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WRONSKIAN MATRICES AND WEIERSTRASS GAP SET FOR A PAIR OF POINTS ON A COMPACT RIEMANN SURFACE

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1. Introduction

Let X be a compact Riemann surface and $P_1, \dots, P_n$ be distinct points on it. Then the Weierstrass semigroup for $P_1, \dots, P_n$ is defined by

$$H(P_1, \dots, P_n) := \left\{ (m_1, \dots, m_n) \in (\mathbf{N}_0)^n \;\middle|\; \begin{array}{l} \text{there exists a meromorphic} \\ \text{function } f \text{ on } X \text{ such that} \\ \mathrm{div}_\infty(f) = m_1 P_1 + \cdots + m_n P_n \end{array} \right\}.$$

Here $\mathbf{N}_0$ denotes the set of non-negative integers, and $\mathrm{div}_\infty(f)$ denotes the polar divisor of the meromorphic function f. It is known that $H(P_1, \dots, P_n)$ becomes a subgroup of the additive semigroup $(\mathbf{N}_0)^n = \mathbf{N}_0 \times \cdots \times \mathbf{N}_0$ (n-times product). The complement of $H(P_1, \dots, P_n)$ in $(\mathbf{N}_0)^n$ is called the Weierstrass gap set (gap set for short) for $P_1, \dots, P_n$, and it is denoted by $G(P_1, \dots, P_n)$:

$$G(P_1, \dots, P_n) := (\mathbf{N}_0)^n \backslash H(P_1, \dots, P_n).$$

These sets are related closely to the concept of base points. In fact, an n-tuple of nonnegative integers $(m_1, \dots, m_n)$ belongs to $H(P_1, \dots, P_n)$ if and only if the effective divisor $m_1 P_1 + \cdots + m_n P_n$ is base-point-free. Because any divisor is base-point-free provided its degree is grater than $2g$ (g denotes the genus of X), the Weierstrass gap set is of finite. We are mainly interested in the cardinality of the Weierstrass gap sets.

The case of $n = 1$ is the classical Weierstrass point theory. It is well known that the gap set for any point always consists of g integers, which are called the gap numbers at the point. The case of $n = 2$ has been studied first by Kim [K], in which he proved that the cardinality of the gap set for a pair of points is bounded from below by $(g^2 + 3g)/2$ and from above by $(3g^2 + g)/2$. The upper bound is attained when and only when both points are hyperelliptic Weierstrass points. The cases of $n = 3, 4$ were studied by Isii [I], in which he obtained similar results to Kim's results (partially for $n = 4$). The cases of $n \geqq 5$ remain to be unknown at present. Balico and Kim [BK] have proposed a conjecture on the range of the cardinality of the gap set for n points. Results by Kim and Ishii support their conjecture.

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When $n = 1$, the gap numbers at a point P can be interpreted as orders of holomorphic 1-forms on X at P. In fact an integer α belongs to $G(P)$ if and only if there exists a holomorphic 1-form ω such that $\mathrm{ord}_P(\omega) = \alpha - 1$. In this article, we study the gap set for a pair of points from this point of view. In section 2, we characterize the Kim's bijection (denoted by μ in this article) between the gap sets for a single point in terms of the orders of holomorphic 1-form (Theorem 2.6). This characterization gives a good basis for the space of holomorphic 1-forms on X with respect to a pair of distinct points. Such a basis is useful to controll the Wronskian matrix associated to an effective divisor supported by the pair of distinct points. Because the dimension of the space of meromorphic functions associated to an effective divisor is computed by the rank of a certain Wronskian matrix (Theorem 2.1), such a basis turns out to be usefull in the study on the gap set for a pair of points. In section 3, we use the method of the Wronskian matrices developed in section 2 in order to obtain the expressions of the cardinality of the Weierstrass gap sets for a pair of distinct points due to Kim and Homma.

Section 4 and 5 will be devoted to investigate a pair of points for which the cardinality of the Weierstrass gap set attains the lowest bound $g(g+3)/2$. In the classical Weierstrass point theory (namely the case $n = 1$), the Weierstrass points are characterized as zeros of a holomorphic section of a certain holomorphic line bundle over X. To be exact, the holomorphic line bundle is the $g(g+1)/2$-times tensor product of the canonical line bundle of X, and the holomorphic section is the Wronskian determinant associated to a basis of the space of holomorphic 1-forms. After making preparations on triangulations of the Wronskian matrices, we construct a family of holomorphic sections of a certain holomorphic line bundle over $X \times X$. Then we show that a necessary and sufficient condition for the Weierstrass gap set for a pair of points to have the lowest cardinality is that at least one of the holomorphic sections in the family does not vanish at the pair of points (Theorem 5.2). As a consequence, we find that the cardinality of the Weierstrass gap set for a pair of points attains the lowest bound on an open and dense subset in $X \times X \backslash \Delta(X)$. This formalism seems to be an analogy to the classical Weierstrass point theory mentioned above.

In the final section 6, we calculate the holomorphic section defined in section 5 in the case where X is a hyperellptic Riemann surface.

2. Wronskian matrices and Weierstrass gap sets

Let X be a compact Riemann surface of genus g and ω be a holomorphic 1-form on X. Taking a local coordinate function z on an open subset U in X, we write $\omega = f\,dz$, where f is a holomorphic function defined on U. Then, for a non-negative integer v, we set

$$\omega^{(v)}(P) = \frac{d^v f}{dz^v}(P) \quad (P \in U).$$

Although this value depends on the choice of a local coordinate function, we use the notation whenever any confusion may not occur.

For holomorphic 1-forms $\omega_1, \dots, \omega_\ell$ (not necessarily linearly independent) on X, a point P in X and a non-negative integer v, we define a $\ell \times v$ matrix by

$$W_{vP}[\omega_1, \dots, \omega_\ell] = \begin{pmatrix} \omega_1(P) & \omega_1'(P) & \cdots & \omega_1^{(v-1)}(P) \\ \omega_2(P) & \omega_2'(P) & \cdots & \omega_2^{(v-1)}(P) \\ \cdots & \cdots & \cdots & \cdots \\ \omega_\ell(P) & \omega_\ell'(P) & \cdots & \omega_\ell^{(v-1)}(P) \end{pmatrix}.$$

For an effective divisor $D = v_1P_1 + \cdots + v_nP_n$, where $P_1, \dots, P_n$ are distinct points in X, we put further

$$W_D[\omega_1, \dots, \omega_\ell] = (W_{v_1P_1}[\omega_1, \dots, \omega_\ell], \dots, W_{v_nP_n}[\omega_1, \dots, \omega_\ell]),$$

which is a matrix with ℓ rows and $\deg D = v_1 + \cdots + v_n$ columns. We call the matrix the *Wronskian matrix* associated to an effective divisor D and holomorphic 1-forms $\omega_1, \dots, \omega_\ell$ on X. When $v_i = 0$ for some i, we understand for the i-th matrix $W_{v_iP_i}[\omega_1, \dots, \omega_\ell]$ to be ommited.

Let $\Omega(X)$ denote the space of holomorphic 1-forms on X and $\omega_1, \dots, \omega_g$ be a basis of $\Omega(X)$. If A is an invertible $g \times g$ matrix, we obtain a new basis $(\omega_1, \dots, \omega_g)A$ of $\Omega(X)$ from $\omega_1, \dots, \omega_g$. The Wronskian matrices associated to these bases are related as

$$(1) \qquad W_D[(\omega_1, \dots, \omega_g)A] = {}^tA \cdot W_D[\omega_1, \dots, \omega_g].$$

The Local coordinate functions chosen to define the Wronskian matrices are fixed in this formula.

The divisor of a meromorphic function f on X is denoted by $\mathrm{div}(f)$. For a divisor D, we denote, as usual, by $h^0(D)$ the dimension of the space consisting of all meromorphic functions f on X which satisfy $\mathrm{div}(f) \geqq -D$. The following formula which express $h^0(D)$ in terms of a Wronskian matrix is one of the consequences of the Riemann-Roch Theorem. This plays essential roles in our study on the Weierstrass gap sets.

THEOREM 2.1. *Let D be an effective divisor on X and $\omega_1, \dots, \omega_g$ be a basis of $\Omega(X)$. Then we have*

$$h^0(D) = \deg D + 1 - \mathrm{rank}\, W_D[\omega_1, \dots, \omega_g].$$

This formula can be found in the Gunning's text [G, Lemma 17 (p. 118)].

By definition, for a divisor D, a point P is a base point of D if and only if an equality $h^0(D) = h^0(D - P)$ holds. Thus Theorem 2.1 implies

COROLLARY 2.2. *Let D be an effective divisor on X and $\omega_1, \dots, \omega_g$ be a basis of $\Omega(X)$. Then for a point P contained in the support of D, the following are mutually equivalent.*

(1) *P is a base point of D.*
(2) rank $W_D[\omega_1, \dots, \omega_g] = \text{rank } W_{D-P}[\omega_1, \dots, \omega_g] + 1$.

Note that any point not contained in the support of D is not a base point of it.

Now let P and Q be distinct points in X. As was mentioned in the introduction, the Weierstrass semigroup $H(P, Q)$ is related to the concenpt of base points of a divisor and some Wronskian matrices as well.

Corollary 2.3. *Let $\omega_1, \dots, \omega_g$ be a basis of $\Omega(X)$. Then for a pair $(m, n) \in \mathbf{N}_0 \times \mathbf{N}_0$, the following are mutually eqivalent.*
(1) *(m, n) belongs to $H(P, Q)$.*
(2) *The effective divisor $mP + nQ$ is base-point-free.*
(3) *$h^0(mP + nQ) = h^0((m-1)P + nQ) = h^0(mP + (n-1)Q)$.*
(4)

$$\begin{aligned}\text{rank } W_{mP+nQ}[\omega_1, \dots, \omega_g] &= \text{rank } W_{(m-1)P+nQ}[\omega_1, \dots, \omega_g] \\ &= \text{rank } W_{mP+(n-1)Q}[\omega_1, \dots, \omega_g].\end{aligned}$$

In [K], Kim have defined a map from $G(P)$ to $G(Q)$, which plays a central role in his study on $G(P, Q)$. For a gap number α at P, we define after Kim (β_α in his notation)

$$\mu(\alpha) := \min\{\beta \geqq 1 \mid (\alpha, \beta) \in H(P, Q)\}.$$

Then $\mu(\alpha)$ is a gap number at Q, and the correspondence $\alpha \mapsto \mu(\alpha)$ defines a bijection between $G(P)$ and $G(Q)$. We want to investigate how the map μ relates to the orders of holomorphic 1-forms on X. To begin with, we show the

Proposition 2.4. *For a gap number α at P and a holomorphic 1-form ω on X which satisfy $\mathrm{ord}_P(\omega) = \alpha - 1$, we have the following inequality*

$$\mathrm{ord}_Q(\omega) \leqq \mu(\alpha) - 1.$$

Proof. Let $\omega_1, \dots, \omega_g$ be a basis of $\Omega(X)$ such that $\omega_1 = \omega$. Suppose we have inequalities $\mathrm{ord}_Q(\omega) \geqq \beta \geqq 1$, then the Wronskian matrices are

$$\begin{aligned}W_{\alpha P+\beta Q}[\omega_1, \dots, \omega_g] &= \left(\begin{array}{c|c} 0 \ \cdots \ 0 \ \omega_1^{(\alpha-1)}(P) & \omega_1(Q) \ \cdots \ \omega_1^{(\beta-1)}(Q) \\ W_{\alpha P}[\omega_2, \dots, \omega_g] & W_{\beta Q}[\omega_2, \dots, \omega_g] \end{array}\right) \\ &= \left(\begin{array}{c|c} 0 \ \cdots \ 0 \ \omega_1^{(\alpha-1)}(P) & 0 \ \cdots\cdots \ 0 \\ W_{\alpha P}[\omega_2, \dots, \omega_g] & W_{\beta Q}[\omega_2, \dots, \omega_g] \end{array}\right),\end{aligned}$$

$$\begin{aligned}W_{(\alpha-1)P+\beta Q}[\omega_1, \dots, \omega_g] &= \left(\begin{array}{c|c} 0 \ \cdots\cdots \ 0 & \omega_1(Q) \ \cdots \ \omega_1^{(\beta-1)}(Q) \\ W_{(\alpha-1)P}[\omega_2, \dots, \omega_g] & W_{\beta Q}[\omega_2, \dots, \omega_g] \end{array}\right) \\ &= \left(\begin{array}{c|c} 0 \ \cdots\cdots \ 0 & 0 \ \cdots\cdots \ 0 \\ W_{(\alpha-1)P}[\omega_2, \dots, \omega_g] & W_{\beta Q}[\omega_2, \dots, \omega_g] \end{array}\right).\end{aligned}$$

Because $\omega_1^{(\alpha-1)}(P) \neq 0$,

$$\begin{aligned}
&\text{rank } W_{\alpha P+\beta Q}[\omega_1, \dots, \omega_g] \\
&\quad = \text{rank}\left(\begin{array}{ccc|c} 0 \quad \cdots\cdots\cdots \quad 0 & \omega_1^{(\alpha-1)}(P) & 0 \quad \cdots\cdots \quad 0 \\ W_{(\alpha-1)P}[\omega_2, \dots, \omega_g] & O & W_{\beta Q}[\omega_2, \dots, \omega_g] \end{array}\right) \\
&\quad = \text{rank } W_{(\alpha-1)P+\beta Q}[\omega_1, \dots, \omega_g] + 1,
\end{aligned}$$

and hence Corollary 2.3 implies that (α, β) does not belong to $H(P, Q)$. Therefore if (α, β) belongs to $H(P, Q)$, then the inequality $\mathrm{ord}_Q(\omega) < \beta$ necessarily holds. Especially we obtain the inequality $\mathrm{ord}_Q(\omega) \leqq \mu(\alpha) - 1$. $\diamondsuit$

Corollary 2.5. *Let P and Q be distinct points in X and the sets of gap numbers at those points be $G(P) = \{\alpha_1, \dots, \alpha_g\}$, $G(Q) = \{\beta_1, \dots, \beta_g\}$. Suppose there exists a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ such that $\mathrm{ord}_P(\omega_i) = \alpha_i - 1$ and $\mathrm{ord}_Q(\omega_i) = \beta_i - 1$ for each $i = 1, \dots, g$. Then $\mu(\alpha_i) = \beta_i$ for each $i = 1, \dots, g$.*

Proof. The assumption and Proposition 2.4 show that $\{\beta_1, \dots, \beta_g\} = \{\mu(\alpha_1), \dots, (\alpha_g)\}$ and $\beta_i \leqq \mu(\alpha_i)$ $(i = 1, \dots, g)$. Therefore we obtain $\mu(\alpha_i) = \beta_i$ for each $i = 1, \dots, g$. $\diamondsuit$

It is obvious that almost all ω do not attain the equality in Proposition 2.4. Next we consider when this is the case.

Theorem 2.6. *For each gap number α at P, there exists a holomorphic 1-form ω whose orders at P and Q are $\alpha - 1$ and $\mu(\alpha) - 1$ respectively.*

Before proceeding to prove the theorem, we note the following lemma which is easy to see.

Lemma 2.7. *For holomorphic 1-forms $\omega_1, \dots, \omega_\ell$ on X and any point P, the following are equivalent.*

(1) *$\omega_1, \dots, \omega_\ell$ are linearly independent over* $\mathbf{C}$.

(2) rank $W_{\nu P}[\omega_1, \dots, \omega_\ell] = \ell$ *for some* $\nu \geqq 1$.

Proof of Theorem 2.6. Let $\alpha_1 = 1 < \alpha_2 < \cdots < \alpha_g (< 2g)$ be the gap numbers at P with $\alpha = \alpha_i$. Then we can take a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ whose orders are given by $\mathrm{ord}_P(\omega_j) = \alpha_j - 1$ $(j = 1, \dots, g)$. In what follows, we shall construct a holomorphic 1-form ω and a positive integer β that satisfy the following:

$$\text{(2)} \qquad \mathrm{ord}_P(\omega) = \alpha - 1, \quad \mathrm{ord}_Q(\omega) = \beta - 1 \quad \text{and} \quad (\alpha, \beta) \in H(P, Q).$$

Then we find $\beta = \mu(\alpha)$ by the definition of $\mu(\alpha)$ and Proposition 2.4, and the assertion of the theorem will be obtained.

Now because $\omega_1, \dots, \omega_g$ are linearly independent, Lemma 2.7 implies that

$$\operatorname{rank} W_{\nu Q}[\omega_i, \omega_{i+1}, \dots, \omega_g] = g - i + 1 > \operatorname{rank} W_{\nu Q}[\omega_{i+1}, \dots, \omega_g] = g - i,$$

for sufficiently large ν, so that the following β exists:

$$\beta := \min\{\nu \geqq 1 \mid \operatorname{rank} W_{\nu Q}[\omega_i, \omega_{i+1}, \dots, \omega_g] > \operatorname{rank} W_{\nu Q}[\omega_{i+1}, \dots, \omega_g]\}.$$

Note that the property $\operatorname{ord}_P(\omega_j) = \alpha_j - 1$ $(j = 1, \dots, g)$ implies the following equalities.

$$\begin{aligned} \operatorname{rank} W_{\alpha_i P + \beta Q}[\omega_1, \dots, \omega_g] &= \operatorname{rank} W_{\beta Q}[\omega_{i+1}, \dots, \omega_g] + i, \\ \operatorname{rank} W_{(\alpha_i - 1)P + \beta Q}[\omega_1, \dots, \omega_g] &= \operatorname{rank} W_{\beta Q}[\omega_i, \omega_{i+1}, \dots, \omega_g] + i - 1, \\ \operatorname{rank} W_{\alpha_i P + (\beta - 1)Q}[\omega_1, \dots, \omega_g] &= \operatorname{rank} W_{(\beta-1)Q}[\omega_{i+1}, \dots, \omega_g] + i. \end{aligned} \tag{3}$$

We first consider the case where $\beta = 1$. In this case, because of the definition of β and

$$W_Q[\omega_i, \omega_{i+1}, \dots, \omega_g] = \begin{pmatrix} \omega_i(Q) \\ W_Q[\omega_{i+1}, \dots, \omega_g] \end{pmatrix},$$

we have $\omega_i(Q) \neq 0$ and $W_{\beta Q}[\omega_{i+1}, \dots, \omega_g] = O$. Therefore $\operatorname{ord}_Q(\omega_i) = 0 = \beta - 1$ and (3) implies

$$\begin{aligned} \operatorname{rank} W_{\alpha_i P + Q}[\omega_1, \dots, \omega_g] &= \operatorname{rank} W_{(\alpha_i - 1)P + \beta Q}[\omega_1, \dots, \omega_g] \\ &= \operatorname{rank} W_{\alpha_i P}[\omega_1, \dots, \omega_g] = i. \end{aligned}$$

Thus $(\alpha, \beta) = (\alpha_i, 1)$ belongs to $H(P, Q)$ by Corollary 2.3, and hence ω_i itself and $\beta = 1$ satisfy (2).

We next consider the case where $\beta > 1$. In this case, by the definition of β, we have the following equalities

$$\operatorname{rank} W_{(\beta-1)Q}[\omega_i, \omega_{i+1}, \dots, \omega_g] = \operatorname{rank} W_{(\beta-1)Q}[\omega_{i+1}, \dots, \omega_g], \tag{4}$$

$$\operatorname{rank} W_{\beta Q}[\omega_i, \omega_{i+1}, \dots, \omega_g] = \operatorname{rank} W_{\beta Q}[\omega_{i+1}, \dots, \omega_g] + 1. \tag{5}$$

Because

$$W_{(\beta-1)Q}[\omega_i, \omega_{i+1}, \dots, \omega_g] = \begin{pmatrix} W_{(\beta-1)Q}[\omega_i] \\ W_{(\beta-1)Q}[\omega_{i+1}, \dots, \omega_g] \end{pmatrix} = \begin{pmatrix} W_{(\beta-1)Q}[\omega_i] \\ W_{(\beta-1)Q}[\omega_{i+1}] \\ \cdots\cdots\cdots\cdots \\ W_{(\beta-1)Q}[\omega_g] \end{pmatrix},$$

(4) implies that the $(\beta - 1)$-dimensional row vector $W_{(\beta-1)Q}[\omega_i]$ can be written as a linear combination of the row vectors $W_{(\beta-1)Q}[\omega_{i+1}], \dots, W_{(\beta-1)Q}[\omega_g]$. Thus we can write

$$W_{(\beta-1)Q}[\omega_i] = c_1 W_{(\beta-1)Q}[\omega_{i+1}] + \cdots + c_{g-i} W_{(\beta-1)Q}[\omega_g], \tag{6}$$

where $c_1, \dots, c_{g-i}$ are complex constants. Then define a holomorphic 1-form ω by

$$\omega := \omega_i - c_1\omega_{i+1} - \cdots - c_{g-i}\omega_g.$$

We shall show that these ω and β constructed as above satisfy (2).

First since $\mathrm{ord}_P(\omega_i) = \alpha_i - 1$, $\mathrm{ord}_P(\omega_{i+1}) = \alpha_{i+1} - 1, \dots, \mathrm{ord}_P(\omega_g) = \alpha_g - 1$ are strictly increasing series, $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(\omega_i) = \alpha_i - 1 = \alpha - 1$.

Second we have $\omega^{(\nu)}(Q) = \omega_i^{(\nu)}(Q) - c_1\omega_{i+1}^{(\nu)}(Q) - \cdots - c_{g-i}\omega_g^{(\nu)}(Q) = 0$ for each $\nu = 0, 1, \dots, \beta - 2$ by the definition of ω and (6), and so $\mathrm{ord}_Q(\omega) \geqq \beta - 1$. If we suppose $\omega^{(\beta-1)}(Q) = 0$, then $\omega_i^{(\beta-1)}(Q) = c_1\omega_{i+1}^{(\beta-1)}(Q) - \cdots - c_{g-i}\omega_g^{(\beta-1)}(Q)$, which implies

$$W_{\beta Q}[\omega_i] = c_1 W_{\beta Q}[\omega_{i+1}] + \cdots + c_{g-i} W_{\beta Q}[\omega_g].$$

However this contradicts to (5), and we have $\mathrm{ord}_Q(\omega) = \beta - 1$.

Finally we show that (α, β) belongs to $H(P, Q)$. To this end, it is sufficient to show the equalities

$$\begin{aligned}
&\mathrm{rank}\ W_{\alpha P+\beta Q}[\omega_1, \dots, \omega_{i-1}, \omega, \omega_{i+1}, \dots, \omega_g] \\
&\quad = \mathrm{rank}\ W_{(\alpha-1)P+\beta Q}[\omega_1, \dots, \omega_{i-1}, \omega, \omega_{i+1}, \dots, \omega_g] \\
&\quad = \mathrm{rank}\ W_{\alpha P+(\beta-1)Q}[\omega_1, \dots, \omega_{i-1}, \omega, \omega_{i+1}, \dots, \omega_g],
\end{aligned}$$

because of Corollary 2.3. On the other hand, the Wronskian matrices associated to a new basis $\omega_1, \dots, \omega_{i-1}, \omega, \omega_i, \dots, \omega_g$ of $\Omega(X)$ still satisfy the same equalities as in (3). Hence we have only to show the equalities

$$\begin{aligned}
\mathrm{rank}\ W_{\beta Q}[\omega, \omega_{i+1}, \dots, \omega_g] &= \mathrm{rank}\ W_{\beta Q}[\omega_{i+1}, \dots, \omega_g] + 1 \\
&= \mathrm{rank}\ W_{(\beta-1)Q}[\omega_{i+1}, \dots, \omega_g] + 1.
\end{aligned}$$

Now by the definition of ω, the matrix $W_{\beta Q}[\omega, \omega_i, \dots, \omega_g]$ is obtained by making use of suitable row operations to the matrix $W_{\beta Q}[\omega_i, \omega_{i+1}, \dots, \omega_g]$, so that they have the same rank. Therefore because of the definiton of β,

$$\begin{aligned}
\mathrm{rank}\ W_{\beta Q}[\omega, \omega_{i+1}, \dots, \omega_g] &= \mathrm{rank}\ W_{\beta Q}[\omega_i, \omega_{i+1}, \dots, \omega_g] \\
&= \mathrm{rank}\ W_{\beta Q}[\omega_{i+1}, \dots, \omega_g] + 1.
\end{aligned}$$

On the other hand, because $\mathrm{ord}_Q(\omega) = \beta - 1$ as we have shown above,

$$\begin{aligned}
\mathrm{rank}\ W_{\beta Q}[\omega, \omega_{i+1}, \dots, \omega_g] &= \mathrm{rank}\begin{pmatrix} 0 & \cdots\cdots\cdots & 0 & \omega^{(\beta-1)}(Q) \\ & & & \omega_{i+1}^{(\beta-1)}(Q) \\ & W_{(\beta-1)Q}[\omega_{i+1}, \dots, \omega_g] & & \vdots \\ & & & \omega_g^{(\beta-1)}(Q) \end{pmatrix} \\
&= \mathrm{rank}\ W_{(\beta-1)Q}[\omega_{i+1}, \dots, \omega_g] + 1. \qquad \diamondsuit
\end{aligned}$$

As an immediate consequence of Proposition 2.4 and Theorem 2.6, we have another interpretation for the map $\mu : G(P) \to G(Q)$ in terms of holomorphic 1-forms:

$$\mu(\alpha) = \max\left\{ \beta \geqq 1 \;\middle|\; \begin{array}{c} \text{there exists } \omega \in \Omega(X) \text{ such that} \\ \mathrm{ord}_P(\omega) = \alpha - 1 \text{ and } \mathrm{ord}_Q(\omega) = \beta - 1 \end{array} \right\}.$$

Now let α be a gap number at P and take a holomorphic 1-form ω with $\mathrm{ord}_P(\omega) = \alpha - 1$ and $\mathrm{ord}_Q(\omega) = \mu(\alpha) - 1$. Then because the degree of a canonical divisor is equal to $2g - 2$,

$$(\alpha - 1) + (\mu(\alpha) - 1) = \mathrm{ord}_P(\omega) + \mathrm{ord}_Q(\omega) \leqq 2g - 2.$$

Hence for each gap number α at P, we have an inequality

$$\alpha + \mu(\alpha) \leqq 2g. \tag{7}$$

PROPOSITION 2.8. *The following are equivalent*:
(1) $\alpha + \mu(\alpha) = 2g$ *for each* $\alpha \in G(P)$.
(2) $h^0(\alpha P + \mu(\alpha)Q) = g + 1$ *for each* $\alpha \in G(P)$.
(3) *Both P and Q are hyperelliptic Weierstrass points in X. Especially X must be a hyperelliptic Riemann surface.*

Here a point in X is called a hyperelliptic Weierstrass point provided its least non-gap number is equal to 2.

Proof. (1) $\Leftrightarrow$ (2): A general result on the range of $h^0(D)$ for divisors D shows the equivalence between (1) and (2). See, for instance, the figure at page 331 in [C].

(1) $\Rightarrow$ (3): Taking summation on $\alpha \in G(P)$ in the equality $\alpha + \mu(\alpha) = 2g$, we obtain

$$\mathrm{wt}(P) + \mathrm{wt}(Q) = g^2 - g = 2\frac{g(g-1)}{2}, \tag{8}$$

where $\mathrm{wt}(P) := \sum_{\alpha \in G(P)} \alpha - g(g+1)/2$ denotes the Weierstrass weight at a point P. In general, it is known that the Weierstrass weight at a point P satisfies the inequalities $1 \leqq \mathrm{wt}(P) \leqq g(g-1)/2$ and the equality $\mathrm{wt}(P) = g(g-1)/2$ holds if and only if P is a hyperelliptic Weierstrass point. Thus (8) implies (3).

(3) $\Rightarrow$ (1): In this case, the gap numbers are $G(P) = G(Q) = \{1, 3, \dots, 2g-1\}$. Then by virture of the inequality (7), the map μ must be determined as $\mu(\alpha) = 2g - \alpha$, because μ is a bijection. ◇

Compare the argument in the proof of Theorem 3.2 in [K].

3. The Cardinality of the Weierstrass gap sets

In this section, by means of the Wronskian matrices, we will derive expressions of the cardinality of the Weierstrass gap set for a pair of distinct points due to Kim and Homma. As a result, we give a necessary and sufficient condition for the Weierstrass gap set for a pair of distinct points to have the lowest cardinality $g(g+3)/2$ in terms of the Wronskian matrix.

Let P and Q be distinct points in X and $\alpha_1 = 1 < \alpha_2 < \cdots < \alpha_g$ be gap numbers at P. By virture of Theorem 2.6, we can take a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ with $\mathrm{ord}_P(\omega_i) = \alpha_i - 1$ and $\mathrm{ord}_Q(\omega_i) = \mu(\alpha_i) - 1$ for all $i = 1, \dots, g$. Then the Wronskian matrix associated to an effective divisor $mP + nQ$ and the basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ is of the form such as

$$W_{mP+nQ}[\omega_1, \dots, \omega_g] = (W_{mP}[\omega_1, \dots, \omega_g], W_{nQ}[\omega_1, \dots, \omega_g])$$

$$= \left(\begin{array}{cccccc|cccccc} A_1 & \cdots & \cdots & \cdots & \cdots & \omega_1^{(m-1)}(P) & 0 & \cdots & 0 & B_1 & \cdots & \omega_1^{(n-1)}(Q) \\ 0 & \cdots & 0 & A_2 & \cdots & \omega_2^{(m-1)}(P) & 0 & \cdots 0 & B_2 & \cdots & \cdots & \omega_2^{(n-1)}(Q) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & A_i & \cdots & \omega_i^{(m-1)}(P) & 0 & \cdots & \cdots & \cdots & \cdots & \omega_i^{(n-1)}(Q) \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 & B_{i+1} & & \omega_{i+1}^{(n-1)}(Q) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & B_g & \cdots & \cdots & \cdots & \omega_g^{(n-1)}(Q) \end{array}\right),$$

where we put $A_i = \omega_i^{(\alpha_i - 1)}(P)$ and $B_i = \omega_i^{(\mu(\alpha_i)-1)}(Q)$ $(i = 1, \dots, g)$. For each non-negative integers m and n, we set

$$\sigma_m := \#\{i \geqq 1 \mid \alpha_i \leqq m\} = \max\{i \geqq 1 \mid \alpha_i \leqq m\},$$
$$\tau_{m,n} := \#\{j \geqq 1 \mid j > \sigma_m \text{ and } \mu(\alpha_j) \leqq n\},$$

where $\#S$ denotes the cardinality of the set S. Then the rank of $W_{mP+nQ}[\omega_1, \dots, \omega_g]$ is equal to $\sigma_m + \tau_{m,n}$, and so Theorem 2.1 implies

$$h^0(mP + nQ) = m + n + 1 - \sigma_m - \tau_{m,n}. \tag{9}$$

In addition, if we set $\sigma_{-1} = \tau_{m,-1} = 0$, $\tau_{-1,n} = \tau_{0,n}$, then we obtain the following equalities:

$$h^0(mP + nQ) - h^0((m-1)P + nQ) = 1 + \sigma_{m-1} - \sigma_m + \tau_{m-1,n} - \tau_{m,n},$$
$$h^0(mP + nQ) - h^0(mP + (n-1)Q) = 1 + \tau_{m,n-1} - \tau_{m,n},$$
$$\sigma_{m-1} - \sigma_m = \begin{cases} -1 & (m \in G(P)), \\ 0 & (m \notin G(P)), \end{cases}$$
$$\tau_{m-1,n} - \tau_{m,n} = \begin{cases} 1 & (m \in G(P) \text{ and } \mu(\alpha_{\sigma_m}) \leqq n), \\ 0 & (m \in G(P) \text{ and } \mu(\alpha_{\sigma_m}) > n), \end{cases}$$
$$\tau_{m,n-1} - \tau_{m,n} = \begin{cases} -1 & (\mu(\alpha_j) = n \text{ for some } j > \sigma_m), \\ 0 & (\mu(\alpha_j) \neq n \text{ for all } j > \sigma_m). \end{cases}$$

Because of these equalities and Corollary 2.3, we find that each $(m,n) \in \mathbf{N}_0 \times \mathbf{N}_0$ belongs to $G(P,Q)$ if and only if $m \in G(P)$ and $\mu(\alpha_{\sigma_m}) > n$, or $\mu(\alpha_j) = n$ for some $j > \sigma_m$. Therefore setting

$$\begin{aligned} G' &= \{(m,n) \in \mathbf{N}_0 \times \mathbf{N}_0 \mid m \in G(P) \text{ and } \mu(\alpha_{\sigma_m}) > n\}, \\ G'' &= \{(m,n) \in \mathbf{N}_0 \times \mathbf{N}_0 \mid \mu(\alpha_j) = n \text{ for some } j > \sigma_m\}, \\ G''_0 &= G'' \cap (G(P) \times \mathbf{N}_0), \\ G''_1 &= G'' \cap (H(P) \times \mathbf{N}_0), \end{aligned}$$

the gap set $G(P,Q)$ is decomposed as $G(P,Q) = G' \cup G'' = G' \cup G''_0 \cup G''_1$. The cardinality of these components are given by

$$\begin{aligned} \#G' &= \sum_{i=1}^{g} \mu(\alpha_i) = \sum_{\beta \in G(Q)} \beta = \mathrm{wt}(Q) + \frac{g(g+1)}{2}, \\ \#G''_0 &= \sum_{i=1}^{g} (g-i) = \frac{g(g-1)}{2}, \\ \#G''_1 &= \sum_{\substack{m \in H(P) \\ 0 \leqq m \leqq 2g-2}} (g - \sigma_m) = \mathrm{wt}(P) + g. \end{aligned}$$

On the other hand, $G' \cap G''_0 = \bigcup_{i=1}^{g} \{(\alpha_i, \mu(\alpha_j)) \mid j > i \text{ and } \mu(\alpha_j) < \mu(\alpha_i)\}$, and so we set

$$t(\mu) := \#\{(\alpha, \alpha') \in G(P) \times G(P) \mid \alpha > \alpha' \text{ and } \mu(\alpha) < \mu(\alpha')\},$$

then $t(\mu) = \#G' \cap G''_0$. Note that $t(\mu)$ is nothing but the Homma's "r" defined in [H]. Summarizing the argument above, we obtain an expression of the cardinality of the gap set $G(P,Q)$:

$$\begin{aligned} \#G(P,Q) &= \#G' + \#G''_0 + \#G''_1 - \#G' \cap G''_0 \\ &= \mathrm{wt}(P) + \mathrm{wt}(Q) - t(\mu) + g(g+1). \end{aligned} \tag{10}$$

This is the Homma's expression of $\#G(P,Q)$ appeared in [H, Theorem 1 (p. 340)].

To obtain the other expression of $\#G(P,Q)$, we note

$$\tau_{\alpha_i, \mu(\alpha_i)} = \#\{j \geqq 1 \mid j > \sigma_{\alpha_i} = i \text{ and } \mu(\alpha_j) < \mu(\alpha_i)\},$$

and hence

$$t(\mu) = \sum_{i=1}^{g} \tau_{\alpha_i, \mu(\alpha_i)} = \sum_{\alpha \in G(P)} \tau_{\alpha, \mu(\alpha)}.$$

Putting $m = \alpha$, $n = \mu(\alpha)$ in the equality (9) and taking summation on $\alpha \in G(P)$, we obtain

$$\sum_{\alpha \in G(P)} h^0(\alpha P + \mu(\alpha)Q) = \mathrm{wt}(P) + \mathrm{wt}(Q) - t(\mu) + \frac{g(g+3)}{2}.$$

Hence (10) implies the another expression of the cardinality of the gap set $G(P,Q)$:

$$\#G(P,Q) = \sum_{\alpha \in G(P)} h^0(\alpha P + \mu(\alpha)Q) + \frac{g(g-1)}{2}.$$

This is the Kim's expression of $\#G(P,Q)$ appeared in [K, Theorem 3.1 (p. 79)].

In consequence of (10), Homma have mentioned the equivalence between the first two in the lemma below.

Lemma 3.1. *Let P and Q be distinct points in X. Then the following are equivalent.*

(1) $\#G(P,Q) = \dfrac{g(g+3)}{2}$ (*the lowest cardinality*)

(2) *Both P and Q are non-Weierstrass points and the map $\mu : G(P) \to G(Q)$ is given by $\mu(i) = g+1-i$ $(i = 1, \dots, g)$.*

(3) *There exists a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ associated to which the Wronskian matrix $W_{gP+gQ}[\omega_1, \dots, \omega_g]$ turns to be of the form*

$$W_{gP+gQ}[\omega_1, \dots, \omega_g] = \left(\begin{array}{cccc|cccc} A_1 & * & * & * & 0 & \cdots & 0 & B_1 \\ 0 & A_2 & * & * & \vdots & \iddots & B_2 & * \\ \vdots & \ddots & \ddots & * & 0 & \iddots & * & * \\ 0 & \cdots & 0 & A_g & B_g & * & * & * \end{array}\right), \tag{11}$$

where $A_1, \dots, A_g$ and $B_1, \dots, B_g$ are non-zero complex numbers.

Proof. We give a proof only for the equivalence between (2) and (3).

If we suppose (2), both P and Q have the same gap numbers $1, 2, \dots, g$. By virture of Theorem 2.6, we can take a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ whose orders are $\mathrm{ord}_P(\omega_i) = i-1$ and $\mathrm{ord}_Q(\omega_1) = \mu(i) - 1 = g - i$ $(i = 1, \dots, g)$. Then the Wronskian matrix $W_{gP+gQ}[\omega_1, \dots, \omega_g]$ turns to be of the form (11).

Conversely, if the Wronskian matrix $W_{gP+gQ}[\omega_1, \dots, \omega_g]$ is of such a form (11), the gap numbers are $G(P) = G(Q) = \{1, 2, \dots, g\}$, that is, both P and Q are non-Weierstrass points and the orders of ω_i are $\mathrm{ord}_P(\omega_i) = i-1$ and $\mathrm{ord}_Q(\omega_i) = g-i$. Then Corollary 2.5 shows $\mu(i) = g+1-i$ $(i = 1, \dots, g)$. ◇

4. Triangulations of the Wronskian matrices

Throughout the present section, we let g be a positive integer and $f_1, \dots, f_g$ be holomorphic functions defined on an open subset U in the complex plane $\mathbf{C}$.

The Wronskian matrix associated to $f_1, \dots, f_g$ is define to be a square $g \times g$ matrix

$$W[f_1, \dots, f_g] = \begin{pmatrix} f_1 & f_1' & \cdots & f_1^{(g-1)} \\ f_2 & f_2' & \cdots & f_2^{(g-1)} \\ \cdots & \cdots & \cdots & \cdots \\ f_g & f_g' & \cdots & f_g^{(g-1)} \end{pmatrix}.$$

Its determinant (called the Wronskian determinant) is denoted by $W(f_1, \dots, f_g)$.

We begin with providing an explicit formula of the upper triangular matrix obtained by making use of a succession of row operations to the Wronskian matrix. The triangulation formulas obtained below will be used in the next section to investigate a pair of distinct points for which the Weierstrass gap set have the lowest cardinality $g(g+3)/2$. For this purpose, Lemma 3.1 suggests that two kinds of triangulations are needed, that is, upper triangulations and lower anti-triangulations. For a square $n \times n$ matrix $A = (a_{ij})$, its *anti-diagonal entry* is an entry a_{ij} with $i + j = n + 1$. The matrix A is called an *upper* (resp. *lower*) *anti-triangular matrix* if $a_{ij} = 0$ for $i + j > n + 1$ (resp. $i + j < n + 1$).

Now define an upper triangular $g \times g$ matrix $W^{\Delta}[f_1, \dots, f_g]$, lower triangular $g \times g$ matrices $L_k[f_1, \dots, f_g]$ $(k = 1, \dots, g-1)$ and $L[f_1, \dots, f_g]$ as follows:

$$W^{\Delta}[f_1, \dots, f_g]_{ij} := \frac{\begin{vmatrix} f_1 & f_1' & \cdots & f_1^{(i-2)} & f_1^{(j-1)} \\ f_2 & f_2' & \cdots & f_2^{(i-2)} & f_2^{(j-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_i & f_i' & \cdots & f_i^{(i-2)} & f_i^{(j-1)} \end{vmatrix}}{W(f_1, \dots, f_{i-1})} \quad (1 \leqq i \leqq j \leqq g),$$

$$L_k[f_1, \dots, f_g] := \begin{pmatrix} 1 & & & & & & & \mathbf{0} \\ 0 & \ddots & & & & & & \\ \vdots & \ddots & \ddots & & & & & \\ 0 & \cdots & 0 & 1 & & & & \\ 0 & \cdots & 0 & -\dfrac{W(f_1, \dots, f_{k-1}, f_{k+1})}{W(f_1, \dots, f_k)} & 1 & & & \\ \vdots & \cdots & \vdots & \vdots & 0 & \ddots & & \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & \underbrace{-\dfrac{W(f_1, \dots, f_{k-1}, f_g)}{W(f_1, \dots, f_k)}}_{k\text{-th column}} & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$L[f_1, \dots, f_g] := L_{g-1}[f_1, \dots, f_g] \cdot \cdots \cdot L_2[f_1, \dots, f_g] \cdot L_1[f_1, \dots, f_g].$$

Here we put $W(f_1, \dots, f_{i-1}) = 1$ when $i = 1$. Then we have the following triangulation formula for the Wronskian matrices:

$$L[f_1, \dots, f_g] \cdot W[f_1, \dots, f_g] = W^{\Delta}[f_1, \dots, f_g].$$

In order to obtain holomorphic objects, we set

$$T_{\mathrm{low}}[f_1, \dots, f_g] := \left(\prod_{k=1}^{g-1} W(f_1, \dots, f_i)\right) L[f_1, \dots, f_g],$$

$$W^{\mathrm{up}}[f_1, \dots, f_g] := \left(\prod_{k=1}^{g-1} W(f_1, \dots, f_i)\right) W^{\Delta}[f_1, \dots, f_g].$$

Then $T_{\mathrm{low}}[f_1, \dots, f_g]$ (resp. $W^{\mathrm{up}}[f_1, \dots, f_g]$) is a lower (resp. upper) triangular matrix with entries in $\mathcal{O}(U)$ (as usual $\mathcal{O}(U)$ denotes the space of holomorphic functions on U), and the following equality holds:

$$T_{\mathrm{low}}[f_1, \dots, f_g] \cdot W[f_1, \dots, f_g] = W^{\mathrm{up}}[f_1, \dots, f_g].$$

We note that the entries of $T_{\mathrm{low}}[f_1, \dots, f_g]$ are given precisely by

$$T_{\mathrm{low}}[f_1, \dots, f_g]_{ij} = (-1)^{i+j} \frac{W(f_1, \dots, \widehat{f_j}, \dots, f_i)}{W(f_1, \dots, f_{i-1})} \prod_{k=1}^{g-1} W(f_1, \dots, f_i). \tag{12}$$

for $1 \leqq j \leqq i \leqq g$. Here the circumflex over a term means that it is to be ommited. We also note that because their i-th diagonal entries ($i = 1, \dots, g$) are given by

$$T_{\mathrm{low}}[f_1, \dots, f_g]_{ii} = \prod_{k=1}^{g-1} W(f_1, \dots, f_k),$$

$$W^{\mathrm{up}}[f_1, \dots, f_g]_{ii} = \frac{W(f_1, \dots, f_{i-1}, f_i)}{W(f_1, \dots, f_{i-1})} \prod_{k=1}^{g-1} W(f_1, \dots, f_k),$$

for a point z in U, $T_{\mathrm{low}}[f_1, \dots, f_g](z)$ is invertible if and only if $\prod_{k=1}^{g-1} W(f_1, \dots, f_i)(z) \neq 0$, while $W^{\mathrm{up}}[f_1, \dots, f_g](z)$ is invertible if and only if $\prod_{k=1}^{g} W(f_1, \dots, f_i)(z) \neq 0$.

The lower anti-triangulation version to the above can be obtained by applying the argument above to $f_g, \dots, f_1$. Namely if we set

$$R := \begin{pmatrix} 0 & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \quad (g \times g \text{ matrix}),$$

$$T^{\mathrm{up}}[f_1, \dots, f_g] := R \cdot T_{\mathrm{low}}[f_g, \dots, f_1] \cdot R = {}^{t}T_{\mathrm{low}}[f_g, \dots, f_1],$$

$$W_{\mathrm{lowan}}[f_1, \dots, f_g] := R \cdot W^{\mathrm{up}}[f_g, \dots, f_1],$$

then $T^{\mathrm{up}}[f_1, \dots, f_g]$ (resp. $W_{\mathrm{lowan}}[f_1, \dots, f_g]$) is a upper (resp. lower anti-) triangular matrix with entries in $\mathcal{O}(U)$, and the following equality holds:

$$T^{\mathrm{up}}[f_1, \dots, f_g] \cdot W[f_1, \dots, f_g] = W_{\mathrm{lowan}}[f_1, \dots, f_g].$$

Their i-th diagonal and anti-diagonal entries ($i = 1, \dots, g$) are given by

$$T^{\mathrm{up}}[f_1, \dots, f_g]_{ii} = \prod_{k=1}^{g-1} W(f_g, \dots, f_{g+1-k}),$$

$$W_{\mathrm{lowan}}[f_1, \dots, f_g]_{i,g+1-i} = \frac{W(f_g, \dots, f_{i+1}, f_i)}{W(f_g, \dots, f_{i+1})} \prod_{k=1}^{g-1} W(f_g, \dots, f_{g+1-k}).$$

For later use, we introduce some notation. For a square $g \times g$ matrix $A = (a_{ij})$, we denote by $\boldsymbol{\delta}^i(A)$ the i-th principal minor ($i = 1, \dots, g$) of A, and moreover we put $\boldsymbol{\delta}_i(A) := \boldsymbol{\delta}^i(RA)$, where R is the anti-diagonal matrix mentioned above. Precisely those are defined by

$$\boldsymbol{\delta}^i(A) = \begin{vmatrix} a_{11} & \cdots & a_{1i} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ii} \end{vmatrix}, \quad \boldsymbol{\delta}_i(A) = \begin{vmatrix} a_{g1} & \cdots & a_{gi} \\ \cdots & \cdots & \cdots \\ a_{g-i+1,1} & \cdots & a_{g-i+1,i} \end{vmatrix}.$$

Then the Wronskian determinants appeared above are denoted as

$$W(f_1, \dots, f_i) = \boldsymbol{\delta}^i(W[f_1, \dots, f_g]) \quad \text{and} \quad W(f_g, \dots, f_{g+1-i}) = \boldsymbol{\delta}_i(W[f_1, \dots, f_g]).$$

Summarizing the argument above, we obtain the following.

Lemma 4.1. *Let $f_1, \dots, f_g$ be holomorphic functions defined on an open subset U in $\mathbf{C}$. Then there exist a lower triangular matrix $T_{\mathrm{low}}[f_1, \dots, f_g]$ and an upper triangular matrix $T^{\mathrm{up}}[f_1, \dots, f_g]$ both with entries in $\mathcal{O}(U)$ which have the following properties. Let z be a point in U.*

(1) *Suppose that $\prod_{i=1}^{g} \boldsymbol{\delta}^i(W[f_1, \dots, f_g](z)) \neq 0$, then $T_{\mathrm{low}}[f_1, \dots, f_g](z)$ is an invertible lower triangular matrix, for which the product $(T_{\mathrm{low}}[f_1, \dots, f_g](z)) \cdot (W[f_1, \dots, f_g](z))$ turns to an invertible upper triangular matrix.*

(2) *Suppose that $\prod_{i=1}^{g} \boldsymbol{\delta}_i(W[f_1, \dots, f_g](z)) \neq 0$, then $T^{\mathrm{up}}[f_1, \dots, f_g](z)$ is an invertible upper triangular matrix, for which the product $(T^{\mathrm{up}}[f_1, \dots, f_g](z)) \cdot (W[f_1, \dots, f_g](z))$ turns to an invertible lower anti-triangular matrix.*

5. The lowest cardinality for $\#G(P, Q)$

Let $\omega_1, \dots, \omega_g$ be a basis of $\Omega(X)$, U and V be open subsets of X with local coordinate functions z and w respectively. We then write $\omega_i = f_i\, dz = h_i\, dw$ ($i = 1, \dots, g$), where f_i and h_i are holomorphic functions defined respectively on U and V.

We now use the triangular matrices constructed in the previous section to define a holomorphic function on $U \times V$, namely we set

$$\psi[\omega_1, \dots, \omega_g](z, w) := \prod_{i=1}^{g} \boldsymbol{\delta}^i (W[f_1, \dots, f_g](z)) \\ \times \prod_{i=1}^{g} \boldsymbol{\delta}_i ((T_{\mathrm{low}}[f_1, \dots, f_g](z)) \cdot (W[h_1, \dots, h_g](w))).$$

Then this function relates to the cardinality of $G(z, w)$ as follows.

Proposition 5.1. *Under the notation as above, suppose we are given distinct points z in U and w in V. Then the condition $\psi(z, w) \neq 0$ implies $\# G(z, w) = g(g+3)/2$.*

Proof. We will show the existence of a basis of $\Omega(X)$ associated to which the Wronskian matrix of the effective divisor $gz + gw$ turns to be of the form (11).

First since $\prod_{i=1}^{g} \boldsymbol{\delta}^i (W[f_1, \dots, f_g](z)) \neq 0$ by the condition $\psi[\omega_1, \dots, \omega_g](z, w) \neq 0$, the lower triangular matrix $T_{\mathrm{low}}[f_1, \dots, f_g](z)$ must be invertible by Lemma 4.1.(1). Thus we can define a new basis $\hat{\omega}_1, \dots, \hat{\omega}_g$ of $\Omega(X)$ by

$$(\hat{\omega}_1, \dots, \hat{\omega}_g) := (\omega_1, \dots, \omega_g) \cdot ({}^t T_{\mathrm{low}}[f_1, \dots, f_g](z)).$$

Write $\hat{\omega}_i = \hat{f}_i \, dz = \hat{h}_i \, dw$ on U and V respectively as before. Then on account of (1), the Wronskian matrix $W[\hat{f}_1, \dots, \hat{f}_g](z)$ is equal to $(T_{\mathrm{low}}[f_1, \dots, f_g](z)) \cdot (W[f_1, \dots, f_g](z))$, which turns to an invertible upper triangular matrix because of Lemma 4.1.(1). Moreover also (1) implies that the Wronskian matrix $W[\hat{h}_1, \dots, \hat{h}_g](w)$ is given by

$$W[\hat{h}_1, \dots, \hat{h}_g](w) = (T_{\mathrm{low}}[f_1, \dots, f_g](z)) \cdot (W[h_1, \dots, h_g](w)).$$

Thus we have $\prod_{i=1}^{g} \boldsymbol{\delta}_i (W[\hat{h}_1, \dots, \hat{h}_g](w)) \neq 0$ by the condition $\psi[\omega_1, \dots, \omega_g](z, w) \neq 0$, and hence Lemma 4.1.(2) implies that $T^{\mathrm{up}}[\hat{h}_1, \dots, \hat{h}_g](w)$ turns to an invertible upper triangular matrix. Thus we can further define a new basis $\hat{\hat{\omega}}_1, \dots, \hat{\hat{\omega}}_g$ of $\Omega(X)$ by

$$(\hat{\hat{\omega}}_1, \dots, \hat{\hat{\omega}}_g) = (\hat{\omega}_1, \dots, \hat{\omega}_g) \cdot ({}^t T^{\mathrm{up}}[\hat{h}_1, \dots, \hat{h}_g](w)).$$

Write $\hat{\hat{\omega}}_i = \hat{\hat{f}}_i \, dz = \hat{\hat{h}}_i \, dw$ on U and V respectively as well. Then also on account of (1), the Wronskian matrix associated to $\hat{\hat{h}}_1, \dots, \hat{\hat{h}}_g$ at w is

$$W[\hat{\hat{h}}_1, \dots, \hat{\hat{h}}_g](w) = (T^{\mathrm{up}}[\hat{h}_1, \dots, \hat{h}_g](w)) \cdot (W[\hat{h}_1, \dots, \hat{h}_g](w)),$$

which turns to an invertible lower anti-triangular matrix by Lemma 4.1.(2).

On the other hand, the Wronskian matrix associated to $\hat{\hat{f}}_1, \dots, \hat{\hat{f}}_g$ at z is

$$W[\hat{\hat{f}}_1, \dots, \hat{\hat{f}}_g](z) = (T^{\mathrm{up}}[\hat{h}_1, \dots, \hat{h}_g](w)) \cdot (W[\hat{f}_1, \dots, \hat{f}_g](z)),$$

which turns to an invertible upper triangular matrix because so are the both factors on the right as we have shown above. Consequently the Wronskian matrix

$$W_{gz+gw}[\hat{\hat{\omega}}_1, \dots, \hat{\hat{\omega}}_g] = (W[\hat{\hat{f}}_1, \dots, \hat{\hat{f}}_g](z), W[\hat{\hat{h}}_1, \dots, \hat{\hat{h}}_g](w))$$

associated to the effective divisor $gz + gw$ turns to be of the form (11). $\diamond$

Next in order to obtain a global object (some holomorpic section of a certain holomorphic line bundle over $X \times X$), we want to find the transition low for $\psi(z, w)$ in changing the local coordinate functions. Let $\tilde{z}$ and $\tilde{w}$ be another local coordinate functions defined respectively on U and V, and write $\omega_i = \tilde{f}_i \, d\tilde{z} = \tilde{h}_i \, d\tilde{w}$ $(i = 1, \dots, g)$ as above. Then the holomorphic function $\tilde{\psi}[\omega_1, \dots, \omega_g](\tilde{z}, \tilde{w})$ on $U \times V$ is defined by using $\tilde{f}_i$ and $\tilde{h}_i$ as well as $\psi[\omega_1, \dots, \omega_g](z, w)$. If we put $\lambda = d\tilde{z}/dz$ and $\chi = d\tilde{w}/dw$, the Wronskian matrices associated to $f_1, \dots, f_g$ and $\tilde{f}_1, \dots, \tilde{f}_g$ (we write down only for these and omit for $h_1, \dots, h_g$ and $\tilde{h}_1, \dots, \tilde{h}_g$) are subjected to the following transition low in changing the local coordinate functions:

$$W[f_1, \dots, f_g](z) = (W[\tilde{f}_1, \dots, \tilde{f}_g](\tilde{z})) \cdot \Lambda(z). \tag{13}$$

Here $\Lambda(z)$ is a $g \times g$ upper triangular matrix of the following form:

$$\Lambda(z) = \begin{pmatrix} \lambda & \lambda' & \lambda'' & \cdots \\ 0 & \lambda^2 & 3\lambda\lambda' & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda^g \end{pmatrix}.$$

As for the Wronskian determinant, (13) implies the transition low

$$W(f_{i_1}, \dots, f_{i_m})(z) = \lambda(z)^{m(m+1)/2} W(\tilde{f}_{i_1}, \dots, \tilde{f}_{i_m})(\tilde{z}),$$

for each sequence $1 \leqq i_1 < \cdots < i_m \leqq g$, so that we have

$$T_{\text{low}}[f_1, \dots, f_g](z) = \lambda(z)^{g(g+1)(g+2)/6} T_{\text{low}}[\tilde{f}_1, \dots, \tilde{f}_g](\tilde{z}).$$

Therefore each factor of $\psi[\omega_1, \dots, \omega_g](z, w)$ is subjected respectively to the following transition lows:

$$\prod_{i=1}^{g} \boldsymbol{\delta}^i (W[f_1, \dots, f_g](z)) = \lambda(z)^{g(g+1)(g+2)/6} \prod_{i=1}^{g} \boldsymbol{\delta}^i (W[\tilde{f}_1, \dots, \tilde{f}_g](\tilde{z})),$$

$$\begin{aligned} \prod_{i=1}^{g} \boldsymbol{\delta}_i ((T_{\text{low}}&[f_1, \dots, f_g](z)) \cdot (W[h_1, \dots, h_g](w)) \\ &= \lambda(z)^{(g-1)g^2(g+1)^2/12} \chi(w)^{g(g+1)(g+2)/6} \\ &\quad \times \prod_{i=1}^{g} \boldsymbol{\delta}_i ((T_{\text{low}}[\tilde{f}_1, \dots, \tilde{f}_g](\tilde{z})) \cdot (W[\tilde{h}_1, \dots, \tilde{h}_g](\tilde{w})). \end{aligned}$$

Consequently, we obtain the following transition low for $\psi[\omega_1, \dots, \omega_g](z, w)$ in changing the local coordinate functions:

$$\text{(14)}\quad \psi[\omega_1, \dots, \omega_g](z, w) = \left(\frac{d\tilde{z}}{dz}(z)\right)^{g(g+1)(g^3+g+4)/12} \left(\frac{d\tilde{w}}{dw}(w)\right)^{g(g+1)(g+2)/6} \tilde{\psi}[\omega_1, \dots, \omega_g](\tilde{z}, \tilde{w}).$$

This means the following. Let K_X be the canonical line bundle of X and $p_i : X \times X \to X$ be the projection to the i-th component, namely $p_i(x_1, x_2) := x_i$ $(i = 1, 2)$. Using the notation above, we put on $U \times V$,

$$\Psi[\omega_1, \dots, \omega_g](z, w) := \psi[\omega_1, \dots, \omega_g](z, w)(dz)^{\otimes g(g+1)(g^3+g+4)/12} \otimes (dw)^{\otimes g(g+1)(g+2)/6}.$$

Then $\Psi[\omega_1, \dots, \omega_g]$ defines a global holomorphic section of the holomorphic line bundle

$$p_1^* K_X^{\otimes g(g+1)(g^3+g+4)/12} \otimes p_2^* K_X^{\otimes g(g+1)(g+2)/6} \to X \times X.$$

These holomorphic sections describe a pair of points for which the cardinality of the Weierstrass gap set attains the lowest bound.

THEOREM 5.2. *For a pair of distinct points P, Q in X, the following are equivalent.*

(1) $\#G(P, Q) = g(g+3)/2$.
(2) *There exists a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ for which $\Psi[\omega_1, \dots, \omega_g](P, Q) \neq 0$.*

Proof. The implication (2) $\Rightarrow$ (1) is due exactly to Proposition 5.1.

Conversely suppose that the cardinality of $G(P, Q)$ is equal to $g(g+3)/2$. Then by virture of Lemma 3.1.(3), we can take a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ associated to which the Wronskian matrix $W_{gP+gQ}[\omega_1, \dots, \omega_g] = (W[f_1, \dots, f_g](P), W[h_1, \dots, h_g](Q))$ is of the form (11), f_i and h_i being locally defined holomorphic functions as above. In this case, $L[f_1, \dots, f_g](P)$ becomes the identity matrix, and we have

$$\psi[\omega_1, \dots, \omega_g](P, Q) = A_1 \cdots A_g \left(\prod_{k=1}^{g-1} A_1 \cdots A_k\right)^{(g^2+g+2)/2} \left(\prod_{i=1}^{g} B_g \cdots B_{g+1-i}\right) \neq 0,$$

because $A_1, \dots A_g$, $B_1, \dots, B_g$ are non-zero complex numbers. $\Diamond$

Therefore if we put

$$Z := \left\{ (P, Q) \in X \times X \backslash \Delta(X) \,\middle|\, \#G(P, Q) = \frac{g^2 + 3g}{2} \right\},$$

this set is discribed as

$$Z := \bigcup_{\substack{\omega_1,\ldots,\omega_g;\\ \text{basis of } \Omega(X)}} \{(P,Q) \in X \times X \backslash \Delta(X) \,|\, \Psi[\omega_1,\ldots,\omega_g](P,Q) \neq 0\}.$$

Especially Z is an open and dense subset in $X \times X \backslash \Delta(X)$.

We remark that Homma [H, Proposition 3 (p. 344)] has proved for Z to be open and dense in X for a smooth curve X in characteristic 0. He used the concept of order-sequences of linear systems on X.

6. Hyperelliptic Riemann surfaces

In this section, we examine the results obtained in the preceding sections in the case where the Riemann surface is hyperelliptic. In [K] and [H], the cardinalities of gap sets for pairs of distinct points on a hyperelliptic Riemann surface have been calculated. In fact, the results are valid for hyperelliptic curves in an arbitrary charastaristic. Those are summarized as the table below (we follow the formalism due to [H]). W-point means Weierstrass point.

P, Q	μ	$t(\mu)$	$\#G(P,Q)$
Both P and Q are W-points	$\mu(\alpha) = 2g - \alpha$	$g(g-1)/2$	$g(3g+1)/2$
P is not a W-point and Q is a W-point	$\mu(\alpha) = 2\alpha - 1$	$g(g-1)/2$	$g(g+1)$
Neither P or Q is not W-points and $\sigma(P) = Q$	$\mu(\alpha) = \alpha$	0	$g(g+1)$
Neither P or Q is not W-points and $\sigma(P) \neq Q$	$\mu(\alpha) = g + 1 - \alpha$	$g(g-1)/2$	$g(g+3)/2$

Here σ denotes the hyperelliptic involution.

In what follows, we will calculate the holomorphic section $\Psi[\omega_1,\ldots,\omega_g]$ defined in the preceding section for a hyperelliptic Riemann surface. Then we will observe that the non-generic loci (namely the first three cases) in the table above appear as the irreducible components of the zero locus of the holomorphic section.

Let $b_1, b_2, \ldots, b_{2g+2}$ be distinct complex numbers and a a non-zero complex number. Then a hyperelliptic Riemann surface Y of genus g is constructed as the affine plane curve in $\mathbf{C}^2$ defined by the equation $y^2 = a(x-b_1)(x-b_2)\cdots(x-b_{2g+2})$ with additional two points at infinity to compactify it. It is then known that the Weierstrass points on Y are $(b_1,0),\ldots,(b_{2g+2},0)$. Moreover the hyperelliptic involution $\sigma : Y \to Y$ is defined by $\sigma(x,y) = (x,-y)$. Holomorphic 1-forms $\omega_\alpha = x^\alpha\, dx/y$ $(\alpha = 0,\ldots,g-1)$ on Y form a basis of $\Omega(Y)$. We put $f_\alpha := x^\alpha/y$ away from $y = 0$.

First we explain how the cardinality of the gap set for a pair of distinct points on Y is computed by means of Wronskian matrices associated to an effective divisor. For instance, let $P=(x,y)$ and $Q=(\xi,\eta)$ be distinct points with $\sigma(P)=Q$. This means that $x=\xi$ and $y=-\eta\neq 0$, and hence $W_{gP}[\omega_0,\dots,\omega_{g-1}]=-W_{gQ}[\omega_0,\dots,\omega_{g-1}]$. It is then straightforward that the Wronskian matrix $W_{gP+gQ}[\omega_0,\dots,\omega_{g-1}]$ can be deformed by certain row operations to the following form:

$$\left(\begin{array}{cccc|cccc}
\dfrac{1}{y} & \left(\dfrac{1}{y}\right)' & \cdots & \left(\dfrac{1}{y}\right)^{(g-1)} & \dfrac{1}{\eta} & \left(\dfrac{1}{\eta}\right)' & \cdots & \left(\dfrac{1}{\eta}\right)^{(g-1)} \\
0 & \dfrac{1}{y} & \cdots & \dfrac{(g-1)!}{(g-2)!}\left(\dfrac{1}{y}\right)^{(g-2)} & 0 & \dfrac{1}{\eta} & \cdots & \dfrac{(g-1)!}{(g-2)!}\left(\dfrac{1}{\eta}\right)^{(g-2)} \\
\vdots & \cdots & \ddots & \vdots & \vdots & \cdots & \ddots & \vdots \\
0 & \cdots & 0 & \dfrac{(g-1)!}{y} & 0 & \cdots & 0 & \dfrac{(g-1)!}{\eta}
\end{array}\right)$$

This together with Corollary 2.5 imply that $G(P)=G(Q)=\{1,2,\dots,g\}$ and $\mu(\alpha)=\alpha$ for any α. Therefore $t(\mu)=0$ and $\#G(P,Q)=g(g+1)$ by (10). When (P,Q) are on the other loci in the table above, the similar but slightly involved calculations also work to obtain $\#G(P,Q)$.

Second we shall calculate the function $\psi[\omega_0,\dots,\omega_{g-1}]((x,y),(\xi,\eta))$. All the calculations are elementary and routine, we write down only a recipe in the following. When $y\neq 0$ and $\eta\neq 0$, we can take x and ξ as the local coordinate functions around there. Then as for the Wronskian determinants, we have

$$W(f_0,\dots,f_{i-1})(x,y)=\left(\prod_{k=0}^{i-1}k!\right)\frac{1}{y^i}, \tag{15}$$

$$W(f_0,\dots,\widehat{f_{j-1}},\dots,f_{i-1})(x,y)=\frac{1}{(i-j)!}\left(\prod_{\substack{k=0\\k\neq j-1}}^{i-1}k!\right)\frac{x^{i-j}}{y^{i-1}}, \tag{16}$$

for $1\leqq j\leqq i\leqq g$. These (15), (16) and (12) imply

$$T_{\mathrm{low}}[f_0,\dots,f_{g-1}](x,y)_{ij}=(-1)^{i+j}\binom{i-1}{j-1}x^{i-j}\Delta^{(g)}(x,y). \tag{17}$$

Here $\Delta^{(g)}(x,y)$ is defined by

$$\Delta^{(g)}(x,y)=\prod_{i=1}^{g-1}(W(f_0,\dots,f_{i-1})(x,y))=\left(\prod_{i=1}^{g-1}\prod_{k=0}^{i-1}k!\right)\frac{1}{y^{g(g-1)/2}},$$

in which the second equality comes from (15). The entries of $W[f_0,\dots,f_{g-1}](\xi,\eta)$ are given by

$$W[f_0,\dots,f_{g-1}](\xi,\eta)_{ij}=\sum_{k=0}^{\min\{i-1,j-1\}}\frac{(i-1)!}{(i-1-k)!}\binom{j-1}{k}\xi^{i-1-k}\left(\frac{1}{\eta}\right)^{(j-1-k)}, \tag{18}$$

where $(1/\eta)^{(n)}$ denotes the n-th derivative of $1/\eta$ by ξ. Thus (17) and (18) imply

$$(19)\quad (T_{\mathrm{low}}[f_0,\dots,f_{g-1}](x,y)\cdot W[f_0,\dots,f_{g-1}](\xi,\eta))_{ij} = \Delta^{(g)} \sum_{k=0}^{\min\{i-1,j-1\}} (-1)^{i-1-k} k! \binom{i-1}{k}\binom{j-1}{k}(x-\xi)^{i-1-k}\left(\frac{1}{\eta}\right)^{(j-1-k)}.$$

Hence the i-th anti-principal minor of $T_{\mathrm{low}}[f_0,\dots,f_{g-1}](x,y)\cdot W[f_0,\dots,f_{g-1}](\xi,\eta)$ is given by

$$(20)\quad \boldsymbol{\delta}_i(T_{\mathrm{low}}[f_0,\dots,f_{g-1}](x,y)\cdot W[f_0,\dots,f_{g-1}](\xi,\eta)) = (-1)^{i(2g-i-1)/2}\left(\prod_{\alpha=1}^{g-1}\prod_{k=0}^{\alpha-1}k!\right)^i\left(\prod_{\beta=1}^{i}(\beta-1)!\right)\frac{(x-\xi)^{i(g-i)}}{(y^{g(g-1)/2}\eta)^i}.$$

Therefore from (15) and (20), we have obtained

$$(21)\quad \psi[\omega_0,\dots,\omega_{g-1}]((x,y),(\xi,\eta)) = C\left(\frac{(x-\xi)^{(g-1)/3}}{y^{(g^2-g+2)/2}\eta}\right)^{g(g+1)/2},$$

where the constant C is given by $C := (\prod_{i=1}^{g-1}\prod_{k=0}^{i-1}k!)^{g(g+1)/2}(\prod_{\alpha=1}^{g}\prod_{k=0}^{\alpha-1}k!)^2$.

Remark 6.1. Both $y\neq 0$ and $\eta\neq 0$ are assumed in the expression (21). When those are not the case, we can also obtain an expression of the function by using the transition low (14) in changing the local coordinate functions. For example, we consider the function at $y=0$ and $\eta\neq 0$. Then we can take y and ξ as the local coordinate functions around $(x,0)$ and (ξ,η) respectively. Then the transition low (14) implies that

$$\begin{aligned}\tilde{\psi}[\omega_0,\dots,\omega_{g-1}]((x,y),(\xi,\eta)) &= \left(\frac{dx}{dy}\right)^{g(g+1)(g^3+g+4)/12}\psi[\omega_0,\dots,\omega_{g-1}]((x,y),(\xi,\eta)) \\ &= C\left(\frac{(x-\xi)^{(g-1)/3}(x')^{(g^3+g+4)/6}}{y^{(g^2-g+2)/2}\eta}\right)^{g(g+1)/2},\end{aligned}$$

where $x' = dx/dy$. Because x'/y is holomorphic at $y=0$, $\tilde{\psi}[\omega_0,\dots,\omega_{g-1}]((x,y),(\xi,\eta))$ is holomorphic at $((x,0),(\xi,\eta))$ $(\eta\neq 0)$.

In consequence of the expression of $\psi[\omega_0,\dots,\omega_{g-1}]$, we find the following.

1. As for the special basis $\omega_0,\omega_1,\dots,\omega_{g-1}$ of $\Omega(Y)$ as above, the cardinality of $\#G(P,Q)$ attains the lowest value $g(g+3)/2$ if and only if $\Psi[\omega_1,\dots,\omega_g](P,Q)\neq 0$.
2. The irreducible components of the zero locus of $\Psi[\omega_0,\dots,\omega_{g-1}]$ are
   (i) $\{\text{Weierstrass point}\}\times Y$,
   (ii) $Y\times\{\text{Weierstrass point}\}$,
   (iii) the graph of the hyperellptic involution σ,
   which have been appeared in the table above.

This seems to suggest a relation between the irreducible components of the zero locus of $\Psi[\omega_0, \dots, \omega_{g-1}]$ and the cardinality of the gap set in general.

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