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A_2-SINGULARITIES OF HYPERSURFACES WITH NON-NEGATIVE SECTIONAL CURVATURE IN EUCLIDEAN SPACE

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Abstract

In a previous work, the authors gave a definition of 'front bundles'. Using this, we give a realization theorem for wave fronts in space forms, like as in the fundamental theorem of surface theory. As an application, we investigate the behavior of principal singular curvatures along A_2-singularities of hypersurfaces with non-negative sectional curvature in Euclidean space.

0. Introduction

It is known that several Gauss-Bonnet formulas hold for closed orientable fronts (wave fronts) in $\boldsymbol{R}^3$ (see [3], [7] and [11]). From these, it is expected that there is an intrinsic formulation of wave fronts, as well as of their realization problem, like as in the fundamental theorem of surface theory.

In this paper, we recall the definitions of *coherent tangent bundles* and *front bundles* given in [11], which is an intrinsic formulation for wave fronts, and give a necessary and sufficient condition for a given front bundle to be realized as a wave front in a space form (cf. Theorem 2.7). As an application, we give a necessary and sufficient condition for a given coherent tangent bundle over a manifold to be realized as a smooth map into a same dimensional space form (cf. Theorem 2.9). Another application of the realization theorem is given in [6] to describe the duality of conformally flat Riemannian manifolds.

Moreover, using this new framework, we show the following assertion, which is a generalization of [7, Theorem 3.1] for 2-dimensional fronts.

Theorem 0.1. *Let M^m be an m-manifold and $f : M^m \to \boldsymbol{R}^{m+1}$ a wave front with the singular set Σ_f. Take an open subset $U(\subset M^m)$ such that $U \cap \Sigma_f$ consists only of A_2-singular points. Then the following hold*:

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(1) *If the sectional curvature K of the induced metric is bounded on $U \backslash \Sigma_f$, then the second fundamental form of f vanishes along $\Sigma_f \cap U$.*
(2) *If K is non-negative on $U \backslash \Sigma_f$, then it is bounded and the singular principal curvatures of f (cf. Definition 1.6) along $U \cap \Sigma_f$ are all non-positive.*

The first assertion of [7, Theorem 5.1] is the same statement as (1). This theorem follows from the corresponding intrinsic version of the statements given in Theorem 3.2, which enable us to prove the similar assertions for wave fronts in the space form of constant curvature c by a suitable modification. As a direct consequence of the theorem, we get the following assertion, which is the second assertion of [7, Theorem 5.1].

COROLLARY 0.2. *Let $f : U \to \boldsymbol{R}^{m+1}$ $(m \geq 3)$ be a front whose singular points are all A_2-points. If the sectional curvature K is positive everywhere on the set of regular points, the sectional curvature of the singular submanifold is non-negative. Furthermore, if $K \geq \delta (> 0)$, then the sectional curvature of the singular submanifold is positive.*

An example satisfying the condition in the theorem and the corollary is given in [7]. In this paper, we shall also give a new such example.

1. Coherent tangent bundles

1.1. Coherent tangent bundles and their singularities. According to [11], we recall a general setting for intrinsic fronts: Let M^m be an oriented m-manifold $(m \geq 1)$. A *coherent tangent bundle* over M^m is a 5-tuple $(M^m, \mathscr{E}, \langle\, , \rangle, D, \varphi)$, where

(1) $\mathscr{E}$ is a vector bundle of rank m over M^m with an inner product $\langle\, , \rangle$,
(2) D is a metric connection on $(\mathscr{E}, \langle\, , \rangle)$,
(3) $\varphi : TM^m \to \mathscr{E}$ is a bundle homomorphism which satisfies

$$D_X \varphi(Y) - D_Y \varphi(X) - \varphi([X, Y]) = 0 \tag{1.1}$$

for vector fields X and Y on M^m.

In this setting, the pull-back of the metric

$$ds_\varphi^2 := \varphi^* \langle\, , \rangle \tag{1.2}$$

is called the *φ-metric*, which is a positive semidefinite symmetric tensor on M^m. A point $p \in M^m$ is called a *φ-singular point* if $\varphi_p : T_p M^m \to \mathscr{E}_p$ is not a bijection, where $\mathscr{E}_p$ is the fiber of $\mathscr{E}$ at p, that is, ds_φ^2 is not positive definite at p. We denote by Σ_φ the set of φ-singular points on M^m. On the other hand, a point $p \in M^m \backslash \Sigma_\varphi$ is called a *φ-regular point*. By (1.1), the pull-back connection of D by φ coincides with the Levi-Civita connection with respect to ds_φ^2 on the set of

φ-regular points. Thus, one can recognize that the concept of coherent tangent bundles is a generalization of Riemannian manifolds (cf. [6]).

A coherent tangent bundle $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi)$ is called *co-orientable* if the vector bundle $\mathscr{E}$ is orientable, namely, there exists a smooth non-vanishing section μ of the determinant bundle of the dual bundle $\mathscr{E}^*$ such that

$$\mu(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m) = \pm 1 \tag{1.3}$$

for any orthonormal frame $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ on $\mathscr{E}$. The form μ is determined uniquely up to a $\pm$-ambiguity. A *co-orientation* of the coherent tangent bundle $\mathscr{E}$ is a choice of μ. An orthonormal frame $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ is called *positive* with respect to the co-orientation μ if $\mu(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m) = +1$.

We give here typical examples of coherent tangent bundles:

Example 1.1 ([11]). Let M^m be an oriented m-manifold and (N^m, g) an oriented Riemannian m-manifold. A C^∞-map $f : M^m \to N^m$ induces a coherent tangent bundle over M^m as follows: Let $\mathscr{E}_f := f^*TN^m$ be the pull-back of the tangent bundle TN^m by f. Then g induces a positive definite metric $\langle\,,\rangle$ on $\mathscr{E}_f$, and the restriction D of the Levi-Civita connection of g gives a connection on $\mathscr{E}_f$ which is compatible with respect to the metric $\langle\,,\rangle$. We set $\varphi_f := df : TM^m \to \mathscr{E}_f$, which gives the structure of the coherent tangent bundle on M^m. A necessary and sufficient condition for a given coherent tangent bundle over an m-manifold to be realized as a smooth map into an m-dimensional space form will be given in Theorem 2.9 in Section 2.

Example 1.2 ([11]). Let (N^{m+1}, g) be an $(m+1)$-dimensional Riemannian manifold. A C^∞-map $f : M^m \to N^{m+1}$ is called a *frontal* if for each $p \in M^m$, there exists a neighborhood U of p and a unit vector field ν along f defined on U such that $g(df(X), \nu) = 0$ holds for any vector field X on U (that is, ν is a unit normal vector field), and the map $\nu : U \to T_1N^{m+1}$ is a C^∞-map, where T_1N^{m+1} is the unit tangent bundle of N^{m+1}. Moreover, if ν can be taken to be an immersion for each $p \in M^m$, f is called a *front* or a *wave front*. We remark that f is a front if and only if f has a lift $L_f : M^m \to P(T^*N^{m+1})$ as a Legendrian immersion, where $P(T^*N^{m+1})$ is the projectified cotangent bundle on N^{m+1} with the canonical contact structure. The subbundle $\mathscr{E}_f$ which consists of the vectors in the pull-back bundle f^*TN^{m+1} perpendicular to ν gives a coherent tangent bundle. In fact, $\varphi_f : TM^m \ni X \mapsto df(X) \in \mathscr{E}_f$ gives a bundle homomorphism. Let ∇ be the Levi-Civita connection on N^{m+1}. Then by taking the tangential part of ∇, it induces a connection D on $\mathscr{E}_f$ satisfying (1.1). Let $\langle\,,\rangle$ be a metric on $\mathscr{E}_f$ induced from the Riemannian metric on N^{m+1}. Then D is a metric connection on $\mathscr{E}_f$. Thus we get a coherent tangent bundle $(M^m, \mathscr{E}_f, \langle\,,\rangle, D, \varphi_f)$. Since the unit tangent bundle can be canonically identified with the unit cotangent bundle, the map $\nu : U \to T_1N^{m+1}$ can be considered as a lift of $L_f|_U$. A frontal f is called *co-orientable* if there is a unit normal vector field ν globally defined on M^m. When N^{m+1} is orientable, the coherent tangent bundle is co-orientable if and only if so is f.

From now on, we assume that $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi)$ is co-orientable, and fix a co-orientation μ on the coherent tangent bundle. (If $\mathscr{E}$ is not co-orientable, one can take a double cover $\pi : \hat{M}^m \to M^m$ such that the pull-back of $\mathscr{E}$ by π is a co-orientable coherent tangent bundle over $\hat{M}^m$.)

DEFINITION 1.3 ([11]). The *signed φ-volume form* $d\hat{A}_\varphi$ and the (*unsigned*) *φ-volume form* dA_φ are defined as

$$d\hat{A}_\varphi := \varphi^*\mu = \lambda_\varphi\, du_1 \wedge \cdots \wedge du_m, \quad dA_\varphi := |\lambda_\varphi|\, du_1 \wedge \cdots \wedge du_m, \tag{1.4}$$

where $(U; u_1, \ldots, u_m)$ is a local coordinate system of M^m compatible with the orientation of M^m, and

$$\lambda_\varphi = \mu(\varphi_1, \ldots, \varphi_m) \quad \left(\varphi_j = \varphi\left(\frac{\partial}{\partial u_j}\right), j = 1, \ldots, m\right). \tag{1.5}$$

We call the function λ_φ the *φ-Jacobian function* on U. The set of φ-singular points on U is expressed as

$$\Sigma_\varphi \cap U := \{p \in U; \lambda_\varphi(p) = 0\}. \tag{1.6}$$

Both $d\hat{A}_\varphi$ and dA_φ are independent of the choice of positively oriented local coordinate system $(U; u_1, \ldots, u_m)$, and give two globally defined m-forms on M^m. ($d\hat{A}_\varphi$ is C^∞-differentiable, but dA_φ is only continuous.) When M^m has no φ-singular points, the two forms coincide up to sign. We set

$$M_\varphi^+ := \{p \in M^m \backslash \Sigma_\varphi; d\hat{A}_\varphi(p) = dA_\varphi(p)\},$$
$$M_\varphi^- := \{p \in M^m \backslash \Sigma_\varphi; d\hat{A}_\varphi(p) = -dA_\varphi(p)\}.$$

The φ-singular set Σ_φ coincides with the boundary $\partial M_\varphi^+ = \partial M_\varphi^-$.

A φ-singular point p $(\in \Sigma_\varphi)$ is called *non-degenerate* if $d\lambda_\varphi$ does not vanish at p. On a neighborhood of a non-degenerate φ-singular point, the φ-singular set consists of an $(m-1)$-submanifold in M^m, called the *φ-singular submanifold*. If p is a non-degenerate φ-singular point, the rank of φ_p is $m-1$. The direction of the kernel of φ_p is called the *null direction*. Let η be the smooth (non-vanishing) vector field along the φ-singular submanifold Σ_φ, which gives the null direction at each point in Σ_φ.

DEFINITION 1.4 (A_2-singular points, [11]). Let $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi)$ be a coherent tangent bundle. A non-degenerate φ-singular point $p \in M^m$ is called an *A_2-singular point* or an *A_2-point of φ* if the null direction $\eta(p)$ is transversal to the singular submanifold.

It should be remarked that for the definition of A_2-singularities, the condition (1.1) is not required (cf. [11]).

We set

$$\lambda'_\varphi := d\lambda_\varphi(\tilde{\eta}), \tag{1.7}$$

where $\tilde{\eta}$ is a vector field on a neighborhood U of p which coincides with η on $\Sigma_\varphi \cap U$. Then p is an A_2-point if and only if the function λ'_φ does not vanish at p (see [10, Theorem 2.4]).

When $m = 2$ and $(M^2, \mathscr{E}, \langle\,,\rangle, D, \varphi)$ comes from a front in 3-manifold as in Example 1.2 (resp. a map into 2-manifold as in Example 1.1), an A_2-point corresponds to a cuspidal edge (resp. a fold) (cf. [9]).

1.2. Singular curvatures. Let $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi)$ be a co-oriented coherent tangent bundle and fix a φ-singular point $p \in \Sigma_\varphi$ which is an A_2-point. Then there exists a neighborhood U of p such that $\Sigma_\varphi \cap U$ consists of A_2-points. Now we define the *singular shape operator* as follows: Since the kernel of φ_p is transversal to Σ_φ at p, $\varphi|_{T(\Sigma_\varphi \cap U)}$ is injective, where U is a sufficiently small neighborhood of p. Then the metric ds^2_φ is positive definite on $\Sigma_\varphi \cap U$. We take an orthonormal frame field $e_1, e_2, \dots, e_{m-1}$ on $\Sigma_\varphi \cap U$ with respect to ds^2_φ. Without loss of generality, we may assume that $(e_1, e_2, \dots, e_{m-1})$ is smoothly extended on U as an orthonormal $(m-1)$-frame field. Then we can take a unique smooth section $\boldsymbol{n} : U \to \mathscr{E}$ (called the *conormal vector field*) so that $(\varphi(e_1), \dots, \varphi(e_{m-1}), \boldsymbol{n})$ gives a positively oriented orthonormal frame field on $\mathscr{E}$. Now, we set

$$S_\varphi(X) := -\mathrm{sgn}(d\lambda_\varphi(\eta(q)))\varphi^{-1}(D_X\boldsymbol{n}) \quad (X \in T_q\Sigma_\varphi, q \in \Sigma_\varphi \cap U), \tag{1.8}$$

where the non-vanishing null vector field η is chosen so that $(e_1, \dots, e_{m-1}, \eta)$ is compatible with respect to the orientation of M^m. It holds that

$$\mathrm{sgn}(d\lambda_\varphi(\eta(q))) = \begin{cases} 1 & \text{if } \eta(q) \text{ points toward } M^+_\varphi, \\ -1 & \text{if } \eta(q) \text{ points toward } M^-_\varphi. \end{cases} \tag{1.9}$$

Since φ is injective on each tangent space of Σ_φ and $D_X\boldsymbol{n} \in \varphi(T\Sigma_\varphi)$, the inverse element $\varphi^{-1}(D_X\boldsymbol{n})$ is uniquely determined. Thus we get a bundle endomorphism $S_\varphi : T\Sigma_\varphi \to T\Sigma_\varphi$ which is called the *singular shape operator* on Σ_φ.

FACT 1.5 ([11]). *The definition of the singular shape operator S_φ is independent of the choice of an orthonormal frame field $e_1, \dots, e_{m-1}$, the choice of an orientation of M^m, and the choice of a co-orientation of $\mathscr{E}$. Moreover, it holds that*

$$ds^2_\varphi(S_\varphi(X), Y) = ds^2_\varphi(X, S_\varphi(Y)) \quad (X, Y \in T_q\Sigma_\varphi, q \in \Sigma_\varphi),$$

namely, S_φ is symmetric with respect to ds^2_φ.

DEFINITION 1.6 ([11]). Let $p \in \Sigma_\varphi$ be an A_2-point of φ. Then

$$\kappa_\varphi(X) := ds^2_\varphi(S_\varphi(X), X)/ds^2_\varphi(X, X), \quad (X \in T_p\Sigma_\varphi \backslash \{0\}) \tag{1.10}$$

is called the *φ-singular normal curvature* at p with respect to the direction X. The eigenvalues of S_φ are called the *φ-singular principal curvatures*, which give the critical values of the singular normal curvature on $T_p\Sigma_\varphi$.

In [11, Theorem 2.13], it was shown that at least one of the φ-singular principal curvatures diverges to $-\infty$ at non-degenerate singular points other than A_2-points.

When $m = 2$, the φ-singular principal curvature is called (simply) the *φ-singular curvature*, which is also denoted by κ_φ. This definition of the singular curvature is the same as in [7, (1.7)] and [8, (1.6)]. More precisely, κ_φ is computed as follows: Let $p \in \Sigma_\varphi$ be an A_2-point of φ. Then the φ-singular set Σ_φ is parametrized by a regular curve $\gamma(t)$ ($t \in I \subset \boldsymbol{R}$) on a neighborhood of p, and $\gamma(t)$ is an A_2-point of φ for each $t \in I$. Since $\dot\gamma(t)$ ($\dot{} = d/dt$) is not a null-direction, $\varphi(\dot\gamma(t)) \neq 0$. Take a section $\boldsymbol{n}(t)$ of $\mathscr{E}$ along γ such that $\{\varphi(\dot\gamma)/|\varphi(\dot\gamma)|, \boldsymbol{n}\}$ gives a positive orthonormal frame field on $\mathscr{E}$ along γ, where $|\varphi(\dot\gamma)| = \langle\varphi(\dot\gamma), \varphi(\dot\gamma)\rangle^{1/2}$. Then we have

$$\kappa_\varphi(t) := \kappa_\varphi(\dot\gamma(t)) = -\mathrm{sgn}(d\lambda_\varphi(\eta(t)))\frac{\langle D_{d/dt}\boldsymbol{n}(t), \varphi(\dot\gamma(t))\rangle}{|\varphi(\dot\gamma(t))|^2}, \tag{1.11}$$

where $\eta(t)$ is a null-vector field along $\gamma(t)$ such that $\{\dot\gamma(t), \eta(t)\}$ is compatible with the orientation of M^2. By (1.9), it holds that

$$\mathrm{sgn}(d\lambda_\varphi(\eta(t))) = \begin{cases} 1 & \text{if } M_\varphi^+ \text{ lies on the left-hand side of } \gamma, \\ -1 & \text{if } M_\varphi^- \text{ lies on the left-hand side of } \gamma. \end{cases} \tag{1.12}$$

2. The realization of frontal bundles

First, we recall a definition of frontal bundles given in [11], and consider a realization problem of them as fronts in space forms.

2.1. Front bundles. Let M^m be an oriented m-manifold and $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi)$ a co-orientable coherent tangent bundle over M^m. If there exists another bundle homomorphism $\psi : TM^m \to \mathscr{E}$ such that $(M^m, \mathscr{E}, \langle\,,\rangle, D, \psi)$ is also a coherent tangent bundle and the pair (φ, ψ) of bundle homomorphisms satisfies a compatibility condition

$$\langle\varphi(X), \psi(Y)\rangle = \langle\varphi(Y), \psi(X)\rangle, \tag{2.1}$$

then $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ is called a *frontal bundle*. The bundle homomorphisms φ and ψ are called the *first homomorphism* and the *second homomorphism*, respectively. We set

$$I(X, Y) := ds_\varphi^2(X, Y) = \langle\varphi(X), \varphi(Y)\rangle,$$

$$II(X, Y) := -\langle\varphi(X), \psi(Y)\rangle,$$

$$III(X, Y) := ds_\psi^2(X, Y) = \langle\psi(X), \psi(Y)\rangle$$

for $X, Y \in T_pM^m$ $(p \in M^m)$, and we call them *the first*, *the second and the third fundamental forms*, respectively. They are all symmetric covariant tensors on M^m.

DEFINITION 2.1 ([11]). A frontal bundle $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ is called a *front bundle* if

$$\mathrm{Ker}(\varphi_p) \cap \mathrm{Ker}(\psi_p) = \{0\} \tag{2.2}$$

holds for each $p \in M^m$.

Example 2.2 ([11]). Let $(N^{m+1}(c), g)$ be an $(m+1)$-dimensional space form, that is, the simply connected complete Riemannian $(m+1)$-manifold of constant curvature c, and denote by ∇ the Levi-Civita connection on $N^{m+1}(c)$. Let $f : M^m \to N^{m+1}(c)$ be a co-orientable frontal. Then there exists a globally defined unit normal vector field ν. Since the coherent tangent bundle $\mathscr{E}_f$ given in Example 1.2 is orthogonal to ν, we can define a bundle homomorphism

$$\psi_f : T_pM^m \ni X \mapsto \nabla_X \nu \in \mathscr{E}_p \quad (p \in M^m).$$

Then $(M^m, \mathscr{E}_f, \langle\,,\rangle, D, \varphi_f, \psi_f)$ is a frontal bundle (we shall prove this in Proposition 2.4 later). Moreover, this is a front bundle in the sense of Definition 2.1 if and only if f is a front, which is equivalent to $I + III$ being positive definite.

Remark 2.3. As seen above, if $f : M^m \to N^{m+1}(c)$ is a front, then

$$(M^m, \mathscr{E}_f, \langle\,,\rangle, D, \varphi_f, \psi_f)$$

is a front bundle. Since $\varphi = \varphi_f$ and $\psi = \psi_f$ have the completely same conditions, the third fundamental form III plays the same role as I by definition. This means that we can reverse the roles of I and III.

When $N^{m+1}(c)$ is the unit sphere S^{m+1} (i.e. $c = 1$), then the unit normal vector field ν along f can be considered as a map $\nu : M^m \to S^{m+1}$ and the third fundamental form of f coincides with the first fundamental form of ν.

When $N^{m+1}(c)$ is the Euclidean space $\boldsymbol{R}^{m+1}$ (i.e. $c = 0$), then the unit normal vector field ν along f can be considered as a map $\nu : M^m \to S^m$ and the third fundamental form of f coincides with the pull-back of the canonical metric of the unit sphere S^m by ν.

Next, we cosider the case that $N^{m+1}(c)$ is the hyperbolic space H^{m+1} (i.e. $c = -1$):

$$H^{m+1} := \{p = (p_0, \dots, p_{m+1}) \in \boldsymbol{R}^{m+2}_1; p \cdot p = -1, p_0 > 0\}, \tag{2.3}$$

where '$\cdot$' is the canonical Lorentzian inner product with singuature $(-, +, \dots, +)$ of the Lorentz-Minkowski space $\boldsymbol{R}^{m+2}_1$. The unit normal vector field ν along f can be considered as a map $\nu : M^m \to S^{m+1}_1$ and the third fundamental form of f coincides with the first fundamental form of ν, where

$$S^{m+1}_1 := \{p \in \boldsymbol{R}^{m+2}_1; p \cdot p = 1\} \tag{2.4}$$

is the de Sitter space.

PROPOSITION 2.4. *Let $f : M^m \to N^{m+1}(c)$ be a co-orientable frontal, and ν a unit normal vector field. Then $(M^m, \mathscr{E}_f, \langle\,,\rangle, D, \varphi_f, \psi_f)$ as in Example 2.2 is a frontal bundle. Moreover, the following identity (i.e. the Gauss equation) holds (We denote by "det" the determinant of matrices.):*

$$\begin{aligned}(2.5)\qquad &\langle R^D(X,Y)\xi,\zeta\rangle \\ &= c\det\begin{pmatrix}\langle\varphi(Y),\xi\rangle & \langle\varphi(Y),\zeta\rangle\\ \langle\varphi(X),\xi\rangle & \langle\varphi(X),\zeta\rangle\end{pmatrix} + \det\begin{pmatrix}\langle\psi(Y),\xi\rangle & \langle\psi(Y),\zeta\rangle\\ \langle\psi(X),\xi\rangle & \langle\psi(X),\zeta\rangle\end{pmatrix},\end{aligned}$$

where $\varphi = \varphi_f$ and $\psi = \psi_f$, X and Y are vector fields on M^m, ξ and ζ are sections of $\mathscr{E}_f$, and R^D is the curvature tensor of the connection D:

$$R^D(X,Y)\xi := D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi.$$

Furthermore, this frontal bundle is a front bundle if and only if f is a front.

Proof. Let R^c be the curvature tensor of $N^{m+1}(c)$. Since

$$\nabla_X \xi = D_X \xi - \langle\psi_f(X),\xi\rangle\nu$$

holds for the Levi-Civita connection ∇ of $N^{m+1}(c)$, we have the following identity:

$$\begin{aligned}(2.6)\qquad R^c(df(X),df(Y))\xi &= R^D(X,Y)\xi - \langle\psi_f(Y),\xi\rangle\psi_f(X) + \langle\psi_f(X),\xi\rangle\psi_f(Y)\\ &\quad - (\langle D_X\psi_f(Y),\xi\rangle - \langle D_Y\psi_f(X),\xi\rangle - \langle\psi_f([X,Y]),\xi\rangle)\nu.\end{aligned}$$

Taking the normal component, we get

$$\langle D_X\psi_f(Y),\xi\rangle - \langle D_Y\psi_f(X),\xi\rangle = \langle\psi_f([X,Y]),\xi\rangle.$$

Since ξ is arbitrary, this proves that $(M^m, \mathscr{E}_f, \langle\,,\rangle, D, \psi_f)$ is a coherent tangent bundle. Moreover,

$$\langle\varphi_f(X),\psi_f(Y)\rangle = g(df(X),\nabla_Y\nu) = \langle\varphi_f(Y),\psi_f(X)\rangle.$$

Hence $(M^m, \mathscr{E}_f, \langle\,,\rangle, D, \varphi_f, \psi_f)$ is a frontal bundle.

On the other hand, taking the tangential component of (2.6), we get

$$R^c(df(X),df(Y))\xi = R^D(X,Y)\xi - \langle\psi_f(Y),\xi\rangle\psi_f(X) + \langle\psi_f(X),\xi\rangle\psi_f(Y).$$

Since $(N^{m+1}(c), g)$ is of constant curvature c, it holds that

$$R^c(df(X),df(Y))\xi = c(\langle\varphi_f(Y),\xi\rangle\varphi_f(X) - \langle\varphi_f(X),\xi\rangle\varphi_f(Y)),$$

and hence we get the Gauss equation (2.5). □

DEFINITION 2.5. For a real number c, a frontal bundle $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ is said to be *c-integrable* if (2.5) holds.

2.2. A realization of frontal bundles. Now, we give the fundamental theorem for frontal bundles. To state the theorem, we define equivalence of frontal bundles:

DEFINITION 2.6. Two frontal bundles over M^m are *isomorphic* or *equivalent* if there exists an orientation preserving bundle isomorphism between them which preserves the inner products, the connections and the bundle maps.

Let $(N^{m+1}(c), g)$ be the $(m+1)$-dimensional space form of constant curvature c.

THEOREM 2.7 (Realization of frontal bundles). *Let $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ be a c-integrable frontal bundle over a simply connected manifold M^m, where c is a real number. Then there exists a frontal $f : M^m \to N^{m+1}(c)$ such that $\mathscr{E}$ is isomorphic to $\mathscr{E}_f$ induced from f as in Proposition* 2.4. *Moreover, such an f is unique up to orientation preserving isometries of $N^{m+1}(c)$.*

Let S_1^{m+1} be the de Sitter space of constant sectional curvature 1. As mentioned in Remark 2.3, S_1^{m+1} can be identified with the hyperquadric in the Lorentz-Minkowski space $\boldsymbol{R}_1^{m+2}$ (see (2.4)). A C^∞-map $f : M^m \to S_1^{m+1}$ is called a *frontal* if there exists a C^∞-map

$$\nu : M^m \to H^{m+1} := \{p = (p_0, \dots, p_{m+1}) \in \boldsymbol{R}_1^{m+2}; p \cdot p = -1, p_0 > 0\}$$

such that $d\nu \cdot f = \nu \cdot df = 0$. Moreover, f is called a (*wave*) *front* if $(f, \nu) : M^m \to \boldsymbol{R}_1^{m+2} \times \boldsymbol{R}_1^{m+2}$ is an immersion. By definition, f is a front if and only if ν also is. Thus, by interchanging the roles of the first homomorphism and the second homomorphism, we get the following

COROLLARY 2.8. *Let $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ be a (-1)-integrable frontal bundle over a simply connected manifold M^m. Then there exists a space-like frontal $\nu : M^m \to S_1^{m+1}$ such that $(M^m, \mathscr{E}, \langle\,,\rangle, D, \psi, \varphi)$ is isomorphic to $\mathscr{E}_\nu$ induced from ν. Moreover, such a ν is unique up to orientation preserving isometries of S_1^{m+1}.*

Proof of Theorem 2.7. To prove Theorem 2.7, we write down the fundamental equations for frontals. Without loss of generality, we may assume that M^m is simply connected domain $U \subset \boldsymbol{R}^m$. First, we consider the case $c = 0$. Let $f : U \to \boldsymbol{R}^{m+1} = N^{m+1}(0)$ be a frontal, where we consider elements in the Euclidean space $\boldsymbol{R}^{m+1}$ as column vectors. Then the unit normal vector field ν can be considered as a map $\nu : U \to S^m \subset \boldsymbol{R}^{m+1}$, and $\nabla\nu = d\nu$, where ∇ is the Levi-Civita connection of $\boldsymbol{R}^{m+1}$. Thus the corresponding frontal bundle is $(U, \mathscr{E}_f, \langle\,,\rangle, D, \varphi := df, \psi := d\nu)$, where $\mathscr{E}_f = \{(p, \boldsymbol{x}) \in U \times \boldsymbol{R}^{m+1}; \boldsymbol{x} \cdot \nu(p) = 0\}$. Take a positively oriented orthonormal frame field (called an *adopted frame field* of f)

$$\mathscr{F} := (\boldsymbol{e}_1, \dots, \boldsymbol{e}_{m+1}) : U \to \mathrm{SO}(m+1) \tag{2.7}$$

of $\boldsymbol{R}^{m+1}$ along f such that $\boldsymbol{e}_{m+1} = \nu$. Since $\nu = \boldsymbol{e}_{m+1}$, $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ is an orthonormal frame field of $\mathscr{E}_f$. Let ω_i^j be the *connection forms* of D with respect to this frame, as 1-forms on U:

$$D\boldsymbol{e}_i = \sum_{l=1}^{m} \omega_i^l \boldsymbol{e}_l, \quad \omega_i^j = -\omega_j^i \quad (i, j = 1, \dots, m). \tag{2.8}$$

Define an $\mathfrak{so}(m)$-valued 1-form Ω by $\Omega = (\omega_i^j)$, where $\mathfrak{so}(m)$ is the Lie algebra of $\mathrm{SO}(m)$. Next, we define $\boldsymbol{R}^m$-valued 1-forms $\boldsymbol{g}$ and $\boldsymbol{h}$ as

$$\boldsymbol{g} := {}^t(g^1, \dots, g^m), \quad \boldsymbol{h} := {}^t(h^1, \dots, h^m) \tag{2.9}$$

with

$$g^j := \langle \varphi, \boldsymbol{e}_j \rangle, \quad h^j := -\langle \psi, \boldsymbol{e}_j \rangle \quad (j = 1, \dots, m),$$

where $\boldsymbol{R}^m$ is considered as a column vector space. Then, by definition, the adapted frame $\mathscr{F}$ in (2.7) satisfies the ordinary differential equation

$$df = \sum_{l=1}^{m} g^l \boldsymbol{e}_l, \quad d\mathscr{F} = \mathscr{F}\tilde{\Omega}, \quad \tilde{\Omega} = \begin{pmatrix} \Omega & -\boldsymbol{h} \\ {}^t\boldsymbol{h} & 0 \end{pmatrix}. \tag{2.10}$$

Next, we consider the case $c > 0$. Without loss of generality, we may assume that $c = 1$. In this case, $N^{m+1}(1)$ can be considered as the unit sphere $S^{m+1} (\subset \boldsymbol{R}^{m+2})$ centered at the origin. Let $f : U \to S^{m+1}$ be a frontal with the unit normal vector field $\nu : U \to S^{m+1}$. Then the coherent tangent bundle $\mathscr{E}_f$ is written as

$$\mathscr{E}_f = \{(p, \boldsymbol{x}) \in U \times \boldsymbol{R}^{m+1}; \boldsymbol{x} \cdot f(p) = \boldsymbol{x} \cdot \nu(p) = 0\} \tag{2.11}$$

where "$\cdot$" is the canonical inner product of $\boldsymbol{R}^{m+2}$. The induced inner product $\langle\, , \rangle$ of $\mathscr{E}_f$ is the restriction of "$\cdot$". Take an $\mathrm{SO}(m+2)$-valued function (*an adopted frame*) $\mathscr{F} := (\boldsymbol{e}_0, \dots, \boldsymbol{e}_{m+1}) : U \to \mathrm{SO}(m+2)$ such that $\boldsymbol{e}_0 := f$, $\boldsymbol{e}_{m+1} := \nu$. Since $d\nu \cdot f = d\nu \cdot \nu = 0$, $d\nu$ is an $\mathscr{E}_f$-valued 1-form, and then it holds that

$$\nabla \nu = d\nu,$$

where ∇ is the Levi-Civita connection of S^{m+1}. Thus, setting $\varphi = df$ and $\psi = d\nu$, we have the frontal bundle. Denoting by ω_i^j $(i, j = 1, \dots, m)$ the connection forms of D with respect to $\{\boldsymbol{e}_j\}$, the adapted frame field $\mathscr{F}$ satisfies

$$d\mathscr{F} = \mathscr{F}\tilde{\Omega}, \quad \tilde{\Omega} = \begin{pmatrix} 0 & -{}^t\boldsymbol{g} & 0 \\ \boldsymbol{g} & \Omega & -\boldsymbol{h} \\ 0 & {}^t\boldsymbol{h} & 0 \end{pmatrix}, \tag{2.12}$$

where $\Omega = (\omega_i^j)$, and $\boldsymbol{g}$ and $\boldsymbol{h}$ are as in (2.9) in the case of $c = 0$.

Finally, we consider the case $c < 0$. We may assume that $c = -1$. Then $N^{m+1}(-1)$ is the hyperbolic space H^{m+1} as in (2.3). Let $f : U \to H^{m+1}$ be a frontal and ν the unit normal vector field. Then ν is a space-like frontal in de

Sitter space S_1^{m+1} as in (2.4), and the coherent tangent bundle is written like as (2.11), using the canonical Lorentzian inner product. Take an $\mathrm{SO}_0(1,m+1)$-valued function (*an adapted frame*) $\mathscr{F} := (\boldsymbol{e}_0, \dots, \boldsymbol{e}_{m+1}) : U \to \mathrm{SO}_0(1,m+1)$ such that $\boldsymbol{e}_0 := f$, $\boldsymbol{e}_{m+1} := \nu$, where $\mathrm{SO}_0(1,m+1)$ is the identity component of the group of linear isometries $\mathrm{O}(1,m+1)$ of $\boldsymbol{R}_1^{m+2}$. Similar to the case of $c > 0$, it holds that $\nabla \nu = d\nu$, and then we can set $\varphi = df$, $\psi = d\nu$. Hence the adapted frame field $\mathscr{F}$ satisfies

$$d\mathscr{F} = \mathscr{F}\tilde{\boldsymbol{\Omega}}, \quad \tilde{\boldsymbol{\Omega}} = \begin{pmatrix} 0 & {}^t\boldsymbol{g} & 0 \\ \boldsymbol{g} & \Omega & -\boldsymbol{h} \\ 0 & {}^t\boldsymbol{h} & 0 \end{pmatrix}, \tag{2.13}$$

as well as the case of $c > 0$, where $\Omega = (\omega_i^j)$, $\boldsymbol{g}$ and $\boldsymbol{h}$ are as in (2.9).

Now, in these three situations, the Gauss equation (2.5) and the Codazzi equation (1.1) for ψ can be considered as the integrability conditions for the differential equations (2.10), (2.12) and (2.13). Thus we get the assertion. □

We give here several applications of the realization theorem.

THEOREM 2.9 (Maps into $N^m(c)$ of an m-manifold). *Let M^m be a simply connected domain on $\boldsymbol{R}^m$ and $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi)$ a coherent tangent bundle over M^m. Assume that for any vector fields X, Y on M^m and a section ξ of $\mathscr{E}$, it holds that*

$$R^D(X,Y)\xi = c(\langle\varphi(Y),\xi\rangle\varphi(X) - \langle\varphi(X),\xi\rangle\varphi(Y)), \tag{2.14}$$

where R^D is the curvature tensor of D. Then there exists a C^∞-map $f : M^m \to N^m(c)$ into the m-dimensional simply connected space form $N^m(c)$ such that $\mathscr{E}$ and $\mathscr{E}_f$ (as in Example 1.1) *are isomorphic.*

Proof. We may set $M^m = U(\subset \boldsymbol{R}^m)$. Consider the trivial bundle map $\mathbf{0} : TM^m \ni X \mapsto \mathbf{0} \in \mathscr{E}$. Then by (2.14), $(U, \mathscr{E}, \langle\,,\rangle, D, \varphi, \mathbf{0})$ is a c-integrable frontal bundle, and then there exists the corresponding frontal $\tilde{f} : U \to N^{m+1}(c)$. Since $\psi = \mathbf{0}$, the image of $\tilde{f}$ lies in a totally geodesic hypersurface of $N^{m+1}(c)$. □

2.3. Applications to surface theory. Now we introduce applications for surface theory. To state them, we rewrite the c-integrability (2.5) for the 2-dimensional case. Let $(M^2, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ be a frontal bundle over a 2-manifold M^2. Take a (local) orthonormal frame field $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ of $\mathscr{E}$, and take a 1-form ω as

$$D\boldsymbol{e}_1 = -\omega\boldsymbol{e}_2, \quad D\boldsymbol{e}_2 = \omega\boldsymbol{e}_1, \tag{2.15}$$

that is, $\omega(= \omega_2^1)$ is the connection form of D with respect to the frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. Then one can easily see that $(M^2, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ is c-integrable if and only if

$$d\omega = c\alpha + \beta \tag{2.16}$$

holds, where α and β are 2-forms on M^2 defined by

$$\alpha(X, Y) = \langle \varphi(X), \boldsymbol{e}_1 \rangle \langle \varphi(Y), \boldsymbol{e}_2 \rangle - \langle \varphi(X), \boldsymbol{e}_2 \rangle \langle \varphi(Y), \boldsymbol{e}_1 \rangle,$$
$$\beta(X, Y) = \langle \psi(X), \boldsymbol{e}_1 \rangle \langle \psi(Y), \boldsymbol{e}_2 \rangle - \langle \psi(X), \boldsymbol{e}_2 \rangle \langle \psi(Y), \boldsymbol{e}_1 \rangle.$$

Remark 2.10. Let K_φ be the Gaussian curvature of the first fundamental form $I = ds_\varphi^2$. Then

$$d\omega = K_\varphi \, d\hat{A}_\varphi \tag{2.17}$$

holds, where $d\hat{A}_\varphi$ is the signed φ-volume form defined in Definition 1.3.

THEOREM 2.11 (Fronts of constant negative extrinsic curvature). *Let U be a simply connected domain of $\boldsymbol{R}^2$ and $c \in \boldsymbol{R}$ a constant. Take a smooth real-valued function $\theta = \theta(u, v)$ on U which satisfies the equation*:

$$\theta_{uv} = (1 - c) \sin \theta, \tag{2.18}$$

where $\theta_{uv} := \partial^2\theta/(\partial u \partial v)$. Then there exists a front $f : U \to N^3(c)$ whose fundamental forms are given by

$$\begin{aligned} I &= \langle \varphi, \varphi \rangle = du^2 + 2 \cos \theta \, dudv + dv^2, \\ II &= -\langle \varphi, \psi \rangle = 2 \sin \theta \, dudv, \\ III &= \langle \psi, \psi \rangle = du^2 - 2 \cos \theta \, dudv + dv^2. \end{aligned} \tag{2.19}$$

In particular, the Gaussian curvature of f is identically $c - 1$ on $U \backslash \Sigma$, where $\Sigma = \{\theta \equiv 0 \pmod{\pi}\}$ is the singular set of f. Conversely, any front $f : U \to N^3(c)$ whose regular set $R_f := U \backslash \Sigma_f$ is dense in U and whose Gaussian curvature is $c - 1$ on R_f is given in this manner.

Proof. Let $\mathscr{E} = U \times \boldsymbol{R}^2$ be the trivial bundle and take the canonical orthonormal frame $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. Define the bundle homomorphisms φ and ψ from TU to $\mathscr{E}$ as

$$\begin{aligned} \varphi &:= \cos \frac{\theta}{2}(du + dv)\boldsymbol{a}_1 - \sin \frac{\theta}{2}(du - dv)\boldsymbol{a}_2, \\ \psi &:= -\sin \frac{\theta}{2}(du + dv)\boldsymbol{a}_1 - \cos \frac{\theta}{2}(du - dv)\boldsymbol{a}_2. \end{aligned} \tag{2.20}$$

Take a connection D of $\mathscr{E}$ as

$$D\boldsymbol{a}_1 = -\omega \boldsymbol{a}_2, \quad D\boldsymbol{a}_2 = \omega \boldsymbol{a}_1, \quad \omega = \frac{1}{2}(\theta_u \, du - \theta_v \, dv). \tag{2.21}$$

One can directly show that (2.16) is equivalent to (2.18). Then we have the corresponding front f. In particular, the fundamental forms of f are given by (2.19). Hence the coordinate system (u, v) of U forms the asymptotic Chebyshev

net of f, and the Gaussian curvature is $c-1$. Moreover, θ is the angle between two asymptotic directions with respect to the first fundamental form.

Conversely, suppose that $f : U \to N^3(c)$ is a front such that the regular set R_f of f is dense in U and f has constant Gaussian curvature $c-1$ on R_f. Then the sum $I + III$ of the first and the third fundamental forms is a flat metric on U because R_f is dense. Since U is simply connected, there is an immersion $\Phi : U \to (\boldsymbol{R}^2; u, v)$ such that $I + III = \Phi^*(2(du^2 + dv^2))$. The asymptotic lines of f on R_f are geodesic lines with respect to the metric $I + III$, and two asymptotic directions are mutually orthogonal with respect to the metric $I + III$. Thus by rotating the coordinate system (u, v), we may assume that the inverse image of u, v-lines by Φ consists of asymptotic lines. Then the fundamental forms are given by (2.19) on $\Phi(R_f)$. Since R_f is a dense set, (2.19) holds on $\Phi(U)$, which proves the assertion. □

In particular, we have the following assertion on the realization of fronts of constant negative curvature -1 in $\boldsymbol{R}^3$ and flat front in S^3, respectively.

COROLLARY 2.12. *Let U be a simply connected domain of $\boldsymbol{R}^2$, and take a smooth real-valued function θ on U which satisfies*

$$\theta_{uv} = \sin\theta \quad (\textit{resp. } \theta_{uv} = 0). \tag{2.22}$$

Then there exists a front $f : U \to \boldsymbol{R}^3$ (resp. S^3) such that the Gaussian curvature of f is identically -1 (resp. 0) on $U \backslash \Sigma$, where $\Sigma = \{\theta \equiv 0 \pmod{\pi}\}$ is the singular set.

THEOREM 2.13 (Fronts of constant positive curvature). *Let U be a simply connected domain of $\boldsymbol{C} = \boldsymbol{R}^2$, and take a smooth real-valued function θ on U which satisfies the sinh-Gordon equation:*

$$\frac{1}{4}(\theta_{uu} + \theta_{vv})(= \theta_{z\bar{z}}) = -\sinh\theta, \tag{2.23}$$

where $z = u + iv$ is the complex coordinate on $\boldsymbol{C} = \boldsymbol{R}^2$. Then there exists a front $f : U \to \boldsymbol{R}^3$ without umbilic points, whose fundamental forms are given by

$$\begin{aligned}
I = \langle \varphi, \varphi \rangle &= dz^2 + 2\cosh\theta\, dz d\bar{z} + d\bar{z}^2, \\
&= 4\{\cosh^2(\theta/2)\, du^2 + \sinh^2(\theta/2)\, dv^2\}, \\
II = -\langle \varphi, \psi \rangle &= 2\sinh\theta\, dz d\bar{z}, \\
&= 4\cosh(\theta/2)\sinh(\theta/2)(du^2 + dv^2), \\
III = \langle \psi, \psi \rangle &= -dz^2 + 2\cosh\theta\, dz d\bar{z} - d\bar{z}^2, \\
&= 4\{\sinh^2(\theta/2)\, du^2 + \cosh^2(\theta/2)\, dv^2\}.
\end{aligned} \tag{2.24}$$

Conversely, any front $f : U \to \boldsymbol{R}^3$ whose regular set $R_f = U \backslash \Sigma_f$ is dense in U and whose Gaussian curvature is 1 on R_f without umbilic points is given in this manner.

Proof. Let $\mathscr{E} = U \times \boldsymbol{R}^2$ be the trivial bundle and take the canonical orthonormal frame $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. Define the bundle homomorphisms φ and ψ as

$$\begin{aligned}
\varphi &:= 2\left[\left(\cosh\frac{\theta}{2}\,du\right)\boldsymbol{a}_1 + \left(\sinh\frac{\theta}{2}\,dv\right)\boldsymbol{a}_2\right] \\
&= \cosh\frac{\theta}{2}(dz + d\bar{z})\boldsymbol{a}_1 - i\sinh\frac{\theta}{2}(dz - d\bar{z})\boldsymbol{a}_2, \\
\psi &:= -2\left[\left(\sinh\frac{\theta}{2}\,du\right)\boldsymbol{a}_1 + \left(\cosh\frac{\theta}{2}\,dv\right)\boldsymbol{a}_2\right] \\
&= -\sinh\frac{\theta}{2}(dz + d\bar{z})\boldsymbol{a}_1 + i\cosh\frac{\theta}{2}(dz - d\bar{z})\boldsymbol{a}_2,
\end{aligned} \tag{2.25}$$

and define a connection D on $\mathscr{E}$ by a connection form

$$\omega = \frac{1}{2}(\theta_v\,du - \theta_u\,dv) = \frac{i}{2}(\theta_z\,dz - \theta_{\bar{z}}\,d\bar{z}). \tag{2.26}$$

One can directly show that (2.16) is equivalent to (2.18). Then we have the corresponding front f. In particular, the fundamental forms of f are given by (2.24). Hence (u, v) forms a curvature line coordinate system, and the Gaussian curvature is 1.

Conversely, suppose that $f : U \to \boldsymbol{R}^3$ is a front such that the regular set R_f of f is dense in U and f has constant Gaussian curvature 1 on R_f. Then $I - III$ gives a flat Lorentzian metric. Since U is simply connected, there is an immersion $\Phi : U \to (\boldsymbol{R}^2; u, v)$ such that $I - III = 4\Phi^*(du^2 - dv^2)$. The curvature lines of f on R_f are geodesic lines with respect to the metric $I - III$, and the two principal directions are orthogonal with respect to $I - III$. Thus by a Lorentzian rotation of the coordinate system (u, v), we may assume that the inverse image of u, v-lines under Φ consists of principal curvature lines. Then the fundamental forms are given by (2.24) on $\Phi(R_f)$. Since R_f is a dense set, (2.24) holds on $\Phi(U)$, which proves the assertion. □

3. A relationship between sectional curvatures and singular principal curvatures

In this section, we investigate a relationship between sectional curvatures (cf. (3.2)) near A_2-singular points of hypersurfaces (as wave fronts) and their singular principal curvatures.

We fix a front bundle $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ over an m-dimensional manifold M^m.

Definition 3.1. When $p \in M^m$ is not a singular point of φ, we define

$$K^{\mathrm{ext}}(X \wedge Y) := \frac{II(X,X)II(Y,Y) - II(X,Y)^2}{I(X,X)I(Y,Y) - I(X,Y)^2} \quad (X, Y \in T_pM^m), \tag{3.1}$$

which is called the *extrinsic curvature* at p with respect to the $X \wedge Y$-plane in T_pM^m.

If a front bundle $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ is induced from a front in $N^{m+1}(c)$, then it holds that

$$K^{\mathrm{ext}}(X \wedge Y) := K(X \wedge Y) + c \quad (X, Y \in T_pM^m), \tag{3.2}$$

where $K(X \wedge Y)$ is the sectional curvature at each φ-regular point p of M^m. Theorem 0.1 given in the introduction is a direct consequence of the following assertion:

THEOREM 3.2. *Let $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ be a front bundle over an oriented m-manifold M^m. Take an A_2-point $p \in M^m$ of φ. Then the following hold*:

(1) *Suppose that K^{ext} is bounded except on the singular set near p. Then $II(X, Y) = 0$ holds for all $X, Y \in T_pM^m$.*

(2) *If K^{ext} does not change sign on a neighborhood of p with the singular set removed, then K^{ext} is bounded on that neighborhood of p with the singular set removed.*

(3) *If K^{ext} is non-negative except on the singular set near p, then the singular principal curvatures at p are all non-positive. Furthermore, if there exists a C^∞-vector field $\tilde{\eta}$ defined on a neighborhood U of p and a constant $\delta > 0$ such that the restriction of $\tilde{\eta}$ on $U \cap \Sigma_\varphi$ gives a null vector field, and $K^{\mathrm{ext}}(X \wedge \tilde{\eta}) \geq \delta$ holds on $U \backslash \Sigma_\varphi$ for each C^∞-vector field X on U satisfying $X \wedge \tilde{\eta} \neq 0$, then the singular principal curvatures are all negative at p.*

When $m = 2$, the assertion has been proved in [7]. The first assertion of [7, Theorem 5.1] is essentially same statement as (1). We shall prove it for general m.

Example 3.3. Consider a front

$$f : M^3 := S^2 \times \boldsymbol{R} \ni (p, t) \mapsto ((a + t^2)p, t^3) \in \boldsymbol{R}^4,$$

where $S^2 := \{(x, y, z) \in \boldsymbol{R}^3; x^2 + y^2 + z^2 = 1\}$ and a is a positive constant. The singular set of f is $\Sigma := S^2 \times \{0\}$, which consists of A_2-points, and $\partial/\partial t$ gives a null vector field. We set $\tilde{\eta} = \partial_t$, which is an extended null vector field. One can easily see that this front satisfies the condition (3) of Theorem 3.2, and all principal curvatures are equal to $-1/a$.

First, we choose a coordinate system around an A_2-singular point:

LEMMA 3.4. *Let $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ be a frontal bundle over an oriented m-manifold M^m, and let $p \in M^m$ be an A_2-singular point of φ. We fix $X \in T_p\Sigma_\varphi \backslash \{0\}$. Then there exists a local coordinate system $(u_1, \dots, u_m)$ of M^m on a neighborhood U of p such that*

(1) *The φ-singular set Σ_φ is parametrized as*

$$\Sigma_\varphi \cap U = \{(u_1, \dots, u_m); u_m = 0\}.$$

(2) $X = \partial_1$ *at* p.
(3) ∂_m *is a null vector field on* $\Sigma_\varphi \cap U$.
(4) *For each* $j = 1, \dots, m-1$, $\langle \varphi_j, D_m \varphi_m \rangle = 0$ *holds at* p.
Here, we denote

$$\partial_j = \frac{\partial}{\partial u_j}, \quad \varphi_j = \varphi(\partial_j), \quad \psi_j = \psi(\partial_j), \quad \textit{and} \quad D_j = D_{\partial_j} \quad (j = 1, \dots, m).$$

Proof. Since p is a non-degenerate singular point, the singular set Σ_φ is a smooth hypersurface on a neighborhood of p. Moreover, the null vector field is transversal to Σ_φ because p is an A_2-point. Then one can choose a coordinate system $(u_1, \dots, u_m)$ around p such that (1)–(3) hold.

We take a new coordinate system $(\tilde{u}_1, \dots, \tilde{u}_m)$ as

$$\begin{cases} \tilde{u}_j := u_j + (u_m)^2 a_j \quad (j = 1, \dots, m-1), \\ \tilde{u}_m := u_m, \end{cases}$$

where a_j $(j = 1, \dots, m-1)$ are constants. Then we have

$$\begin{cases} \dfrac{\partial}{\partial \tilde{u}_j} = \dfrac{\partial}{\partial u_j} \quad (j = 1, \dots, m-1), \\ \dfrac{\partial}{\partial \tilde{u}_m} = -2u_m \left(\displaystyle\sum_{j=1}^{m-1} a_j \frac{\partial}{\partial u_j} \right) + \dfrac{\partial}{\partial u_m}, \end{cases}$$

and thus

$$\varphi\left(\frac{\partial}{\partial \tilde{u}_m}\right) = \varphi\left(\frac{\partial}{\partial u_m}\right) - 2u_m \sum_{j=1}^{m-1} a_j \varphi\left(\frac{\partial}{\partial u_j}\right).$$

Since $\partial/\partial u_m = \partial/\partial \tilde{u}_m$ at p, we have that

$$D_{\partial/\partial \tilde{u}_m} \varphi\left(\frac{\partial}{\partial \tilde{u}_m}\right) = D_m \varphi_m - 2 \sum_{j=1}^{m-1} a_j \varphi_j$$

at p. If we set $h_{ij} := \langle \varphi_i, \varphi_j \rangle$, then (4) is equivalent to the equations

$$2 \sum_{j=1}^{m-1} a_j h_{jk} = \langle D_m \varphi_m, \varphi_k \rangle \quad (k = 1, 2, \dots, m-1). \tag{3.3}$$

Since $(h_{jk})_{j,k=1,\dots,m-1}$ is a non-singular matrix, we can choose $a_1, \dots, a_{m-1}$ so that (3.3) holds, and $(\tilde{u}_1, \dots, \tilde{u}_m)$ satisfies (1)–(4). $\square$

COROLLARY 3.5. *Let* $(u_1, \dots, u_m)$ *be a coordinate system as in Lemma* 3.4, *and assume* $(M^m, \mathscr{E}, \langle\,,\rangle, D, \varphi, \psi)$ *a front bundle. Then both* $D_m\varphi_m$ *and* ψ_m *are non-zero vectors perpendicular to* φ_j $(j = 1, \dots, m-1)$ *at* p. *In particular,* $D_m\varphi_m$ *is proportional to* ψ_m *at* p.

Proof. By (3) of Lemma 3.4, $\varphi_m = 0$ holds on Σ_φ. Since $p \in \Sigma_\varphi$ is a non-degenerate singular point, $d\lambda_\varphi(p) \neq 0$. Then by (1), it holds that $\partial_m\lambda_\varphi(p) \neq 0$:

$$\partial_m\lambda_\varphi = \partial_m\mu(\varphi_1, \dots, \varphi_m) = \mu(\varphi_1, \dots, \varphi_{m-1}, D_m\varphi_m) \neq 0 \quad (\text{at } p).$$

Hence $\{\varphi_1, \dots, \varphi_{m-1}, D_m\varphi_m\}$ is linearly independent at p. That is, $D_m\varphi_m$ is a non-zero vector which is perpendicular to $\{\varphi_1, \dots, \varphi_{m-1}\}$ at p.

On the other hand, by (2.1), we have

$$\langle \varphi_j, \psi_m \rangle = \langle \psi_j, \varphi_m \rangle = 0 \quad (j = 1, \dots, m-1)$$

on Σ_φ. Thus $\psi_m(p)$ is perpendicular to $\{\varphi_1(p), \dots, \varphi_{m-1}(p)\}$, that is, proportional to $D_m\varphi_m$ at p. Here, by (2.2), $\varphi_m(p) = 0$ implies $\psi_m(p) \neq 0$. Thus we have the conclusion. □

Proof of (1) *and* (2) *of Theorem* 3.2. Let $(u_1, \dots, u_m)$ be the local coordinate system on a neighborhood of p as in Lemma 3.4, and set

$$h_1 := \langle \varphi_1, \varphi_1 \rangle \langle \varphi_m, \varphi_m \rangle - \langle \varphi_1, \varphi_m \rangle^2,$$
$$h_2 := \langle \varphi_1, \psi_1 \rangle \langle \varphi_m, \psi_m \rangle - \langle \varphi_m, \psi_1 \rangle^2$$

on a neighborhood of p. Then $K^{\mathrm{ext}}(\partial_1 \wedge \partial_m) = h_2/h_1$ on $U \backslash \Sigma_\varphi$. Since $\varphi_m = 0$ on the φ-singular set $U \backslash \Sigma_\varphi = \{u_m = 0\}$,

$$h_1 = 0, \quad \frac{\partial h_1}{\partial u_m} = 0, \quad h_2 = 0,$$

whenever $u_m = 0$. Then there exist smooth functions $\tilde{h}_1$ and $\tilde{h}_2$ on a neighborhood of p such that

$$h_1 = (u_m)^2 \tilde{h}_1 \quad \text{and} \quad h_2 = u_m \tilde{h}_2.$$

Since $\varphi_m = 0$ on $\{u_m = 0\}$, and since $\{\varphi_1, D_m\varphi_m\}$ are linearly independent, as seen in the proof of Corollary 3.5,

$$\begin{aligned} 2\tilde{h}_1|_{u_m=0} &= \left.\frac{\partial^2}{\partial u_m^2}\right|_{u_m=0} h_1 \\ &= \langle \varphi_1, \varphi_1 \rangle \langle D_m\varphi_m, D_m\varphi_m \rangle - \langle \varphi_1, D_m\varphi_m \rangle^2 \\ &=: |\varphi_1 \wedge D_m\varphi_m|^2 > 0 \end{aligned}$$

holds on the singular set near p. On the other hand, we have

$$\tilde{h}_2|_{u_m=0} = \frac{\partial}{\partial u_m}\bigg|_{u_m=0} h_2 = \langle \varphi_1, \psi_1 \rangle \langle D_m \varphi_m, \psi_m \rangle.$$

We assume that $K^{\mathrm{ext}}(\partial_1 \wedge \partial_m)$ is bounded on $U \backslash \Sigma_\varphi$. Then $h_2/h_1 = \tilde{h}_2/(u_m \tilde{h}_1)$ is bounded on $U \backslash \Sigma_\varphi$. Thus, $\tilde{h}_2$ must vanish on the singular set near p. Here, $\langle D_m \varphi_m, \psi_m \rangle \neq 0$ holds on a neighborhood of p because of Corollary 3.5. Thus we have $\langle \varphi_1, \psi_1 \rangle = -II(X, X) = 0$ on a singular set near p. Here, since $II(\partial_m, \partial_m) = -\langle \varphi_m, \psi_m \rangle = 0$ and X is an arbitrary vector on $T_p \Sigma_\varphi$, $II(Y, Y) = 0$ holds for all $Y \in T_p M^m$. Since II is a symmetric 2-tensor, we have (1).

On the other hand, if K^{ext} is unbounded on $U \backslash \Sigma_\varphi$, the function $\tilde{h}_2$ does not vanish at p. Then $K^{\mathrm{ext}}(\partial_1 \wedge \partial_m) = (1/u_m)(\tilde{h}_2/\tilde{h}_1)$ changes sign at Σ_φ. This implies (2). □

Proof of (3) *of Theorem* 3.2. We use the same notations as in the proof of the previous parts. Then it holds that

$$\begin{aligned} \frac{\partial \tilde{h}_2}{\partial u_m}\bigg|_{u_m=0} &= \frac{\partial^2}{\partial u_m^2}\bigg|_{u_m=0} (\langle \varphi_1, \psi_1 \rangle \langle \varphi_m, \psi_m \rangle - \langle \varphi_m, \psi_1 \rangle^2) \\ &= (\partial_m \langle \varphi_1, \psi_1 \rangle)(\partial_m \langle \varphi_m, \psi_m \rangle) - (\partial_m \langle \varphi_m, \psi_1 \rangle)^2, \end{aligned}$$

because $\varphi_m = 0$ and $\langle \varphi_1, \psi_m \rangle = \langle \varphi_m, \psi_m \rangle = 0$ on the singular set. Thus,

$$\lim_{\substack{q \to p, \\ q \notin \Sigma_\varphi}} K^{\mathrm{ext}}(\partial_1 \wedge \partial_m)_q = \frac{\partial_m \langle \varphi_1, \psi_1 \rangle \partial_m \langle \varphi_m, \psi_m \rangle - (\partial_m \langle \varphi_m, \psi_1 \rangle)^2}{|\varphi_1 \wedge D_m \varphi_m|^2}. \tag{3.4}$$

Here, the assumption of the theorem implies that the value (3.4) is greater than or equal to δ (≥ 0). We consider the case that $\delta > 0$. Then it holds that

$$(\partial_m \langle \varphi_1, \psi_1 \rangle)(\partial_m \langle \varphi_m, \psi_m \rangle) > 0 \quad \text{at } p \tag{3.5}$$

because of (3.4). (If $\delta = 0$, then the left-hand side of (3.5) is non-negative.) Since $\varphi_m = 0$ and $\langle \varphi_1, \psi_m \rangle = 0$ on the singular set Σ_φ, we have

$$\begin{aligned} \partial_m \langle \varphi_1, \psi_1 \rangle &= \langle D_m \varphi_1, \psi_1 \rangle + \langle \varphi_1, D_m \psi_1 \rangle = \langle D_1 \varphi_m, \psi_1 \rangle + \langle \varphi_1, D_1 \psi_m \rangle \\ &= \partial_1 \langle \varphi_m, \psi_1 \rangle - \langle \varphi_m, D_1 \psi_1 \rangle + \partial_1 \langle \varphi_1, \psi_m \rangle - \langle D_1 \varphi_1, \psi_m \rangle \\ &= -\langle D_1 \varphi_1, \psi_m \rangle \end{aligned}$$

at p. Since $D_m \varphi_m$ is proportional to ψ_m by Corollary 3.5, this is written as

$$\begin{aligned} \partial_m \langle \varphi_1, \psi_1 \rangle &= -\langle D_1 \varphi_1, \psi_m \rangle \\ &= -\frac{\langle D_1 \varphi_1, D_m \varphi_m \rangle \langle D_m \varphi_m, \psi_m \rangle}{|D_m \varphi_m|^2} \quad \text{at } p. \end{aligned} \tag{3.6}$$

On the other hand,

$$\partial_m \langle \varphi_m, \psi_m \rangle = \langle D_m \varphi_m, \psi_m \rangle \tag{3.7}$$

holds at p. By (3.5), (3.6) and (3.7), we have

$$\langle D_1 \varphi_1, D_m \varphi_m \rangle < 0 \tag{3.8}$$

at p. Next, we compute the φ-singular normal curvature $\kappa_\varphi(\partial_1)$ with respect to the direction ∂_1 at p. Let

$$\boldsymbol{n} = \frac{\varphi_1 \wedge \cdots \wedge \varphi_{m-1}}{|\varphi_1 \wedge \cdots \wedge \varphi_{m-1}|},$$

which is the unit conormal vector field such that $\{\varphi_1, \dots, \varphi_{m-1}, \boldsymbol{n}\}$ is positively oriented. Then

$$\kappa_\varphi(\partial_1) = -\varepsilon \frac{\langle D_1 \boldsymbol{n}, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle} = \varepsilon \frac{\langle \boldsymbol{n}, D_1 \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle},$$

where

$$\begin{aligned} \varepsilon &= \operatorname{sgn}(\partial_m \lambda_\varphi) = \operatorname{sgn}(\partial_m \mu(\varphi_1, \dots, \varphi_m)) = \operatorname{sgn} \mu(\varphi_1, \dots, D_m \varphi_m) \\ &= \operatorname{sgn}\langle \varphi_1 \wedge \cdots \wedge \varphi_{m-1}, D_m \varphi_m \rangle = \operatorname{sgn}\langle \boldsymbol{n}, D_m \varphi_m \rangle. \end{aligned}$$

Here, by Corollary 3.5, $D_m \varphi_m$ is perpendicular to $\{\varphi_1, \dots, \varphi_{m-1}\}$, that is, it is proportional to $\boldsymbol{n}$. Thus, (3.8) yields

$$\operatorname{sgn}(\kappa_\varphi(\partial_1)) = \operatorname{sgn}(\langle D_m \varphi_m, \boldsymbol{n} \rangle \langle D_1 \varphi_1, \boldsymbol{n} \rangle) = \operatorname{sgn}\langle D_m \varphi_m, D_1 \varphi_1 \rangle < 0$$

at p. (When $\delta = 0$, $\kappa_\varphi(\partial_1)$ is non-positive.) Hence we have the conclusion. □

Proof of Corollary 0.2. For a front bundle induced by a front in $\boldsymbol{R}^{m+1}$ (see Example 2.2), the sectional curvature of the singular set spanned by two singular principal directions is equal to the product of the two singular principal curvatures by the Gauss equation (2.5). Thus, we have Corollary 0.2 in the introduction. □

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