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r-STABLE SPACELIKE HYPERSURFACES IN CONFORMALLY STATIONARY SPACETIMES

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Abstract

In this paper we study the r-stability of closed spacelike hypersurfaces with constant r-th mean curvature in conformally stationary spacetimes of constant sectional curvature. In this setting, we obtain a characterization of r-stability through the analysis of the first eigenvalue of an operator naturally attached to the r-th mean curvature. As an application, we treat the case in which the spacetime is the de Sitter space.

1. Introduction

The notion of stability concerning hypersurfaces of constant mean curvature of Riemannian ambient spaces was first studied by Barbosa and do Carmo in [4], and by Barbosa, do Carmo and Eschenburg in [5], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations.

In the Lorentz context, in 1993 Barbosa and Oliker [7] obtained an analogous result, proving that constant mean curvature spacelike hypersurfaces in Lorentz manifolds are also critical points of the area functional for variations that keep the volume constant. They also computed the second variation formula and showed, for the de Sitter space $\mathbf{S}_1^{n+1}$, that spheres maximize the area functional for volume-preserving variations.

More recently, Liu and Junlei [15] have characterized the r-stable closed spacelike hypersurfaces with constant scalar curvature in the de Sitter space.

The natural generalization of mean and scalar curvatures for an n-dimensional hypersurface are the r-th mean curvatures H_r, for $r = 1, \dots, n$.

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In fact, H_1 is just the mean curvature and H_2 defines a geometric quantity which is related to the scalar curvature.

In [10], some of the authors have studied the problem of strong stability (that is, stability with respect to not necessarily volume-preserving variations) for spacelike hypersurfaces with constant r-th mean curvature in a Generalized Robertson-Walker (GRW) spacetime of constant sectional curvature, giving a characterization of r-maximal and spacelike slices.

Here, motivated by these works, we consider closed spacelike hypersurfaces with constant r-th mean curvature in a wider class of Lorentz manifolds, the so-called *conformally stationary spacetimes*, in order to obtain a relation between r-stability and the spectrum of a certain elliptic operator naturally attached to the r-th mean curvature of the hypersurfaces. Our approach is based on the use of the Newton transformations P_r and their associated second order differential operators L_r (cf. Section 2). More precisely, we prove the following result.

THEOREM 1.1. *Let $\overline{M}_c^{n+1}$ be a conformally stationary Lorentz manifold with constant curvature c. Suppose that $\overline{M}_c^{n+1}$ has a closed conformal vector field V and a Killing vector field W. Let $x : M^n \to \overline{M}_c^{n+1}$ be a closed spacelike hypersurface, with constant, positive $(r+1)$-th mean curvature H_{r+1} such that*

$$\lambda = c(n-r)\binom{n}{r}H_r - nH_1\binom{n}{r+1}H_{r+1} + (r+2)\binom{n}{r+2}H_{r+2}$$

is constant. Assume also that $Div_{\overline{M}}\, V$ does not vanish on M^n. Then x is r-stable if and only if λ is the first eigenvalue of L_r on M^n.

As an application of the previous result, we obtain the following corollary in the de Sitter space.

COROLLARY 1.2. *Let $x : M^n \to \mathbf{S}_1^{n+1}$ be a closed spacelike hypersurface, contained in the chronological future (or past) of an equator of $\mathbf{S}_1^{n+1}$, with positive constant $(r+1)$-th mean curvature such that*

$$\lambda = (n-r)\binom{n}{r}H_r - nH\binom{n}{r+1}H_{r+1} + (r+2)\binom{n}{r+2}H_{r+2}$$

is constant. Then x is r-stable if and only if λ is the first eigenvalue of L_r on M^n.

Finally, it should be said that, although every GRW is a conformally stationary spacetime, the converse only holds locally, and just for conformally stationary spacetimes of constant sectional curvature. Moreover, our techniques and results are quite different from those of [10], and in particular do not coincide with them in case our conformally stationary spacetime happens to be a GRW of constant sectional curvature.

2. Preliminaries

Let $\overline{M}^{n+1}$ denote a time-oriented Lorentz manifold with Lorentz metric $\bar{g} = \langle\, , \rangle$, volume element $d\overline{M}$ and semi-Riemannian connection $\overline{\nabla}$. In this context, we consider spacelike hypersurfaces $x : M^n \to \overline{M}^{n+1}$, namely, isometric immersions from a connected, n-dimensional orientable Riemannian manifold M^n into $\overline{M}$. We let ∇ denote the Levi-Civita connection of M^n.

If $\overline{M}$ is time-orientable and $x : M^n \to \overline{M}^{n+1}$ is a spacelike hypersurface, then M^n is orientable (cf. [13]) and one can choose a globally defined unit normal vector field N on M^n having the same time-orientation of $\overline{M}$. Such an N is named the *future-pointing Gauss map* of M^n. In this setting, let A denote the shape operator of M with respect to N, so that at each $p \in M^n$, A restricts to a self-adjoint linear map $A_p : T_pM \to T_pM$.

For $1 \le r \le n$, let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p; this way one gets n smooth functions $S_r : M^n \to \mathbf{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in M^n$ and $\{e_k\}$ is a basis of T_pM formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbf{R}[X_1, \dots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$.

For $1 \le r \le n$, one defines the r-th mean curvature H_r of x by

$$\binom{n}{r} H_r = (-1)^r S_r = \sigma_r(-\lambda_1, \dots, -\lambda_n).$$

Also, for $0 \le r \le n$, the r-th Newton transformation P_r on M^n is defined by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = (-1)^r S_r I + A P_{r-1}. \tag{2.1}$$

A trivial induction shows that

$$P_r = (-1)^r (S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r),$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM diagonalizing A at $p \in M^n$ also diagonalize all of the P_r at p.

Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i\rangle^{\perp} \subset T_p\Sigma$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1}(-1)^k S_k(A_i)t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1\le j_1<\cdots<j_k\le n\\ j_1,\ldots,j_k\neq i}} \lambda_{j_1}\cdots\lambda_{j_k}.$$

With the above notations, it is also immediate to check that $P_r e_i = (-1)^r S_r(A_i)e_i$, and hence (cf. Lemma 2.1 of [6])

$$\begin{aligned} \operatorname{tr}(P_r) &= (-1)^r(n-r)S_r = b_r H_r;\\ \operatorname{tr}(AP_r) &= (-1)^r(r+1)S_{r+1} = -b_r H_{r+1};\\ \operatorname{tr}(A^2P_r) &= (-1)^r(S_1S_{r+1} - (r+2)S_{r+2}), \end{aligned} \tag{2.2}$$

where $b_r = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the second order linear differential operator $L_r : \mathscr{D}(M) \to \mathscr{D}(M)$, given by

$$L_r(f) = \operatorname{tr}(P_r \operatorname{Hess} f).$$

For instance, when $r = 0$, L_r is simply the Laplacian operator.

According to [3], if $\overline{M}^{n+1}$ is of constant sectional curvature, then P_r is divergence-free and, consequently,

$$L_r(f) = \operatorname{div}(P_r\nabla f).$$

If x is as above, a *variation* of it is a smooth mapping

$$X : M^n \times (-\varepsilon,\varepsilon) \to \overline{M}^{n+1}$$

satisfying the following conditions:

(1) For $t \in (-\varepsilon,\varepsilon)$, the map $X_t : M^n \to \overline{M}^{n+1}$ given by $X_t(p) = X(t,p)$ is a spacelike immersion such that $X_0 = x$.

(2) $X_t|_{\partial M} = x|_{\partial M}$, for all $t \in (-\varepsilon,\varepsilon)$.

In all that follows, we let dM_t denote the volume element of the metric induced on M by X_t and N_t the unit normal vector field along X_t.

The *variational field* associated to the variation X is the vector field $\left.\frac{\partial X}{\partial t}\right|_{t=0}$. Letting $f = -\left\langle \frac{\partial X}{\partial t}, N_t\right\rangle$, we get

$$\frac{\partial X}{\partial t} = fN_t + \left(\frac{\partial X}{\partial t}\right)^{\top}, \tag{2.3}$$

where $\top$ stands for tangential components.

The *balance of volume* of the variation X is the function $\mathscr{V}:(-\varepsilon,\varepsilon)\to\mathbf{R}$ given by

$$\mathscr{V}(t)=\int_{M\times[0,t]}X^*(d\overline{M}),$$

and we say X is *volume-preserving* if $\mathscr{V}$ is constant.

From now on, we will consider only closed spacelike hypersurfaces $x:M^n\to\overline{M}^{n+1}$. The following lemma is well known and can be found in [10].

Lemma 2.1. *Let $\overline{M}^{n+1}$ be a time-oriented Lorentz manifold and $x:M^n\to\overline{M}^{n+1}$ a closed spacelike hypersurface. If $X:M^n\times(-\varepsilon,\varepsilon)\to\overline{M}^{n+1}$ is a variation of x, then*

$$\frac{d\mathscr{V}}{dt}=\int_M f\,dM_t.$$

In particular, X is volume-preserving if and only if $\int_M f\,dM_t=0$ for all t.

We remark that Lemma 2.2 of [5] remains valid in the Lorentz context, i.e., if $f_0:M\to\mathbf{R}$ is a smooth function such that $\int_M f_0\,dM=0$, then there exists a volume-preserving variation of M whose variational field is f_0N.

In order to extend [6] to the Lorentz setting, we define the *r-area functional* $\mathscr{A}_r:(-\varepsilon,\varepsilon)\to\mathbf{R}$ associated to the variation X by

$$\mathscr{A}_r(t)=\int_M F_r(S_1,S_2,\dots,S_r)\,dM_t,$$

where $S_r=S_r(t)$ and F_r is recursively defined by setting $F_0=1$, $F_1=-S_1$ and, for $2\le r\le n-1$,

$$F_r=(-1)^rS_r-\frac{c(n-r+1)}{r-1}F_{r-2}.$$

We notice that if $r=0$, the functional $\mathscr{A}_0$ is the classical area functional.

The next results were proved in [10] and the first of them is the Lorentz analogue of Proposition 4.1 of [6].

Lemma 2.2. *Let $x:M^n\to\overline{M}_c^{n+1}$ be a closed spacelike hypersurface of the time-oriented Lorentz manifold $\overline{M}_c^{n+1}$ with constant curvature c, and let $X:M^n\times(-\varepsilon,\varepsilon)\to\overline{M}_c^{n+1}$ be a variation of x. Then,*

$$\frac{\partial S_{r+1}}{\partial t}=(-1)^{r+1}[L_rf+c\,\mathrm{tr}(P_r)f-\mathrm{tr}(A^2P_r)f]+\left\langle\left(\frac{\partial X}{\partial t}\right)^{\top},\nabla S_{r+1}\right\rangle. \tag{2.4}$$

PROPOSITION 2.3. *Under the hypotheses of Lemma* 2.2, *if* X *is a variation of* x, *then*

$$\mathscr{A}_r'(t) = \int_M [(-1)^{r+1}(r+1)S_{r+1} + c_r] f \, dM_t, \tag{2.5}$$

where $c_r = 0$ *if* r *is even and* $c_r = -\dfrac{n(n-2)(n-4)\cdots(n-r+1)}{(r-1)(r-3)\cdots 2}(-c)^{(r+1)/2}$ *if* r *is odd.*

In order to characterize spacelike immersions of constant $(r+1)$-th mean curvature, let λ be a real constant and $\mathscr{J}_r : (-\varepsilon, \varepsilon) \to \mathbf{R}$ be the *Jacobi functional* associated to the variation X, i.e.,

$$\mathscr{J}_r(t) = \mathscr{A}_r(t) - \lambda \mathscr{V}(t).$$

As an immediate consequence of (2.5) we get

$$\mathscr{J}_r'(t) = \int_M [b_r H_{r+1} + c_r - \lambda] f \, dM_t,$$

where $b_r = (r+1)\binom{n}{r+1}$. Therefore, if we choose $\lambda = c_r + b_r \overline{H}_{r+1}(0)$, where

$$\overline{H}_{r+1}(0) = \frac{1}{\mathscr{A}_0(0)} \int_M H_{r+1}(0) \, dM$$

is the mean of the $(r+1)$-th curvature $H_{r+1}(0)$ of M, we arrive at

$$\mathscr{J}_r'(t) = b_r \int_M [H_{r+1} - \overline{H}_{r+1}(0)] f \, dM_t.$$

Hence, a standard argument (cf. [4]) shows that M is a critical point of $\mathscr{J}_r$ for all variations of x if and only if M has constant $(r+1)$-th mean curvature.

We wish to study spacelike immersions $x : M^n \to \overline{M}^{n+1}$ that maximize $\mathscr{A}_r$ for all volume-preserving variations X of x. The above dicussion shows that M must have constant $(r+1)$-th mean curvature and, for such an M, one is naturally lead to compute the second variation of $\mathscr{A}_r$. This motivates the following

DEFINITION 2.4. Let $\overline{M}_c^{n+1}$ be a time-oriented Lorentz manifold of constant curvature c, and $x : M^n \to \overline{M}^{n+1}$ be a closed spacelike hypersurface having constant $(r+1)$-th mean curvature. We say that x is r-stable if $\mathscr{A}_r''(0) \le 0$, for all volume-preserving variations of x.

Remark 2.5. Let $x : M^n \to \overline{M}_c^{n+1}$ be a closed spacelike hypersurface with constant $(r+1)$-th mean curvature and denote by $\mathscr{G}$ the set of differential functions $f : M^n \to \mathbf{R}$ with $\int_M f \, dM_t = 0$. Just as [15] we can establish the following criterion for stability: *x is r-stable if and only if* $\mathscr{J}_r''(0) \le 0$, *for all* $f \in \mathscr{G}$.

The sought formula for the second variation of $\mathscr{J}_r$ appears, as stated below, in Proposition 2.5 of [10].

Proposition 2.6. *Let $x : M^n \to \overline{M}_c^{n+1}$ be a closed spacelike hypersurface of the time-oriented Lorentz manifold $\overline{M}_c^{n+1}$, having constant $(r+1)$-mean curvature H_{r+1}. If $X : M^n \times (-\varepsilon, \varepsilon) \to \overline{M}_c^{n+1}$ is a variation of x, then $J_r''(0)$ is given by*

$$\mathscr{J}_r''(0)(f) = (r+1)\int_M [L_r(f) + \{c\,\mathrm{tr}(P_r) - \mathrm{tr}(A^2P_r)\}f]f\,dM. \tag{2.6}$$

3. A characterization of r-stable spacelike hypersurfaces

As in the previous section, let $\overline{M}^{n+1}$ be a Lorentz manifold. A vector field V on $\overline{M}^{n+1}$ is said to be *conformal* if

$$\mathscr{L}_V\langle\,,\rangle = 2\psi\langle\,,\rangle \tag{3.1}$$

for some function $\psi \in C^\infty(\overline{M})$, where $\mathscr{L}$ stands for the Lie derivative of the Lorentz metric of $\overline{M}$. The function ψ is called the *conformal factor* of V.

Since $\mathscr{L}_V(X) = [V, X]$ for all $X \in \mathscr{X}(\overline{M})$, it follows from the tensorial character of $\mathscr{L}_V$ that $V \in \mathscr{X}(\overline{M})$ is conformal if and only if

$$\langle\overline{\nabla}_X V, Y\rangle + \langle X, \overline{\nabla}_Y V\rangle = 2\psi\langle X, Y\rangle, \tag{3.2}$$

for all $X, Y \in \mathscr{X}(\overline{M})$. In particular, V is a Killing vector field relatively to $\bar{g}$ if and only if $\psi \equiv 0$. Observe that the function ψ can be characterized as

$$\psi = \frac{1}{n+1}\,Div_{\overline{M}}\,V.$$

An interesting particular case of a conformal vector field V is that in which $\overline{\nabla}_X V = \psi X$ for all $X \in \mathscr{X}(\overline{M})$; in this case we say that V is closed, an allusion to the fact that its dual 1-form is closed.

Any Lorentz manifold $\overline{M}^{n+1}$, possessing a globally defined, timelike conformal vector field is said to be a *conformally stationary spacetime*.

In what follows we need a formula first derived in [3]. As stated below, it is the Lorentz version of the one stated and proved in [8].

Lemma 3.1. *Let $\overline{M}_c^{n+1}$ be a conformally stationary Lorentz manifold having constant curvature c and a conformal vector field V. Let also $x : M^n \to \overline{M}_c^{n+1}$ be a spacelike hypersurface of $\overline{M}_c^{n+1}$ and N be the future-pointing Gauss map on M^n. If $\eta = \langle V, N\rangle$, then*

$$\begin{aligned} L_r(\eta) = {} & \{\mathrm{tr}(A^2P_r) - c\,\mathrm{tr}(P_r)\}\eta - b_rH_rN(\psi) \\ & + b_rH_{r+1}\psi + \frac{b_r}{r+1}\langle V, \nabla H_{r+1}\rangle, \end{aligned} \tag{3.3}$$

where $\psi : \overline{M}^{n+1} \to \mathbf{R}$ *is the conformal factor of* V, H_j *is the j-th mean curvature of* M^n *and* ∇H_j *stands for the gradient of* H_j *on* M^n.

In particular, we obtain the following

COROLLARY 3.2. *Let* $\overline{M}_c^{n+1}$ *be a conformally stationary Lorentz manifold having constant curvature* c *and a Killing vector field* W. *Let also* $x : M^n \to \overline{M}_c^{n+1}$ *be a spacelike hypersurface having constant* $(r+1)$*-th mean curvature* H_{r+1}, N *be the future-pointing Gauss map on* M^n *and* $\eta = \langle W, N \rangle$, *then*

$$L_r(\eta) + \{c\,\mathrm{tr}(P_r) - \mathrm{tr}(A^2 P_r)\}\eta = 0.$$

In particular, if $x : M^n \to \overline{M}_c^{n+1}$ *is a closed spacelike hypersurface with constant* $(r+1)$*-th mean curvature such that* $\lambda = c\,\mathrm{tr}(P_r) - \mathrm{tr}(A^2 P_r)$ *is constant, then* λ *is an eigenvalue of the operator* L_r *in* M^n *with eigenfunction* η.

Remark 3.3. Assuming that the conformal vector field V is closed and such that $Div_{\overline{M}}\, V$ does not vanish on M^n, then there exists an elliptic point in M^n (cf. Corollary 5.5 of [2]). Moreover, if M^n has an elliptic point and $H_{r+1} > 0$ on M for some $2 \le r \le n-1$, then L_r is elliptic (cf. Lemma 3.3 of [3]); in the case $r = 1$, the hypothesis $H_2 > 0$ garantees the ellipticity of L_1 without the additional assumption on the existence of an elliptic point (cf. Lemma 3.2 of [3]). In fact, although the manifolds considered in [3] are GRW's, a careful inspection on the proofs of these results of [3] will easily convince the reader that they remain valid in our setting.

We can now state and prove our main result.

THEOREM 3.4. *Let* $\overline{M}_c^{n+1}$ *be a conformally stationary Lorentz manifold with constant curvature* c. *Suppose that* $\overline{M}_c^{n+1}$ *has a closed conformal vector field* V *and a Killing vector field* W. *Let* $x : M^n \to \overline{M}_c^{n+1}$ *be a closed spacelike hypersurface, with positive constant* $(r+1)$*-th mean curvature* H_{r+1} *such that*

$$\lambda = c(n-r)\binom{n}{r}H_r - nH_1\binom{n}{r+1}H_{r+1} + (r+2)\binom{n}{r+2}H_{r+2}$$

is constant. Assume also that $Div_{\overline{M}}\, V$ *does not vanish on* M^n. *Then* x *is r-stable if and only if* λ *is the first eigenvalue of* L_r *on* M^n.

Proof. From Remark 3.3 the operator L_r is elliptic. On the other hand, by using the formulas (2.2), it is easy to show that $\lambda = c\,\mathrm{tr}(P_r) - \mathrm{tr}(A^2 P_r)$. Therefore, since that λ is constant and W is a Killing field on $\overline{M}_c^{n+1}$, Corollary 3.2 guarantees that λ is in the spectrum of L_r.

Let λ_1 be the first eigenvalue of L_r on M^n. If $\lambda = \lambda_1$, then the variational characterization of λ_1 gives

$$\lambda = \min_{f \in \mathcal{G} \setminus \{0\}} \frac{-\int_M f L_r(f)\, dM}{\int_M f^2\, dM}.$$

It follows that, for any $f \in \mathcal{G}$,

$$\begin{aligned} \mathcal{J}_r''(0)(f) &= (r+1) \int_M \{ f L_r(f) + \lambda f^2 \}\, dM \\ &\leq (r+1)(-\lambda + \lambda) \int_M f^2\, dM = 0, \end{aligned}$$

and x is r-stable.

Now suppose that x is r-stable, so that $\mathcal{J}_r''(0)(f) \leq 0$ for all $f \in \mathcal{G}$. Let f be an eigenfunction associated to the first eigenvalue λ_1 of L_r. As was already observed, there exists a volume-preserving variation of M whose variational field is fN. Consequently, by (2.6) we get

$$0 \geq \mathcal{J}_r''(0)(f) = (r+1)(-\lambda_1 + \lambda) \int_M f^2\, dM$$

and therefore $\lambda_1 = \lambda$, since that $\lambda_1 \leq \lambda$. □

4. Applications to GRW spacetimes

A particular class of conformally stationary spacetimes is that of *generalized Robertson-Walker* spacetimes, or *GRW* for short (cf. [2]), namely, warped products $\overline{M}^{n+1} = -I \times_\phi F^n$, where $I \subseteq \mathbf{R}$ is an interval with the metric $-ds^2$, F^n is an n-dimensional Riemannian manifold and $\phi : I \to \mathbf{R}$ is positive and smooth. For such a space, let $\pi_I : \overline{M}^{n+1} \to I$ denote the canonical projection onto I. Then the vector field

$$V = (\phi \circ \pi_I) \frac{\partial}{\partial s}$$

is a conformal, timelike and closed, with conformal factor $\psi = \phi'$, where the prime denotes differentiation with respect to s. Moreover (cf. [12]), for $s_0 \in I$, the (spacelike) leaf $M^n_{s_0} = \{s_0\} \times F^n$ is totally umbilical, with umbilicity factor $-\dfrac{\phi'(s_0)}{\phi(s_0)}$ with respect to the future-pointing unit normal vector field N.

If $\overline{M}^{n+1} = -I \times_\phi F^n$ is a GRW and $x : M^n \to \overline{M}^{n+1}$ is a complete spacelike hypersurface of $\overline{M}^{n+1}$, such that $\phi \circ \pi_I$ is limited on M^n, then $\pi_F|_M : M^n \to F^n$ is necessarily a covering map (cf. [2]). In particular, if M^n is closed then F^n is automatically closed.

Also, recall (cf. [13]) that a GRW as above has constant sectional curvature c if and only if F has constant sectional curvature k and the warping function ϕ satisfies the ODE

$$\frac{\phi''}{\phi} = c = \frac{(\phi')^2 + k}{\phi^2}.$$

In this setting, from Theorem 3.4 we obtain the following

Corollary 4.1. *Let $x : M^n \to -I \times_\phi F^n$ be a closed spacelike hypersurface with constant $(r+1)$-th mean curvature $H_{r+1} > 0$. Suppose also that $-I \times_\phi F^n$ is of constant curvature c, has a Killing vector field and ϕ' does not vanish on M^n. If*

$$\lambda = c(n-r)\binom{n}{r}H_r - nH\binom{n}{r+1}H_{r+1} + (r+2)\binom{n}{r+2}H_{r+2}$$

is constant, then x is r-stable if and only if λ is the first eigenvalue of L_r on M^n.

A particular example of GRW spacetime is de Sitter space. More precisely, let $\mathbf{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space ($n \geq 2$), that is, the real vector space $\mathbf{R}^{n+2}$, endowed with the Lorentz metric

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all $v, w \in \mathbf{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbf{S}_1^{n+1}$ as the following hyperquadric of $\mathbf{L}^{n+2}$

$$\mathbf{S}_1^{n+1} = \{p \in \mathbf{L}^{n+2} : \langle p, p \rangle = 1\}.$$

From the above definition it is easy to show that the metric induced from $\langle\, , \rangle$ turns $\mathbf{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature 1.

Choose a unit timelike vector $a \in \mathbf{L}^{n+2}$, then $V(p) = a - \langle p, a \rangle p$, $p \in \mathbf{S}_1^{n+1}$ is a conformal and closed timelike vector field. It foliates the de Sitter space by means of umbilical round spheres $M_\tau = \{p \in \mathbf{S}_1^{n+1} : \langle p, a \rangle = \tau\}$, $\tau \in \mathbf{R}$. The level set given by $\{p \in \mathbf{S}_1^{n+1} : \langle p, a \rangle = 0\}$ defines a round sphere of radius one which is a totally geodesic hypersurface in $\mathbf{S}_1^{n+1}$. We will refer to that sphere as the equator of $\mathbf{S}_1^{n+1}$ determined by a. This equator divides the de Sitter space into two connected components, the chronological future which is given by

$$\{p \in \mathbf{S}_1^{n+1} : (a, p) < 0\},$$

and the chronological past, given by

$$\{p \in \mathbf{S}_1^{n+1} : (a, p) > 0\}.$$

In the context of warped products, the de Sitter space can be thought of as the following GRW

$$\mathbf{S}_1^{n+1} = -\mathbf{R} \times_{\cosh s} \mathbf{S}^n,$$

where $\mathbf{S}^n$ means Riemannian unit sphere. We observe that there is a lot of possible choices for the unit timelike vector $a \in \mathbf{L}^{n+2}$ and, hence, a lot of ways to describe $\mathbf{S}_1^{n+1}$ as such a GRW (cf. [12], Section 4). We notice that in this model, the equator of $\mathbf{S}_1^{n+1}$ is the slice $\{0\} \times \mathbf{S}^n$ and, consequently, $\phi'(s) = \sinh s$ vanishes only on this slice. Finally, the vector field

$$V = \phi'(s)\frac{\partial}{\partial s} = (\sinh s)\frac{\partial}{\partial s}$$

is conformal, timelike and closed in $\mathbf{S}_1^{n+1}$.

In order to rewrite Theorem 3.4 for the case of closed spacelike hypersurfaces immersed in de Sitter space, we recall some facts.

(a) Killing vector fields in de Sitter space $\mathbf{S}_1^{n+1}$ can be constructed by fixing two vectors u and v in the Lorentz-Minkowski space $\mathbf{L}^{n+2}$ and a non-zero constant $k \in \mathbf{R}$, and considering the vector field $W = k\{\langle u, \cdot\rangle v - \langle v, \cdot\rangle u\}$. Geometrically, $W(x)$ determines an orthogonal direction to the position vector x on the subspace spanned by u and v (cf. Example 1 of [11]).

(b) Let $x : M^n \to \mathbf{S}_1^{n+1}$ be a closed spacelike hypersurface with positive constant $(r+1)$-th mean curvature. Assuming that M^n is contained in the chronological future (or past) of the equator of $\mathbf{S}_1^{n+1}$ then $Div_{\overline{M}}\, V$ does not vanish on M^n. Also, there exists an elliptic point in M^n (cf. Theorem 7 of [1]) and, if $H_{r+1} > 0$ on M for some $2 \le r \le n-1$, then, for all $1 \le j \le r$, the operator L_j is elliptic (cf. Lemma 3.3 of [3]). In the case of L_1, it is sufficient to require that $R < c$ (cf. Lemma 3.2 of [3]).

We can now state the following corollary of Theorem 3.4.

Corollary 4.2. *Let $x : M^n \to \mathbf{S}_1^{n+1}$ be a closed spacelike hypersurface, contained in the chronological future* (*or past*) *of an equator of $\mathbf{S}_1^{n+1}$, with positive constant $(r+1)$-th mean curvature such that*

$$\lambda = (n-r)\binom{n}{r}H_r - nH_1\binom{n}{r+1}H_{r+1} + (r+2)\binom{n}{r+2}H_{r+2}$$

is constant. Then x is r-stable if and only if λ is the first eigenvalue of L_r on M^n.

Remark 4.3. We observe that the round spheres of $\mathbf{S}_1^{n+1}$ are r-stable (cf. [9], Proposition 2).

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References

[1] J. A. Aledo, L. J. Alías and A. Romero, Integral formulas for compact space-like hypersurfaces in de Sitter space: Applications to the case of constant higher order mean curvature, J. of Geom. and Physics **31** (1999), 195–208.

[2] L. J. Alías, A. Brasil Jr. and A. G. Colares, Integral formulae for spacelike hypersurfaces in conformally stationary spacetimes and applications, Proc. Edinburgh Math. Soc. **46** (2003), 465–488.

[3] L. Alias and A. G. Colares, Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker spacetimes, Math. Proc. of the Cambridge Phil. Soc. **143** (2007), 703–729.

[4] J. L. M. Barbosa and M. do Carmo, Stability of hypersurfaces with constant mean curvature, Math. Z. **185** (1984), 339–353.

[5] J. L. M. Barbosa, M. do Carmo and J. Eschenburg, Stability of hypersurfaces of constant mean curvature in Riemannian manifolds, Math. Z. **197** (1988), 123–138.

[6] J. L. M. Barbosa and A. G. Colares, Stability of hypersurfaces with constant r-mean curvature, Ann. Global Anal. Geom. **15** (1997), 277–297.

[7] J. L. M. Barbosa and V. Oliker, Spacelike hypersurfaces with constant mean curvature in Lorentz spaces, Matem. Contemporânea **4** (1993), 27–44.

[8] A. Barros and P. Sousa, Compact graphs over a sphere of constant second order mean curvature, Proc. Amer. Math. Soc. **137** (2009), 3105–3114.

[9] A. Brasil Jr. and A. G. Colares, Stability of spacelike hypersurfaces with constant r-mean curvature in de Sitter space, Proceedings of the XII Fall Workshop on Geometry and Physics, Publ. R. Soc. Mat. Esp. **7**, R. Soc. Mat. Esp., Madrid, 2004, 139–145.

[10] F. Camargo, A. Caminha, M. da Silva and H. de Lima, On the r-stability of spacelike hypersurfaces, J. of Geom. and Physics **60** (2010), 1402–1410.

[11] H. F. de Lima, Spacelike hypersurfaces with constant higher order mean curvature in de Sitter space, J. of Geom. and Physics **57** (2007), 967–975.

[12] S. Montiel, Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes, Math. Ann. **314** (1999), 529–553.

[13] B. O'Neill, Semi-Riemannian geometry, with applications to relativity, New York, Academic Press, 1983.

[14] Y. Xin, Minimal submanifolds and related topics, World scientific publishing co., Singapore, 2003.

[15] L. Ximin and D. Junlei, Stable space-like hypersurfaces in the de Sitter Space, Arch. Mathematicum (BRNO) **40** (2004), 111–117.

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