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DEHN TWISTS COMBINED WITH PSEUDO-ANOSOV MAPS

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Abstract

Let S be a Riemann surface of type (p, n) with $3p + n > 4$ and $n \geq 1$. Let a be a puncture of S. We show that for any Dehn twist t_c along a simple closed geodesic c on S, there exists a sequence $\{f_m\}$ of pseudo-Anosov maps of S such that for sufficiently large integers m, the products $f_m \circ t_c^k$ are pseudo-Anosov for all integers k. As a corollary, we prove that for a multi-twist M_2 on $\tilde{S}$ along two disjoint simple closed geodesics, there are infinitely many pseudo-Anosov maps of S that are isotopic to M_2 as a is filled in.

1. Introduction

By the Nielsen–Thurston classification of surface homeomorphisms [19, 5, 6], a non-periodic irreducible map f of a surface S onto itself is isotopic to a pseudo-Anosov map f_0, by which we mean that there is a pair of transverse measured foliations $\{\mathcal{F}_+, \mathcal{F}_-\}$ on S invariant under f_0 such that

$$f_0(\mathcal{F}_+) = \lambda \mathcal{F}_+ \quad \text{and} \quad f_0(\mathcal{F}_-) = \frac{1}{\lambda} \mathcal{F}_-$$

for a fixed real number $\lambda > 1$. It is well known that $\lambda = \lambda(f_0)$ is an algebraic number and is called the dilatation of f_0 in literature. By abuse of language, f is also called pseudo-Anosov, or we simply call the isotopy class of f_0 a pseudo-Anosov mapping class.

When a pseudo-Anosov map is combined with a Dehn twist t_c along a simple closed geodesic c, the resulting map is not necessarily a pseudo-Anosov map. For example, we can take two filling simple closed geodesics α, β on S, then by Thurston [19], for any positive integers m_1 and m_2, the map $f = t_\beta^{-m_2} \circ t_\alpha^{m_1}$ is pseudo-Anosov. However, if we choose $c = \alpha$, then $f \circ t_c^k$ fails to be pseudo-Anosov for at least one integer $k = -m_1$.

There are several articles that deal with the combination of Dehn twists and pseudo-Anosov maps on S. In [13] Long and Morton proved that for

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a pseudo-Anosov map $f : S \to S$ and the Dehn twist t_c along a simple closed geodesic c, the products $f \circ t_c^k$ are pseudo-Anosov for all but at most a finite number of integer values of k. Fathi [10] elaborated that $f \circ t_c^k$ are pseudo-Anosov for all but at most seven consecutive values of k, and the location of the gap in the set of integers $\mathbf{Z}$ depends on the map f and c. It was shown recently in Boyer *et al.* [8] that the number "seven" can be improved to "six".

It is desirable to obtain some pseudo-Anosov maps f so that $f \circ t_c^k$ are pseudo-Anosov for all integers k. Let S be an analytically finite Riemann surface of type (p, n) with at least one puncture a. Assume that $3p + n > 4$. Set $\tilde{S} = S \cup \{a\}$. It was shown in Kra [11] that the set $\mathscr{F}_0$ of pseudo-Anosov maps on S that are isotopic to the identity on $\tilde{S}$ is not empty and contains infinitely many elements. In [22] we obtained certain pseudo-Anosov maps $f \in \mathscr{F}_0$ such that $f \circ t_c^k$ are pseudo-Anosov for all integers k. In this article we will obtain infinitely many pseudo-Anosov maps $f \notin \mathscr{F}_0$ with the same property.

Let $\alpha \subset S$ be a simple closed geodesic so that $\tilde{\alpha}$ is a non-trivial geodesic, where and throughout the article $\tilde{\alpha}$ denotes the geodesic homotopic to α as a is filled in if α is also viewed as a curve on $\tilde{S}$. Choose $\xi \in \mathscr{F}_0$. According to Masur–Minsky [15], for any large integer n, α and $\beta := \xi^n(\alpha)$ fill S (in fact, by an author's recent result [23], n can be chosen to be ≥ 3). Thus by Thurston [19] again, $t_\beta^{-m_2} \circ t_\alpha^{m_1}$ is pseudo-Anosov for any positive integers m_1 and m_2. Since ξ is isotopic to the identity on $\tilde{S}$, we see that $\tilde{\alpha} = \tilde{\beta}$, and thus $t_\beta^{-m_2} \circ t_\alpha^{m_1} \in \mathscr{F}_0$ if and only if $m_1 = m_2$.

The aim of this article is to prove the following result.

Theorem A. *There exist simple closed geodesics c disjoint from α and an integer N such that for all integers $m_1, m_2 \geq N$, the products*

$$(t_\beta^{-m_2} \circ t_\alpha^{m_1}) \circ t_c^k \tag{1.1}$$

are pseudo-Anosov for all integers k.

Remark. A direct consequence of the theorem is that for any simple closed geodesic c, there are pseudo-Anosov maps f of forms $t_\beta^{-m_2} \circ t_\alpha^{m_1}$ such that $f \circ t_c^k$ are pseudo-Anosov for all integers k.

To obtain a geodesic c in Theorem A, we let $\mathscr{C}$ be an a-punctured cylinder on S disjoint from α (usually there are infinitely many such a-punctured cylinders on S). According to the discussion in §2.4 and §3.3, one of the two boundary components of $\mathscr{C}$ can take a role of c in the theorem.

Theorem A does not cover the main result in [22]. Although $t_\beta^{-m_2} \circ t_\alpha^{m_1}$ are elements of $\mathscr{F}_0$ whenever $m_1 = m_2$, the set of pseudo-Anosov mapping classes of forms $t_\beta^{-m_2} \circ t_\alpha^{m_1}$ is a subset of $\mathscr{F}_0$. As a matter of fact, if $f_0 \in \mathscr{F}_0$ is such that $\lambda(f_0)$ is the minimum value among all dilatations of $f \in \mathscr{F}_0$, then by Proposition 6.1 of [24], in most cases, f_0 is not of the form $t_\beta^{-m_2} \circ t_\alpha^{m_1}$.

We believe that (1.1) are pseudo-Anosov maps for all non-zero integers m_1, m_2 and all integers k. Our argument is valid only for large integers m_1 and m_2, and provides no information on how to determine the smallest values of m_1 and m_2 so that (1.1) remains pseudo-Anosov. As we will see later, these values are determined by the relative position of α, β and c. See also §2.6.

A product of Dehn twists along a curve system is called a multi twist. As a direct consequence of Theorem A, we have the following result.

THEOREM B. *Assume that* $3p + n > 5$. *Let* M_j, $j = 1$ *or* 2, *denote an arbitrary multi twist along* j *disjoint loops on* $\tilde{S}$. *Then there exist* (*infinitely many*) *pseudo-Anosov maps isotopic to* M_j *as* a *is filled in.*

This article is organized as follows. In Section 2, we briefly review some notions and facts in Teichmüller theory. In Section 3, a Dehn twist along a simple loop on S is linked to a mapping class τ that can act on the fiber space $F(\tilde{S})$ over the Teichmüller space $T(\tilde{S})$ in a fiber preserving way. We then reduce the main theorem to the study of interactions of various such automorphisms. Section 4 and Section 5 are devoted to the proof of Theorem 3.5.1 and Theorem 3.5.2. In Section 6, we prove Theorem B and also discuss some other applications of Theorem A.

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2. Notation and background

§2.1. Teichmüller spaces and Bers fiber spaces. Let $\tilde{S}$ be a fixed Riemann surface of type $(p, n-1)$ that was introduced in Section 1. Let $\tilde{S}_1$ be a Riemann surface of the same type $(p, n-1)$. Denote by $(\tilde{S}_1, f_1)$ a marked Riemann surface, where $f_1 : \tilde{S} \to \tilde{S}_1$ is a quasiconformal homeomorphism. The Teichmüller space $T(\tilde{S})$ is defined as a set of marked Riemann surfaces $(\tilde{S}_1, f_1)$ quotient by an equivalent relation "$\sim$", where $(\tilde{S}_1, f_1) \sim (\tilde{S}_2, f_2)$ if and only if there is a conformal map $h : \tilde{S}_1 \to \tilde{S}_2$ such that $h \circ f_1$ is isotopic to f_2. We denote by $[\tilde{S}_1, f_1]$ the equivalence class of the marked surface $(\tilde{S}_1, f_1)$.

Every marked surface $(\tilde{S}_1, f_1)$ defines a new conformal structure μ_1 on $\tilde{S}$ via pullbacks. Two conformal structures μ_1 and μ_2 are called equivalent if and only if $(\tilde{S}_1, f_1) \sim (\tilde{S}_2, f_2)$. Let $[\mu]$ denote the equivalence class of a conformal structure μ on $\tilde{S}$. Thus points $[\tilde{S}_1, f_1]$ in $T(\tilde{S})$ can also be identified with $[\mu]$. It is well known that $T(\tilde{S})$ is homeomorphic to a cell in $\mathbf{R}^{6p-8+2n}$ (Abikoff [1]), and can be endowed with a complex structure so as to become a $(3p - 4 + n)$-dimensional complex manifold (Ahlfors–Bers [2]).

Let $\mathbf{H}$ be the hyperbolic plane $\{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. Associated to each point $[\mu] \in T(\tilde{S})$, there is a Jordan domain $w^{\mu}(\mathbf{H})$ depending holomorphically on $[\mu]$, where $w^\mu : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is a quasiconformal map such that (i) $w^\mu(0) = 0$, $w^\mu(1) = 1$, $w^\mu(\infty) = \infty$; (ii) w^μ is conformal off $\mathbf{H}$; and (iii) the Beltrami coefficient

$$\frac{\partial_z w^\mu(z)}{\partial_{\bar{z}} w^\mu(z)} = \mu(z), \quad \text{for } z \in \mathbf{H}.$$

The total space $F(\tilde{S})$ is called the Bers fiber space over $T(\tilde{S})$. Every point in $F(\tilde{S})$ can be written as $([\mu], z)$ where $[\mu] \in T(\tilde{S})$ and $z \in w^\mu(\mathbf{H})$. The projection $\pi : F(\tilde{S}) \to T(\tilde{S})$ that sends $([\mu], z)$ to $[\mu]$ is holomorphic. For more information, we refer to Bers [4] and Kra [11].

§2.2. Mapping class groups. The group of isotopy classes of self-maps of $\tilde{S}$ forms the mapping class group and is denoted by $\mathrm{Mod}_{\tilde{S}}$. This tells us that each element $\chi \in \mathrm{Mod}_{\tilde{S}}$ is represented by a self-map $w : \tilde{S} \to \tilde{S}$ that can be lifted to a map $\hat{w} : \mathbf{H} \to \mathbf{H}$ with $\hat{w} G \hat{w}^{-1} = G$, where G is the covering group of the universal covering map $\varrho : \mathbf{H} \to \tilde{S}$. The map $\hat{w}$ determines an equivalence class $[\hat{w}]$ that consists of all possible lifts $\hat{w}' : \mathbf{H} \to \mathbf{H}$ of self-maps of $\tilde{S}$ with

$$\hat{w} g (\hat{w})^{-1} = \hat{w}' g (\hat{w}')^{-1} \quad \text{for every } g \in G. \tag{2.1}$$

Condition (2.1) is equivalent to that $\hat{w}|_{\hat{\mathbf{R}}} = \hat{w}'|_{\hat{\mathbf{R}}}$. The group $\mathrm{mod}(\tilde{S})$ consists of equivalence classes $[\hat{w}]$ for all $w : \tilde{S} \to \tilde{S}$. Elements $[\hat{w}]$ act on $F(\tilde{S})$ by the formula

$$[\hat{w}]([\mu], z) = ([\nu], w^\nu \hat{w} (w^\mu)^{-1}(z)),$$

where ν is the Beltrami coefficient of $w^\mu \circ \hat{w}^{-1}$. In this way, $G \cong \pi_1(\tilde{S}, a)$ is regarded as a normal subgroup of $\mathrm{mod}(\tilde{S})$ with $\mathrm{mod}(\tilde{S})/G$ being isomorphic to $\mathrm{Mod}_{\tilde{S}}$. The Bers isomorphism (Theorem 9 of [4])

$$\varphi : F(\tilde{S}) \to T(S)$$

defines an isomorphism φ^* of $\mathrm{mod}(\tilde{S})$ onto the group of mapping classes on S fixing the puncture a. In particular, from [4, 7], $\varphi^*(G)$ is the subgroup of the mapping class group Mod_S that consists of mapping classes of S fixing a and projecting to the trivial mapping class on $\tilde{S}$ as a is filled in.

In the sequel, we use the notation $[\hat{w}]^*$ to denote the mapping class on S obtained from $[\hat{w}]$ under the isomorphism φ^*.

§2.3. Mapping classes and their projections under forgetful maps. Assume that $[\hat{w}]^*$ is a reducible mapping class. That is, there is a representative f of $[\hat{w}]^*$ such that f keeps a curve system $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ invariant. Here by a curve system Γ we mean that all elements in Γ are non-trivial disjoint geodesics and for elements $\gamma_i, \gamma_j \in \Gamma$ with $i \neq j$, we have $\gamma_i \neq \gamma_j$ (the assumption that $3p + n > 4$ guarantees that there exist curve systems on S).

Now suppose that $[\hat{w}]^*$ projects to a pseudo-Anosov mapping class χ on $\tilde{S}$. We claim that the only possible reducible mapping classes on S projecting to χ are those elements $[\hat{w}]^*$ so that $\hat{w}$ fixes a parabolic fixed point of G. In other words, if $\hat{w}$ does not fix any parabolic fixed point of G, then $[\hat{w}]^*$ is a pseudo-Anosov mapping class of S projecting to χ. Indeed, assume that $[\hat{w}]^*$ is reduced by a curve system $\Gamma = \{\gamma_1, \dots, \gamma_s\}$ with $s \geq 2$. Let $\mathscr{P}$ denote the set of simple closed geodesics that bound twice punctured disks enclosing a. Note that any two geodesics in $\mathscr{P}$ intersect, there is at least one geodesic γ in Γ that is not an element of $\mathscr{P}$, which means that $\tilde{\gamma}$ is non-trivial. Thus χ is reducible. It follows that $s = 1$ and $\gamma_1 \in \mathscr{P}$. We conclude that $[\hat{w}]^*(\gamma_1) = \gamma_1$. By Lemma 5.1 and Lemma 5.2 of [20], $\hat{w}$ fixes the fixed point of a parabolic element of G.

More generally, a similar argument yields that for any reducible mapping class $[\hat{w}]^*$, if the corresponding curve system $\Gamma = \{\gamma_1, \dots, \gamma_s\}$ contains an element of $\mathscr{P}$, then the same lemmas as in [20] can be applied to conclude that $\hat{w}$ fixes the fixed point of a parabolic element of G.

In particular, if $\tilde{S}$ is compact, then no twice punctured disks exist on S. This implies that all mapping classes on S projecting to a pseudo-Anosov mapping class χ are pseudo-Anosov.

§2.4. Dehn twists and their lifts to a universal covering space. Let $\hat{c} \subset \mathbf{H}$ be a geodesic such that $\tilde{c} = \varrho(\hat{c})$ is a non-trivial simple closed geodesic on $\tilde{S}$. Let D, D^* be the components of $\mathbf{H} - \{\hat{c}\}$. The Dehn twist $t_{\tilde{c}}$ can be lifted to a map $\tau : \mathbf{H} \to \mathbf{H}$ with respect to D in the following way.

Let $g \in G$ be a primitive simple hyperbolic element such that $g(D) = D$. This says that $\hat{c}$ is the axis of g. Throughout the article we use the symbol $A_g = \hat{c}$ to denote the axis of g and assume that A_g is oriented as shown in Figure 1, which is consistent with the Dehn twist $t_{\tilde{c}}$. We take an earthquake g-shift on D and leave D^* fixed. In Figure 1, the arrow underneath A_g indicates the direction of the shift that is consistent with that of A_g. We then define a lift τ of $t_{\tilde{c}}$ via G-invariance. An equivalent description for τ is given in [25].

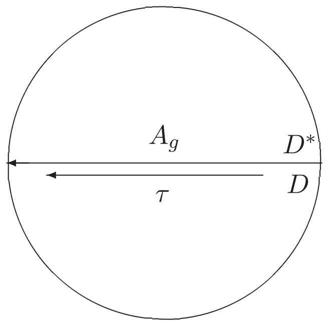

FIGURE 1

The construction of τ gives rise to a collection $\mathscr{U}_\tau$ of layered half planes in $\mathbf{H}$ in a partial order defined by inclusion. There are infinitely many disjoint maximal elements of $\mathscr{U}_\tau$, and if we denote by Ω_τ their complement in $\mathbf{H}$, then

$\Omega_\tau \subset D^*$ and the restriction $\tau|_{\Omega_\tau}$ is the identity. Also, τ keeps each maximal element of $\mathscr{U}_\tau$ invariant. In Figure 1, D is one of the maximal element of $\mathscr{U}_\tau$. Note that D^* contains infinitely many maximal elements of $\mathscr{U}_\tau$.

Obviously, the map τ constructed in this way is a quasiconformal mapping whose Beltrami coefficient $\dfrac{\partial_{\bar{z}}\tau(z)}{\partial_z\tau(z)}$ for $z \in \mathbf{H}$ is supported on the set

$$\mathscr{N} = \bigcup \{N_{h(A_g)}, \text{ for all } h \in G\}, \tag{2.2}$$

where $N_{h(A_g)}$ is an arbitrarily thin "crescent" neighborhood of $h(A_g)$. Thus τ sends any geodesic d in D disjoint from all $h(A_g)$, $h \in G$, to a geodesic in D. In other words, τ sends the half-plane D' in D whose geodesic boundary $\partial D' = d$ to a half-plane D'' in D with $D' \cap D'' = \emptyset$.

As discussed in §2.2, the equivalence class of τ determines an element $[\tau]$ of $\mathrm{mod}(\tilde{S})$. By Lemma 3.2 of [21], $[\tau]^*$ is represented by a Dehn twist t_c along a simple closed geodesic c on S. Note that if τ is a lift of $t_{\tilde{c}}$ with respect to D, then $g^{-1}\tau$ is also a lift of $t_{\tilde{c}}$ (but is with respect to D^*). By Lemma 3.2 of [21] again, $[g^{-1}\tau]^*$ is represented by the Dehn twist t_{c_0} along another simple closed geodesic c_0. Since g commutes with τ, a calculation shows that $t_c \circ t_{c_0}^{-1} = \tau^*([g^{-1}\tau]^*)^{-1} = g^*$ and $t_{c_0} \circ t_c^{-1} = g^*$. But g^* projects to the trivial mapping class on $\tilde{S}$. It follows that c is disjoint from c_0 and is freely homotopic to c_0 as a is filled in. This tells us that c_0 and c are the boundary components of an a-punctured cylinder on S.

Lemma 3.2 of [21] also says that for every simple closed geodesic $c \subset S$, we can obtain a map τ constructed above such that $[\tau]^* = t_c$.

§2.5. Iterates of half-planes under τ. Theorem 4.3.10 of Beardon [3] states that for any loxodromic Möbius transformation h, we let X, Y denote its attracting and repelling fixed points, respectively. Then for any small neighborhoods U_X, U_Y of X, Y, respectively, there is an integer N, which depends only on U_Y and U_X and is independent of choices of $z \in \mathbf{C} - U_Y$, such that $h^m(z) \in U_X$ for all $m \geq N$.

In our application, $h = g \in G$ is a hyperbolic element keeping D invariant, where D is a maximal element of $\mathscr{U}_\tau$. In this situation X, Y are attracting and repelling fixed points of g lying in $\mathbf{S}^1$. For any half-plane $D' \subset D$ with $\partial D'$, the geodesic boundary of D' in $\mathbf{H}$, projecting to a simple closed geodesic $\varrho(\partial D')$, the half-planes $g^m(D')$ are all disjoint and shrink to X as $m \to +\infty$, which means that $g^m(D') \subset U_X$ for large m and the Euclidean area of $g^m(D')$ is smaller than that of D'.

Now we proceed to examine the iteration of D' under τ^m. As mentioned in §2.4, we further assume that either $\varrho(\partial D') = \tilde{c}$ or $\varrho(\partial D')$ is disjoint from $\tilde{c}$. First, we observe from the construction that for any integer $m \neq 0$, $\tau^m(\partial D') \cap \partial D' = \emptyset$. Second, based upon the result mentioned above and by the construction of τ (see (2.2)), the regions $\tau^m(D')$ are all half-planes and the sequence $\{\tau^m(D')\}$ uniformly shrinks to the attracting fixed point X of g as $m \to +\infty$, as long as D' stays away from a small neighborhood of the repelling fixed point of g. Thus the Euclidean area of $\tau^m(D')$ shrinks to zero as $m \to +\infty$. This convergence property for the lift τ will be implicitly applied several times in the article below.

In the sequel, we call the attracting (resp. repelling) fixed point of g the attracting (resp. repelling) endpoint of D with respect to τ.

§2.6. Depths of parabolic fixed points of G. Let $T \in G$ be a parabolic element and let x be its fixed point. We need the following lemma whose proof was given in [21].

LEMMA. *There are only finitely many elements of $\mathscr{U}_\tau$ that can cover x.*

According to the lemma, every parabolic fixed point x of G is associated with a positive integer $\varepsilon_\tau(x)$ that is the number of elements of $\mathscr{U}_\tau$ covering x. The integer $\varepsilon_\tau(x)$ is called the depth of x with respect to τ throughout the rest of the article. Note that $\varepsilon_\tau(x) = \varepsilon_\tau(\tau(x))$ for all parabolic fixed points of G. Moreover, if $\varepsilon_\tau(x) = 0$, then x lies outside of all maximal elements of $\mathscr{U}_\tau$. In this case, T commutes with τ and the geodesic c on S determined by $t_c = [\tau]^*$ is disjoint from the boundary of the twice punctured disk determined by T^* (Theorem 2 of [11, 16]).

3. Reduction of Theorem A

§3.1. Pseudo-Anosov maps represented by Dehn twists. Let $\tilde{\alpha}_0 \subset \tilde{S}$ be a non-trivial simple closed geodesic. Then $\tilde{\alpha}_0$ can be viewed as a curve on S whose geodesic representative is denoted by α. Choose an element $\xi \in \mathscr{F}_0$. Then ξ is pseudo-Anosov and is isotopic to the identity on $\tilde{S}$. For sufficiently large integer k, the geodesic representative β in the homotopy class of $\xi^k(\alpha)$ together with α fills S. We must have that $\tilde{\alpha} = \tilde{\alpha}_0$. Here we recall that $\tilde{\alpha}$ denotes the geodesic on $\tilde{S}$ homotopic to α on $\tilde{S}$ if α is viewed as a curve on $\tilde{S}$. Since ξ^k is isotopic to the identity on $\tilde{S}$, we obtain $\tilde{\alpha} = \tilde{\beta}$. Write

$$f = t_\beta^{-m_2} \circ t_\alpha^{m_1}, \quad m_1, m_2 \in \mathbf{Z}^+. \tag{3.1}$$

Since $\tilde{\alpha} = \tilde{\beta}$, f is isotopic to the identity on $\tilde{S}$ when $m_1 = m_2$. In this case, we let $m = m_1 = m_2$. By Theorem 10 of Bers [4] (see also Theorem 4.2 and 4.3 of Birman [7]), there is non-trivial element $g_m \in G$ such that g_m^* is represented by f. As usual, we write $g_m^* = f$. On the other hand, by Thurston's theorem [19], f is pseudo-Anosov for every non-zero integer m. It follows from Kra (Theorem 2 of [11]) that all g_m are essential hyperbolic in the sense that their axes A_{g_m}, which are denoted by A_m in the sequel, project to filling closed geodesics $\varrho(A_m)$ under the universal covering map $\varrho : \mathbf{H} \to \tilde{S}$.

§3.2. Dehn twists interpreted as elements of $\mathrm{mod}(\tilde{S})$**.** Let α, $\tilde{\alpha}$, β and $\tilde{\beta}$ be given in §3.1. Let $\{\varrho^{-1}(\tilde{\alpha})\}$ denote the collection of all disjoint geodesics $\hat{\alpha}$ in $\mathbf{H}$ with $\varrho(\hat{\alpha}) = \tilde{\alpha}$. Recall that $\tilde{\alpha} = \tilde{\beta}$. The set $\{\varrho^{-1}(\tilde{\alpha})\}$ coincides with the set

$\{\varrho^{-1}(\tilde{\beta})\}$. By Lemma 3.2 of [21], we can choose geodesics $\hat{\alpha}, \hat{\beta} \in \{\varrho^{-1}(\tilde{\alpha})\}$, a component D_1 of $\mathbf{H} - \{\hat{\alpha}\}$ and a component D_2 of $\mathbf{H} - \{\hat{\beta}\}$ so that the lifts τ_1 and τ_2 of $t_{\tilde{\alpha}}$ (notice that $t_{\tilde{\alpha}}$ is the same as $t_{\tilde{\beta}}$) with respect to D_1 and D_2, respectively, satisfy

$$[\tau_1]^* = t_\alpha \quad \text{and} \quad [\tau_2]^* = t_\beta. \tag{3.2}$$

In addition, since (α, β) fills S, t_α does not commute with t_β. We claim that $\Omega_{\tau_1} \cap \Omega_{\tau_2} = \emptyset$ (see §2.4 for the definition of Ω_{τ_i}, $i = 1, 2$). Otherwise, suppose $\Omega_{\tau_1} \cap \Omega_{\tau_2} \neq \emptyset$. Then since geodesics in $\{\varrho^{-1}(\tilde{\alpha})\}$ (note that this set coincides with $\{\varrho^{-1}(\tilde{\beta})\}$) are all disjoint, for any $D_1 \in \mathscr{U}_{\tau_1}$ and any $D_1 \in \mathscr{U}_{\tau_2}$, either $D_1 \subset D_2$, or $D_2 \subset D_1$, or D_1 and D_2 are disjoint. By the construction, τ_1 must commute with τ_2, thus from (3.2), t_α commutes with t_β. This is a contradiction. We conclude that $\Omega_{\tau_1} \cap \Omega_{\tau_2} = \emptyset$ and thus that there exist maximal elements $D_1 \in \mathscr{U}_{\tau_1}$ and $D_2 \in \mathscr{U}_{\tau_2}$ such that

$$D_1 \cap D_2 \neq \emptyset, \quad \partial D_1 \cap \partial D_2 = \emptyset, \quad \text{and} \quad D_1 \cup D_2 = \mathbf{H}. \tag{3.3}$$

In Figure 2 below, D_1 and D_2 are so chosen that (3.2) and (3.3) hold. That is, D_1 is the region below $\hat{\alpha}$, and D_2 is the region above $\hat{\beta}$. The arrows below $\hat{\alpha}$ and above $\hat{\beta}$ indicate the motion of τ_1 and τ_2^{-1} in D_1 and D_2, respectively.

§3.3. Illustration for Figure 3. Now we can choose a simple closed geodesic $c \subset S$ so that c is disjoint from α (this implies that $\tilde{c}$ is disjoint from $\tilde{\alpha}$). Since $\{\alpha, \beta\}$ fills S and $\tilde{\alpha} = \tilde{\beta}$, c must intersect β but $\tilde{c}$ is disjoint from $\tilde{\beta}$. Thus $\{\varrho^{-1}(\tilde{c})\}$ is disjoint from $\{\varrho^{-1}(\tilde{\alpha})\}$. As discussed in §3.2, there is a lift τ of $t_{\tilde{c}}$ such that

$$[\tau]^* = t_c. \tag{3.4}$$

For convenience, we let $\mathscr{U}_1$ and $\mathscr{U}_2$ denote the collections of maximal elements of τ_1 and τ_2, and let Ω_1, Ω_2 be the complements in $\mathbf{H}$ of maximal elements of τ_1 and τ_2, respectively. Since $\tilde{\alpha} = \tilde{\beta}$ and $\tilde{c}$ is disjoint from $\tilde{\alpha}$, all boundary geodesics of maximal elements of $\mathscr{U}_\tau$, $\mathscr{U}_1$, and $\mathscr{U}_2$ are mutually disjoint.

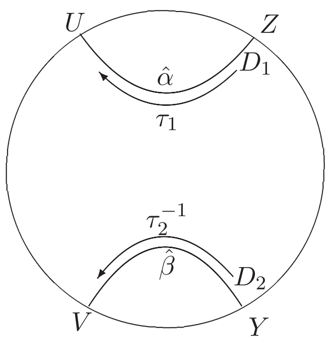

FIGURE 2

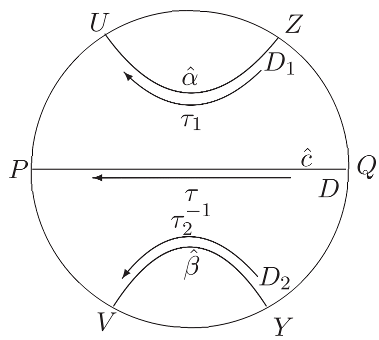

FIGURE 3

Let D_1, D_2 be chosen from §3.2. By assumption, c is disjoint from α, which means that t_c commutes with t_α, or via the Bers isomorphism, as elements of $\mathrm{mod}(\tilde{S})$, τ commutes with τ_1. It is easily shown that $\Omega_\tau \cap \Omega_1 \neq \emptyset$. We see that either (i) D_1 is contained in a maximal element D of $\mathscr{U}_\tau$ or (ii) D_1 contains infinitely many maximal elements $D_{\tau i}$ of $\mathscr{U}_\tau$. From the discussion of §2.4, we may further choose c (by replacing τ with $g^{-1}\tau$ if necessary, where $g \in G$ is hyperbolic and keeps $\hat{c}$ invariant) so that (ii) occurs. Note that t_c does not commute with t_β. By the same argument in §3.2, $\Omega_\tau \cap \Omega_2 = \emptyset$. Therefore, among those maximal elements $D_{\tau i}$, there is a maximal element, denoted by D, such that $D \cap D_2 \neq \emptyset$, $\partial D \cap \hat{\beta} = \emptyset$, and $D \cup D_2 = \mathbf{H}$. Set $\hat{c} = \partial D$. In Figure 3, $\hat{c}$ lies in the region $D_1 \cap D_2$, and D is the region below $\hat{c}$.

§3.4. Notation and convention. We refer to Figure 3. Write $\hat{\alpha} \cap \partial\mathbf{H} = \{U, Z\}$, $\hat{\beta} \cap \partial\mathbf{H} = \{V, Y\}$, and $\hat{c} \cap \partial\mathbf{H} = \{P, Q\}$. For each pair $\{U, P\}$, say, of the adjacent labeling points in $\{U, P, V, Y, Q, Z\}$, we use (U, P) or (P, U) to denote the open unoriented circular arcs in $\mathbf{S}^1$ connecting U and P without containing any other labeling points. We also use (U, P, V) to denote the open unoriented circular arcs in $\mathbf{S}^1$ connecting U and V passing through P, and so on.

§3.5. Reduction of Theorem A. From (3.2) and (3.4) we see that

$$[\tau_2^{-m_2}\tau_1^{m_1}\tau^k]^* = f \circ t_c^k,$$

where f is defined as in (3.1). For simplicity, we denote by $\zeta_k = \tau_2^{-m_2}\tau_1^{m_1}\tau^k$. Then it is readily seen that $[\zeta_k] \in \mathrm{mod}(\tilde{S})$. The proof of Theorem A can be reduced to prove the following two theorems.

THEOREM 3.5.1. *For all sufficiently large integers m_1 and m_2 and any integer k, the map ζ_k does not fix any parabolic fixed points of G.*

Remark. Once Theorem 3.5.1 is proved, then by the discussion of §2.3, the corresponding curve system does not contain any curve that bounds a twice punctured disk enclosing a. That is, all geodesics in the curve system are non-trivial. See also §3.6 for more detailed discussion.

Now suppose that $[\zeta_k]^*$ is reduced by a curve system

$$\Gamma_k = \{\gamma_{k1}, \ldots, \gamma_{ks_k}\}, \tag{3.5}$$

where all γ_{ki} are mutually disjoint geodesics and all $\tilde{\gamma}_{ki}$ are non-trivial disjoint geodesics on $\tilde{S}$. By Lemma 3.2 of [22], $[\zeta_k^2]^*$ keeps each curve γ_{ki} in Γ_k invariant. Let γ_k be any element of Γ_k. Then $[\zeta_k^2]^*(\gamma_k) = \gamma_k$ and $\tilde{\gamma}_k$ is non-trivial. Let τ_k be the lift of the Dehn twist $t_{\tilde{\gamma}_k}$ along $\tilde{\gamma}_k$ so that $[\tau_k]^* = t_{\gamma_k}$. Then τ_k gives rise to a collection of disjoint maximal half-planes that is denoted by $\mathscr{U}_k$. For simplicity we write $t_k = t_{\gamma_k}$.

THEOREM 3.5.2. *Let $\tilde{\gamma}_k$ be as above. A maximal element D_k of $\mathscr{U}_k$ can be selected so that $\zeta_k^2(D_k)$ fails to be a maximal element of $\mathscr{U}_k$.*

§3.6. Proof of Theorem A. If $k = 0$, then by Thurston's theorem [19] and the definition (3.1), $f \circ t_c^k = f$ is pseudo-Anosov for all positive integers m_1 and m_2. So we assume that $k \neq 0$. Suppose that $[\zeta_k]^* = f \circ t_c^k$ is not pseudo-Anosov for some k and some large integers m_1 and m_2. Then by the Nielsen–Thurston classification of surface homeomorphisms, $[\zeta_k]^*$ is either reducible or periodic. Since $[\zeta_k]^*$ projects to a non-trivial multi-twist, $[\zeta_k]^*$ cannot be periodic. We conclude that $[\zeta_k]^*$ is reducible. That is, there is a curve system (3.5) (depending on k) such that

$$[\zeta_k]^* | \Gamma_k = \Gamma_k.$$

If there is a loop γ_{k1}, say, in Γ_k that bounds a twice punctured disk enclosing a, then γ_{k1} is the only one such loop in Γ_k. Thus $[\zeta_k]^*(\gamma_{k1}) = \gamma_{k1}$. By Lemma 5.1 of [20], $\zeta_k = \tau_2^{-m_2}\tau_1^{m_1}\tau^k$ would fix a parabolic fixed point of G, which contradicts that ζ_k fixes no parabolic fixed points of G according to Theorem 3.5.1.

If there is no γ_{ki} in Γ_k such that γ_{ki} bounds a twice punctured disk enclosing a, then all $\tilde{\gamma}_{ki}$ are non-trivial loops on $\tilde{S}$. By Lemma 3.2 of [22] again, $[\zeta_k^2]^*$ keeps each loop γ_{ki} invariant. That is, $(f \circ t_c^k)^2(\gamma_{ki}) = \gamma_{ki}$ for $i = 1, \ldots, s_k$.

We claim that there is an element γ_{k1}, say, of Γ_k such that $\tilde{\gamma}_{k1}$ and $\tilde{c}$ are disjoint but γ_{k1} and c intersect and they form a bigon near the puncture a.

Indeed, since $[\zeta_k^2]^*$ keeps every $\gamma_{ki} \in \Gamma_k$ invariant, the projection of $[\zeta_k^2]^*$ must keep $\tilde{\gamma}_{ki}$ invariant. Note that as a is filled in, the map f^2 is isotopic to either the identity on $\tilde{S}$ (if $m_1 = m_2$) or the Dehn twist $t_{\tilde{\alpha}}^{2(m_1-m_2)}$ (if $m_1 \neq m_2$), on $\tilde{S}$, the map $[\zeta_k^2]^*$ is isotopic to the non-trivial Dehn twist $t_{\tilde{c}}^k$ or the multi-twist $t_{\tilde{\alpha}}^{2(m_1-m_2)} \circ t_{\tilde{c}}^k$. It follows that $t_{\tilde{c}}^k$ or $t_{\tilde{\alpha}}^{2(m_1-m_2)} \circ t_{\tilde{c}}^k$ keeps every $\tilde{\gamma}_{ki}$ invariant. This tells us that one of the following conditions must be satisfied:

(1) $m_1 \neq m_2$ and $\tilde{\gamma}_{ki}$ is disjoint from $\tilde{\alpha}$ and $\tilde{c}$,
(2) $m_1 \neq m_2$, and some curve, say $\tilde{\gamma}_{k1} = \tilde{\alpha}$ or $\tilde{c}$,
(3) $m_1 = m_2$ and $\tilde{\gamma}_{ki}$ is disjoint from $\tilde{c}$, or
(4) $m_1 = m_2$ and some curve, say, $\tilde{\gamma}_{k1} = \tilde{c}$.

In any one of these cases, $\tilde{\gamma}_{k1}$ is either disjoint from $\tilde{c}$ or $\tilde{\gamma}_{k1} = \tilde{c}$. If γ_{k1} is also disjoint from c, then the same argument of Lemma 3.3 of [22] will lead to a contradiction. So we can find a geodesic γ_{k1} such that γ_{k1} and c intersect. We remark that if (3) or (4) occurs, then $\tau_2^{-m_2}\tau_1^{m_1}$ is an element of G. As a consequence, for any half-plane D contained in $D_1 \cap D_2$, where D_1, D_2 are shown in Figure 2, $\tau_2^{-m_2}\tau_1^{m_1}(D) \subset D_1 \cap D_2$ is also a half-plane.

For simplicity we write $\gamma_k = \gamma_{k1}$. We conclude that $\tilde{\gamma}_k$ is disjoint from $\tilde{c}$ but γ_k intersects c and they form a bigon near the puncture a. Since $[\zeta_k^2]^*$ fixes γ_k,

$$(f \circ t_c^k)^2 \circ t_k \circ (f \circ t_c^k)^{-2} = t_k. \tag{3.6}$$

Via the Bers isomorphism $\varphi^* : \mathrm{mod}(\tilde{S}) \to \mathrm{Mod}_S^a$, we then obtain the following equality:

$$\zeta_k^2 \tau_k \zeta_k^{-2} = \tau_k. \tag{3.7}$$

Let $D_k \in \mathscr{U}_k$ be any maximal element. Then τ_k keeps D_k invariant, and no points on $D_k \cap \mathbf{S}^1$ are fixed by τ_k except for the endpoints. Hence $\zeta_k^2 \tau_k \zeta_k^{-2}$ sends $\zeta_k^2(D_k)$ to itself and does not fix any point in $\zeta_k^2(D_k) \cap \mathbf{S}^1$. From (3.7), τ_k sends $\zeta_k^2(D_k)$ to itself and does not fix any point in $\zeta_k^2(D_k) \cap \mathbf{S}^1$. This implies that $\zeta_k^2(D_k)$ is also a maximal element of $\mathscr{U}_k$. This contradicts Theorem 3.5.2, and hence the proof of Theorem A is complete. □

4. Proof of Theorem 3.5.1

Let x be the fixed point of a parabolic element $T \in G$. Obviously, x can not be an endpoint of any element of $\mathscr{U}_1$, $\mathscr{U}_2$, and $\mathscr{U}_\tau$. Otherwise, T would share its fixed point with a hyperbolic element of G, and this would also contradict that G is discrete. We refer to Figure 3.

If $x \in (V, Y)$, then since $\tilde{\beta}$ is simple, $\tau^k(\hat{\beta}) \cap \hat{\beta} = \emptyset$. We see that

$$y = \tau^k(x) \in (P, V) \quad \text{and} \quad \tau_2^{-m_2}\tau_1^{m_1}\tau^k(x) = \tau_2^{-m_2}\tau_1^{m_1}(y) \in (P, V).$$

It follows that $\zeta_k(x) = \tau_2^{-m_2}\tau_1^{m_1}(y) \neq x$. Similar computation also yields that $\zeta_k^2(x) \neq x$.

If $x \in (U, Z)$, and x is not covered by any maximal element of $\mathscr{U}_\tau$, then $\tau^k(x) = x$. Since τ_1 keeps D_1 invariant, τ_1 keeps (U, Z) invariant as well. So $\tau_1^{m_1}(x) \in (U, Z)$. Since $\tilde{\alpha}$ is simple, $\tau_2^{-m_2}$ sends (U, Z) to an interval in (U, P, V) disjoint from (U, Z). It follows that

$$\zeta_k(x) = \tau_2^{-m_2}\tau_1^{m_1}\tau^k(x) \neq x.$$

Similarly we have $\zeta_k^2(x) \neq x$. If $x \in (U, Z)$ and x is covered by a maximal element $D' \in \mathscr{U}_\tau$. Since α is disjoint from c, $D' \subset \mathbf{H} - D_1$. This means that $\tau^k(x) \in (U, Z)$ and $\tau_1^{m_1}\tau^k(x) \in (U, Z)$. Since $\tilde{\alpha}$ is a simple closed geodesic,

$$\tau_2^{-m_2}(\mathbf{H} - D_1) \cap (\mathbf{H} - D_1) = \emptyset.$$

It follows that $\zeta_k(x) = \tau_2^{-m_2}\tau_1^{m_1}\tau^k(x) \neq x$ and that $\zeta_k^2(x) \neq x$. Similar argument yields that $\zeta_k^2(x) \neq x$ if $x \in (P, U) \cup (Q, Y)$.

It remains to settle the case that $x \in (P, V) \cup (Q, Z)$. Suppose that $x \in (P, V)$ and that $\zeta_k^2(x) = x$. In this case, x stays away from the point Z (where we recall that Z is the repelling endpoint of D_1 with respect to τ_1). Let

$$\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_r \ni x, \quad r \geq 1,$$

be elements of $\mathscr{U}_\tau$ covering $x \in (P, V)$. We see that $\Delta_1 = D$. From Lemma 2.6, $r < \infty$, and thus $\varepsilon_\tau(x) = r$. Observe that $\varepsilon_\tau(x) = \varepsilon_\tau(\tau^k(x))$.

Suppose that $x \in (P, V)$ and stays away from a neighborhood $U_\delta \cap (P, V)$ of P, where δ is a small positive number, and U_δ is a small neighborhood of the point P in the interval (P, V). Let

$$D = \Delta_1 = \Delta_1' \supset \Delta_2' \supset \cdots \supset \Delta_r' \ni \tau^k(x)$$

be the corresponding elements of $\mathscr{U}_\tau$ covering $\tau^k(x)$. Then the ratio

$$0 < \frac{\operatorname{diam}(\Delta_r')}{\operatorname{diam}(\Delta_r)} < C \tag{4.1}$$

for a constant $C > 1$, where $\operatorname{diam}(\Delta_r')$ and $\operatorname{diam}(\Delta_r)$ denote the Euclidean diameters of Δ_r' and Δ_r.

For sufficiently large m_1, $\Delta_r'' = \tau_1^{m_1}(\Delta_r')$ shrinks to U (recall that U is the attracting endpoint of D_1 with respect to τ_1, see Figure 3). This means that the Euclidean diameter $\operatorname{diam}(\Delta_r'')$ decreases to zero as $m_1 \to +\infty$.

Also, we observe that $\tau_2^{-m_2}(\Delta_r'')$ shrinks to V, which is the attracting endpoint of D_2 (with respect to τ_2^{-1}), and $\operatorname{diam} \tau_2^{-m_2}(\Delta_r'')$ decreases to zero as $m_2 \to +\infty$. Note that (P, V) is far from Z and Y, by the discussion of §2.5, we conclude that for a large integer m_2, which does not depend on any point $x \in (P, V) - U_\delta$, the Euclidean diameter of $\tau_2^{-m_2}(\Delta_r'') = \tau_2^{-m_2}\tau_1^{m_1}(\Delta_r')$ is smaller than that of Δ_r. Suppose that $\zeta(x) = x$, we then have $\tau_2^{-m_2}(\Delta_r'') \subset \Delta_r$. Since $\tau_2^{-m_2}(\Delta_r'') \in \mathscr{U}_\tau$, we conclude that

$$\varepsilon_\tau(\tau_2^{-m_2}\tau_1^{m_1}\tau^k(x)) = \varepsilon_\tau(\zeta_k(x)) \geq \varepsilon_\tau(x) + 1.$$

It follows that $\zeta_k(x) \neq x$. Similar argument yields that $\zeta_k^2(x) \neq x$. This is a contradiction.

If $x \in U_\delta \cap (P, V)$, then usually, we do not have (4.1) (for instance, k could be certain negative integer). But we observe, by using the same discussion of §2.5, that $\tau_2^{-m_2}(y)$ shrinks to V uniformly for any point $y \in \mathbf{S}^1$ staying away from a fixed neighborhood of Y. Since $\tau_1^{m_1}\tau^k(x)$ shrinks to U, $\tau_1^{m_1}\tau^k(x)$ stay away from Y. We thus conclude that $\tau_2^{-m_2}\tau_1^{m_1}\tau^k(x) \neq x$ for large integers m_1 and m_2 that are independent of $x \in U_\varepsilon \cap (P, V)$. By repeating the computation one shows that $\zeta_k^2(x) \neq x$.

By taking the inverse of ζ_k^2 and using the same argument as above, we can settle the case that $x \in (Q, Z)$. In this case, x stays away from V, the repelling endpoint of D_2 (with respect to τ_2). This proves Theorem 3.5.1. □

5. Proof of Theorem 3.5.2

In this section, we handle the case that $[\zeta_k^2]^* = (f \circ t_c^k)^2$ is reduced by a single geodesic γ_k with $\tilde{\gamma}_k$ being non-trivial on $\tilde{S}$. The regions D, D_1, D_2 and the geodesics $\hat{c}$, $\hat{\alpha}$, $\hat{\beta}$ are drawn in Figure 3. From §3.5 and §3.6, we know that the projection $\tilde{\gamma}_k$ of γ_k is disjoint from $\tilde{c}$ ($\tilde{\gamma}_k$ is disjoint from both $\tilde{\alpha}$ and $\tilde{c}$ if $m_1 \neq m_2$). Hence all boundary geodesics of elements of $\mathscr{U}_k$ are disjoint from $\hat{c}$. Note that D is the component of $\mathbf{H} - \{\hat{c}\}$ below $\hat{c}$. See Figure 3. By selecting a subsequence if needed, there are two cases to consider.

Case 1. D is not included in any element of $\mathscr{U}_k$. In this case, D contains infinitely many maximal elements of $\mathscr{U}_k$. Since c intersects γ_k, there is a maximal element $D_k \in \mathscr{U}_k$ such that

$$D_k \cap D \neq \emptyset, \quad \partial D_k \cap \partial D = \emptyset, \quad \text{and} \quad D \cup D_k = \mathbf{H}.$$

(Otherwise, we have $\Omega_k \cap \Omega_\tau \neq \emptyset$ and this tells us that t_c commutes with t_{γ_k} and thus that c is disjoint from γ_k. This is a contradiction.) Write $\sigma_k = \partial D_k \subset \mathbf{H}$ the boundary geodesic of D_k. Then σ_k is disjoint from $\hat{c} = \partial D$ and $\sigma_k \subset D$ stays away from the point Z. Since $\hat{c} \subset D_k$, and since $\tilde{\gamma}_k$ is a simple curve, for any integer $k \neq 0$, $\tau^k(\sigma_k) \subset D$ and $\tau^k(\sigma_k) \cap \sigma_k = \emptyset$. Therefore, $\tau_1^{m_1}\tau^k(\sigma_k)$ shrinks to the point U. Thus $\tau_2^{-m_2}\tau_1^{m_1}\tau^k(\sigma_k)$ shrinks to the point V. Similar calculations show that $(\tau_2^{-m_2}\tau_1^{m_1}\tau^k)^2(\sigma_k)$ shrinks to the point V also. Here and below, we use the same discussion of §2.5 and conclude that the integers m_1 and m_2 are fixed and are independent of choices of k.

Since both $(\tau_2^{-m_2}\tau_1^{m_1}\tau^k)^2(D_k)$ and D_k contain the region $\mathbf{H} - D$, we have

$$(\tau_2^{-m_2}\tau_1^{m_1}\tau^k)^2(D_k) \cap D_k \neq \emptyset. \tag{5.1}$$

To see that for sufficiently large integers m_1 and m_2, $\tau_2^{-m_2}\tau_1^{m_1}\tau^k(\sigma_k) \neq \sigma_k$, we denote by $\mathrm{diam}(\sigma_k)$ the Euclidean diameter of σ_k.

If $\sigma_k \subset D$ and $\mathrm{diam}(\sigma_k) \to 0$ as $k \to +\infty$ or $k \to -\infty$, and the ratio

$$\frac{\mathrm{diam}(\tau^k(\sigma_k))}{\mathrm{diam}(\sigma_k)}$$

is unbounded above (which occurs when σ_k shrinks to P or Q), then since P or Q stays away from V, $\tau_2^{-m_2}\tau_1^{m_1}\tau^k(\sigma_k)$ is disjoint from σ_k.

Otherwise, we have $\sigma_k \subset D$ and there is a constant $C > 1$

$$\frac{\mathrm{diam}(\tau^k(\sigma_k))}{\mathrm{diam}(\sigma_k)} < C \tag{5.2}$$

for all integers k. It follows from (5.2) that for sufficiently large integers m_1 and m_2, $\mathrm{diam}(\tau_2^{-m_2}\tau_1^{m_1}\tau^k(\sigma_k))$ is smaller than $\mathrm{diam}(\tau^k(\sigma_k))/C$. We thus obtain

$$\mathrm{diam}(\tau_2^{-m_2}\tau_1^{m_1}\tau^k(\sigma_k)) < \frac{\mathrm{diam}(\tau^k(\sigma_k))}{C} < \mathrm{diam}(\sigma_k).$$

In particular, we obtain $\tau_2^{-m_2}\tau_1^{m_1}\tau^k(\sigma_k) \neq \sigma_k$. Similarly, we can show that $(\tau_2^{-m_2}\tau_1^{m_1}\tau^k)^2(\sigma_k) \neq \sigma_k$. Together with (5.1), we conclude that the half plane $(\tau_2^{-m_2}\tau_1^{m_1}\tau^k)^2(D_k)$ cannot be a maximal element of $\mathscr{U}_k$.

Case 2. D is included in an element D_k of $\mathscr{U}_k$. In this case, we consider the inverse map ζ_k^{-2} of ζ_k^2. Let $D_k' = \mathbf{H} - D_k$. Note that both D_k and D_k' share the common boundary geodesic σ_k. The region D_k' stays away from V. Hence $\tau_2^{m_2}(D_k')$ shrinks to the point Y uniformly, and thus $\tau_1^{-m_1}\tau_2^{m_2}(D_k')$ shrinks to the point Z uniformly. This implies that $\mathrm{diam}(\tau_1^{-m_1}\tau_2^{m_2}(D_k'))$ and thus also $\mathrm{diam}((\tau_1^{-m_1}\tau_2^{m_2})^2(D_k'))$ are small. Since both D_k and $(\tau^{-k}\tau_1^{-m_1}\tau_2^{m_2})^2(D_k)$ contain the region D,

$$D_k \cap ((\tau^{-k}\tau_1^{-m_1}\tau_2^{m_2})^2(D_k)) \neq \emptyset. \tag{5.3}$$

We need to show that $D_k \neq (\tau^{-k}\tau_1^{-m_1}\tau_2^{m_2})^2(D_k)$. Suppose for the contrary, we assume $D_k = (\tau^{-k}\tau_1^{-m_1}\tau_2^{m_2})^2(D_k)$. By hypothesis, $\tau^k(D_k') = D_k'$, we obtain

$$\tau_2^{-m_2}\tau_1^{m_1}(D_k') = \tau^{-k}\tau_1^{-m_1}\tau_2^{m_2}(D_k'). \tag{5.4}$$

By examining the actions of $\tau_2^{-m_2}\tau_1^{m_1}$ and $\tau^{-k}\tau_1^{-m_1}\tau_2^{m_2}$ on D_k', we see that (5.4) cannot hold. It follows that $(\tau^{-k}\tau_1^{-m_1}\tau_2^{m_2})^2(D_k) \neq D_k$.

This fact together with (5.3) tells us that $(\tau^{-k}\tau_1^{-m_1}\tau_2^{m_2})^2(D_k)$ is not a maximal element of $\mathscr{U}_k$. Hence $[\zeta_k^{-2}]^*$ cannot be reduced by the geodesic γ_k. This completes the proof of Theorem 3.5.2. □

6. Some remarks

Theorem A has some interesting applications.

§6.1. Proof of Theorem B. (1) First we consider the case that $j = 1$. We may assume that $M_1 = t_{\tilde{c}}$. By Theorem A, there is a large integer N such that for any $m \geq N$, $t_\beta^{-m} \circ t_\alpha^m \circ t_c$ are pseudo-Anosov. Since $\tilde{\alpha} = \tilde{\beta}$, all the mapping classes $t_\beta^{-m} \circ t_\alpha^m \circ t_c$ project to M_1 as a is filled in. This proves (1).

(2) $j = 2$. We set $M_2 = t_{\tilde{\alpha}}^{k_1} \circ t_{\tilde{c}}^{k_2}$, where $k_1, k_2 \in \mathbf{Z} - \{0\}$. For a positive integer s, we consider the following map ζ_s:

$$\zeta_s = t_\beta^{-s} \circ t_\alpha^{s+k_1} \circ t_c^{k_2}.$$

When s is chosen so large that $s, s + k_1 \geq N$, we can apply Theorem A again to conclude that $\zeta_s \in \mathrm{Mod}_S$ is pseudo-Anosov. Since $\tilde{\alpha} = \tilde{\beta}$, all the mapping classes ζ_s project to M_2 as a is filled in. This proves (2). □

§6.2. Generalizations. To proceed, we let N be as in Theorem A, let $m_1, m_2 \geq N$, and set $f = t_\beta^{-m_2} \circ t_\alpha^{m_1}$. Theorem A can be extended to the following result:

Corollary. *For any integers s_i and positive integers r_i, the finite products*

$$\prod_i (f^{r_i} \circ t_c^{s_i}) \tag{6.1}$$

are pseudo-Anosov maps.

Proof. The argument of Theorem 3.5.1 and Theorem 3.5.2 is valid not only for $\zeta_k = \tau_2^{-m_2}\tau_1^{m_1}\tau^k$, but also for any finite product

$$\prod_i ((\tau_2^{-m_2}\tau_1^{m_1})^{r_i}\tau^{s_i})$$

for any positive integers r_i and any integers s_i. Thus the argument of Theorem A (§3.6) can be carried over to the general case. □

§6.3. Examples. Let $A = \{\alpha_1, \dots, \alpha_m\}$ and $B = \{\beta_1, \dots, \beta_l\}$ be two families of disjoint simple closed geodesics on S so that $\{A, B\}$ fills S. It was shown in Thurston [19] (see also [9], [14], [17, 18]) that any word consisting of positive multi twists t_A along elements of A and negative multi twists t_B^{-1} along elements of B represents a pseudo-Anosov mapping class. For an extensive account of the group $\langle t_A, t_B \rangle$ generated by positive multi twists t_A and t_B, we refer to Leininger [12]. As a consequence of Theorem A (or Corollary 6.2), we are able to provide some pseudo-Anosov maps with mixed multi twists in the case that A and B contains no more than two curves. For any geodesic c on S, we recall that $\tilde{c} \subset \tilde{S}$ is the geodesic on $\tilde{S}$ homotopic to c as a is filled in.

Corollary. *Let $A = \{\alpha_1, \alpha_2\}$ and $B = \{\beta\}$. Assume that $\{\alpha_1, \beta\}$ fills S and $\tilde{\alpha}_1 = \tilde{\beta}$. Then for any $s_i, r_i, q_i \in \mathbf{Z}^+$ with r_i, q_i sufficiently large, the finite products*

$$\prod_i t_\beta^{-q_i} \circ (t_{\alpha_1}^{r_i} \circ t_{\alpha_2}^{-s_i}) \tag{6.2}$$

are pseudo-Anosov maps.

Proof. By associativity, (6.2) are finite products by terms

$$(t_\beta^{-q_i} \circ t_{\alpha_1}^{r_i}) \circ t_{\alpha_2}^{-s_i}.$$

Since $\tilde{\alpha}_1 = \tilde{\beta}$, $t_\beta^{-q_i} \circ t_{\alpha_1}^{r_i}$ projects to the Dehn twist $t_{\tilde{\alpha}_1}^{r_i - q_i}$ (if $r_i \neq q_i$), or the identity (if $r_i = q_i$). Hence it can be denoted by f. We see that (6.2) is a special form of (6.1), and this particularly implies that (6.2) are pseudo-Anosov maps. □

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