** (2001), 383–394.

[Pe-Se-Za] V. Petrov, N. Semenov and K. Zainoulline, J-invariant of linear algebraic groups, Ann. Sci. Ec. Norm Super. **41** (2008), 1023–1053.

[Ra] D. Ravenel, Complex cobordism and stable homotopy groups of spheres, Pure and applied mathematics **121**, Academic Press, 1986.

[Ro1] M. Rost, Some new results on Chowgroups of quadrics, preprint, 1990.

[Ro2] M. Rost, On the basic correspondence of a splitting variety, preprint, 2006.

[Se] N. Semenov, Motivic decomposition of a compactification of a Merkurjev-Suslin variety, J. Reine Angew. Math. **617** (2008), 153–167.

[Su-Jo] A. Suslin and S. Joukhovitski, Norm variety, J. Pure and Appl. Algebra **206** (2006), 245–276.

[Tod] H. Toda, On the cohomology ring of some homogeneous spaces, J. Math. Kyoto Univ. **15** (1975), 185–199.

[To-Wa] H. Toda and T. Watanabe, The integral cohomology ring of F_4/T and E_6/T, J. Math. Kyoto Univ. **14** (1974), 257–286.

[To1] B. Totaro, The Chow ring of classifying spaces, Algebraic K-theory, University of Washington, Seattle, 1997, Proc. of Symposia in Pure Math. **67** (1999), 248–281.

[To2] B. Totaro, The torsion index of E_8 and other groups, Duke Math. J. **129** (2005), 219–248.

[To3] B. Totaro, The torsion index of the spin groups, Duke Math. J. **129** (2005), 249–290.

[Vi] A. Vishik, Motives of quadrics with applications to the theory of quadratic forms, Geometric methods in algebraic theory of quadratic forms, Lecture note in math. **1835** (2004), 25–101.

[Vi-Ya] A. Vishik and N. Yagita, Algebraic cobordisms of a Pfister quadric, J. London Math. Soc. **76** (2007), 586–604.

[Vi-Za] A. Vishik and K. Zainoulline, Motivic splitting lemma, Doc. Math. **13** (2008), 81–96.

[Vo1] V. Voevodsky, Motivic cohomology with $\mathbf{Z}/2$ coefficient, Publ. Math. IHES **98** (2003), 59–104.

[Vo2] V. Voevodsky, Voevodsky's Seattle lectures: K-theory and motivic cohomology, Noted by C. Weibel, Algebraic K-theory, University of Washington, Seattle, 1997, Proc. of Symposia in Pure Math. **67** (1999), 283–303.

[Vo3] V. Voevodsky, Reduced power operations in motivic cohomology, Publ. Math. IHES **98** (2003), 1–57.

[Vo4] V. Voevodsky, On motivic cohomology with $\mathbf{Z}/l$-coefficients, Ann. of Math. **174** (2011), 401–438.

[Ya1] N. Yagita, Algebraic cobordism of simply connected Lie groups, Math. Proc. Camb. Phil. Soc. **139** (2005), 243–260.

[Ya2] N. Yagita, Applications of Atiyah-Hirzebruch spectral sequence for motivic cobordism, Proc. London Math. Soc. **90** (2005), 783–816.

[Ya3] N. Yagita, Chow rings of excellent quadrics, J. Pure Appl. Algebra **212** (2008), 2440–2449.

[Ya4] N. Yagita, Algebraic *BP*-theory and norm varieties, http://hopf.math.purdue.edu/cgi-bin/generate?/Yagita/abp, 2006.

Nobuaki Yagita
Department of Mathematics
Faculty of Education
Ibaraki University
Mito, Ibaraki
Japan
E-mail: yagita@mx.ibaraki.ac.jp