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# ISOTROPIC IMMERSIONS OF THE CAYLEY PROJECTIVE PLANE AND CAYLEY FRENET CURVES

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## Abstract

We investigate parallel isotropic immersions of an open submanifold of either the Cayley projective plane $\mathbf{C}ayP^2(\alpha)$ or its noncompact dual into a real space form $\tilde{M}^n(\tilde{c})$, and give a characterization of the first standard embedding of $\mathbf{C}ayP^2(\alpha)$ into $\tilde{M}^n(\tilde{c})$ in terms of a particular class of Frenet curves of order 2.

## 1. Introduction

Let $f : M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold $M$ into an ambient Riemannian manifold $\tilde{M}$. In order to study the properties of the immersion $f$ it is one of natural ways to examine the extrinsic shape of curves in the submanifold $M$.

A smooth curve $\gamma$ in $M$ parametrized by its arc-length is called a *Frenet curve of proper order* 2 *of curvature* $\kappa$ if there exist a smooth unit vector field $V$ along $\gamma$ and a positive smooth function $\kappa$ satisfying the following system of ordinary differential equations

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa V \quad \text{and} \quad \nabla_{\dot{\gamma}} V = -\kappa\dot{\gamma}. \tag{1.1}$$

We call a Frenet curve of proper order 2 with positive constant curvature $k$ a *circle* of curvature $k$. We regard a geodesic as a circle of null curvature. K. Nomizu and K. Yano proved that a submanifold $M$ is an extrinsic sphere of $\tilde{M}$ (that is, a totally umbilical submanifold with parallel mean curvature vector) if and only if every circle of curvature $k$ in $M$ is also a circle in $\tilde{M}$ for some positive constant $k$ ([9]). Motivated by their result, in [13] we gave characterizations of an extrinsic sphere and every totally geodesic submanifold in a Riemannian manifold in terms of a Frenet curve of proper order 2. In [8], S. Maeda and the author characterized all totally geodesic Kähler immersions of Kähler manifolds into an ambient Kähler manifold and all parallel isometric immersions of a complex space form into a real space form by using a particular class of Frenet curves.

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Along this context, in the preceding paper [14] the author established a theorem which provides a characterization of the first standard minimal immersion of the Cayley projective plane $\mathbf{C}ayP^2(c)$ into a real space form by observing the extrinsic shape of some Frenet curves of order 2 in $\mathbf{C}ayP^2(c)$. However, there was a gap in the proof of the theorem. He stated that the parallelism of the second fundamental form implies the rigidity of the parallel isotropic immersion and it is possible to use the classification theorem of complete parallel submanifolds in a real space form as a local theorem (see page 15 in [14]). His comment is true, but he did not give a full detail of the proof. Additionally, there was a slight deficiency of the precision in his theorem, because he did not describe one of examples which he should have done (see page 13 in [14] and our Theorem 2).

We have two aims of the present paper. One of those is to bring the above local rigidity theorem to completion. That is, we shall prove the following.

THEOREM 1. *Let $M$ be a connected open submanifold of either the Cayley projective plane $\mathbf{C}ayP^2(\alpha)$ of maximal sectional curvature $\alpha(>0)$ or Cayley hyperbolic plane $\mathbf{C}ayH^2(\alpha)$ of minimal sectional curvature $\alpha(<0)$. Let $f$ be a full parallel isotropic immersion of $M$ into a real space form $\tilde{M}^{16+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$. Then the immersion $f$ is constant isotropic and we have $\alpha > 0$, $p = 9$ or $10$. Moreover, $f$ is locally congruent to either*

(1) *the first standard minimal immersion $f_1 : \mathbf{C}ayP^2(\alpha) \to S^{25}(3\alpha/4)$ or*

(2) *a parallel immersion defined by $f_2 \circ f_1 : \mathbf{C}ayP^2(\alpha) \xrightarrow{f_1} S^{25}(3\alpha/4) \xrightarrow{f_2} \tilde{M}^{26}(\tilde{c})$, where $f_1$ is given above, $f_2$ is a totally umbilical immersion and $3\alpha/4 \geq \tilde{c}$.*

To prove Theorem 1, we take a different way from that mentioned in [14] and utilize the result of Y. Agaoka and E. Kaneda [3]. The proof of Theorem 1 will be given in §4. Preparatorily, in §3 we study the properties of isotropic immersions of the Cayley plane into a real space form by examining the structure of the first normal space.

Using the above theorem, we can deal with not only $\mathbf{C}ayP^2(\alpha)$ but also $\mathbf{C}ayH^2(\alpha)$. The other aim of this paper is to establish the following new theorem which fills up the deficiency in [14] and is an improvement of that:

THEOREM 2. *Let $M$ be a connected open submanifold of $\mathbf{C}ayP^2(\alpha)$ $(\alpha > 0)$ or $\mathbf{C}ayH^2(\alpha)$ $(\alpha < 0)$ and $f$ an isometric immersion of $M$ into a real space form $\tilde{M}^{16+p}(\tilde{c})$. Assume that there exists a positive smooth function $\kappa(s)$ satisfying that $f$ maps every Cayley Frenet curve $\gamma = \gamma(s)$ of curvature $\kappa(s)$ in $M$ to a plane curve in $\tilde{M}^{16+p}(\tilde{c})$. Then the open submanifold $M$ must be in $\mathbf{C}ayP^2(\alpha)$ $(\alpha > 0)$ and the immersion $f$ is locally congruent to one of the following examples*:

(1) *the first standard minimal immersion $f_1 : \mathbf{C}ayP^2(\alpha) \to S^{25}(3\alpha/4)$,*

(2) *a parallel immersion defined by $f_2 \circ f_1 : \mathbf{C}ayP^2(\alpha) \xrightarrow{f_1} S^{25}(3\alpha/4) \xrightarrow{f_2} \tilde{M}^{16+p}(\tilde{c})$, where $f_1$ is as above, $f_2$ is a totally umbilical immersion and $3\alpha/4 \geq \tilde{c}$.*

*In addition, the Cayley Frenet curve $\gamma$ is a Cayley circle.*

For the notions of Cayley Frenet curves and Cayley circles in the Cayley plane, we refer to §5. The proof of Theorem 2 will be given in §6.

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## 2. Preliminaries

In this section we review a few fundamental equations in submanifold theory and prepare some lemmas. Let $M$ and $\tilde{M}$ be Riemannian manifolds and $f : M \to \tilde{M}$ an isometric immersion. We identify a vector $X$ of $M$ with a vector $f_*(X)$ of $\tilde{M}$ throughout this paper. The Riemannian metrics on $M$, $\tilde{M}$ are denoted by the same notation $\langle , \rangle$. The pull back $f^{-1}T\tilde{M}$ of the tangent bundle $T\tilde{M}$ of $\tilde{M}$ is orthogonally decomposed into the sum of tangent bundle $TM$ of $M$ and normal bundle $NM$: $f^{-1}T\tilde{M} = TM \oplus NM$. We denote by $\nabla$ and $\tilde{\nabla}$ the covariant differentiations of $M$ and $\tilde{M}$, respectively. Then the formulae of Gauss and Weingarten are

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma_f(X, Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$$

for vector fields $X$, $Y$ of $M$ and a normal vector field $\xi$, where $\nabla^\perp$ denotes the covariant differentiation in the normal bundle $NM$. The tensors $\sigma = \sigma_f$ and $A_\xi$ are called the second fundamental form of $f$ and the shape operator in the direction of $\xi$, respectively. We define the covariant differentiation $\nabla'$ of the second fundamental form $\sigma$ of $f$ with respect to the connection in $TM \oplus NM$ by

$$(\nabla'_X \sigma)(Y, Z) = \nabla^\perp_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for vector fields $X$, $Y$, $Z$ of $M$. An isometric immersion $f$ is said to be *parallel* if its second fundamental form satisfies $\nabla'\sigma = 0$.

Let $\{e_1, \ldots, e_n\}$ be a local field of orthonormal frames on $M$, where $n = \dim M$. Then the mean curvature vector field $\mathfrak{h} = \mathfrak{h}_f$ of $f$ is defined by $\mathfrak{h} = (1/n)\sum_{i=1}^n \sigma(e_i, e_i)$. If $\mathfrak{h} = 0$, the immersion $f$ is said to be *minimal*. We say the mean curvature vector field $\mathfrak{h}$ of $f$ is parallel if $\nabla^\perp \mathfrak{h} = 0$. It is said to be *totally umbilical* if $\sigma(X, Y) = \langle X, Y\rangle \mathfrak{h}$ for all vector fields $X$, $Y$ on $M$. If there exists a function $\mu$ on the submanifold $M$ such that $\langle \sigma(X, Y), \mathfrak{h}\rangle = \mu\langle X, Y\rangle$ for any vector fields $X$, $Y$ on $M$, then the immersion $f$ is said to be *pseudo umbilical*. It is clear that every minimal isometric immersion is pseudo umbilical and that for a pseudo umbilical immersion we have $\mu = \|\mathfrak{h}\|^2$.

A *real space form* $\tilde{M}^m(\tilde{c})$ is an $m$-dimensional Riemannian manifold of constant sectional curvature $\tilde{c}$, which is locally congruent to either a Euclidean space $\mathbf{R}^m$, a standard sphere $S^m(\tilde{c})$ or a real hyperbolic space $H^m(\tilde{c})$ according as the curvature $\tilde{c}$ is zero, positive or negative. In case that the ambient manifold is a real space form $\tilde{M}^m(\tilde{c})$, the equation of Gauss for an isometric immersion $f : M \to \tilde{M}^m(\tilde{c})$ can be written as

$$\langle R(X,Y)Z,W\rangle = \tilde{c}(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle) + \langle\sigma(X,W),\sigma(Y,Z)\rangle - \langle\sigma(X,Z),\sigma(Y,W)\rangle, \tag{2.1}$$

for vector fields $X$, $Y$, $Z$, $W$ of $M$, where $R$ represents the curvature tensors for $\nabla$.

Next, we recall the notion of isotropic immersions in the sense of B. O'Neill. An isometric immersion $f : M \to \tilde{M}$ is said to be ($\lambda_x$-)*isotropic* at $x \in M$ if there exists a nonnegative constant $\lambda_x$ such that $\|\sigma(X,X)\| = \lambda_x$ for every unit tangent vector $X \in T_xM$. If there exists a nonnegative constant $\lambda$ satisfying that $\|\sigma(X,X)\| = \lambda$ for every point $x \in M$ and for every unit tangent vector $X \in T_xM$, then $f$ is called a *constant isotropic* immersion whose isotropy constant is $\lambda$. Note that a totally umbilical immersion is isotropic, but not *vice versa*. We have the following lemma (see [12]).

LEMMA 1. *Let $M$, $M'$ be Riemannian manifolds and $\tilde{M}$ a pseudo Riemannian manifold. Let $f' : M \to M'$ be an isometric immersion, $f'' : M' \to \tilde{M}$ a totally umbilical immersion whose mean curvature vector $\mathfrak{h}_{f''}$ is parallel and let $f = f'' \circ f'$ be the composition of $f'$ and $f''$. Then*:

(1) *The mean curvature vector $\mathfrak{h}_f$ of $f$ is parallel if and only if $\mathfrak{h}_{f'}$ of $f'$ is parallel.*
(2) *$f$ is constant $\lambda$-isotropic if and only if $f'$ is constant $\lambda'$-isotropic, where $\lambda^2 = \lambda'^2 + \|\mathfrak{h}_{f''}\|^2$.*
(3) *$f$ is parallel if and only if $f'$ is parallel.*

The *first normal space* at the point $x$ of $M$ is defined as the subspace $N_x^1(M)$ of $N_xM$ spanned by the image of the second fundamental form at $x$, that is,

$$N_x^1(M) = Span_{\mathbf{R}}\{\sigma(X,Y); X, Y \in T_xM\} \subset N_xM,$$

where $Span_{\mathbf{R}}\{*\}$ denotes the real vector space spanned by $\{*\}$. The *discriminant* $\Delta_x$ at $x \in M$ is given as

$$\Delta_x = K(X,Y) - \tilde{K}(X,Y),$$

where $K(X,Y)$ (resp. $\tilde{K}(X,Y)$) represents the sectional curvature of the plane spanned by orthonormal vectors $X, Y \in T_xM$ for $M$ (resp. for $\tilde{M}$).

The following two lemmas are due to B. O'Neill ([10]):

LEMMA 2. *For an isometric immersion $f : M \to \tilde{M}$ the following conditions are mutually equivalent*:

(1) *$f$ is $\lambda_x$-isotropic at $x \in M$ for some $\lambda_x(\geq 0)$.*
(2) *$\langle\sigma(X,X),\sigma(X,Y)\rangle = 0$ for an arbitrary orthogonal pair $X, Y \in T_xM$.*
(3) $\langle\sigma(X,Y),\sigma(Z,W)\rangle + \langle\sigma(X,Z),\sigma(W,Y)\rangle + \langle\sigma(X,W),\sigma(Y,Z)\rangle$
$= \lambda_x^2(\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle)$
*for some $\lambda_x(\geq 0)$ and for any vectors $X, Y, Z, W \in T_xM$.*

LEMMA 3. *Let $x$ be an arbitrary fixed point of $M$, $V$ an $r$-dimensional subspace of $T_xM$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $V$, where $1 < r \leq \dim M$. Let $N(V)$ be a subspace of $N_x^1(M)$ given by $N(V) = Span_{\mathbf{R}}\{\sigma(X, Y); X, Y \in V\}$. Suppose that $f$ is $\lambda_x(>0)$-isotropic at the point $x \in M$ and the restriction $\Delta_x|_V$ of the discriminant $\Delta_x$ to $V$ is constant on $V$. Then we have*

$$-\frac{r+2}{2(r-1)}\lambda_x^2 \leq \Delta_x|_V \leq \lambda_x^2.$$

*Moreover,*

(1) $\Delta_x|_V = \lambda_x^2 \Leftrightarrow \dim N(V) = 1$,

(2) $\Delta_x|_V = -\{(r+2)/2(r-1)\}\lambda_x^2 \Leftrightarrow \sum_{i=1}^{r} \sigma(e_i, e_i) = 0 \Leftrightarrow \dim N(V) = \{r(r+1)/2\} - 1$,

(3) $-\{(r+2)/2(r-1)\}\lambda_x^2 < \Delta_x|_V < \lambda_x^2 \Leftrightarrow \dim N(V) = r(r+1)/2$.

## 3. Isotropic immersions of the Cayley plane

Let $\mathbf{C}ay$ denote the set of Cayley numbers, which is an 8-dimensional non-associative division algebra over the real numbers ([6]). It has multiplicative identity 1 and a positive definite symmetric bilinear form $\langle , \rangle$. Let $\{u_0 = 1, u_1, \ldots, u_7\}$ be an orthonormal basis of $\mathbf{C}ay$ with respect to the form $\langle , \rangle$. The multiplication of Cayley numbers is completely determined by the multiplication table given below:

| | $u_0$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ |
|---|---|---|---|---|---|---|---|---|
| $u_0$ | $u_0$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ |
| $u_1$ | $u_1$ | $-u_0$ | $u_3$ | $-u_2$ | $u_5$ | $-u_4$ | $u_7$ | $-u_6$ |
| $u_2$ | $u_2$ | $-u_3$ | $-u_0$ | $u_1$ | $-u_6$ | $u_7$ | $u_4$ | $-u_5$ |
| $u_3$ | $u_3$ | $u_2$ | $-u_1$ | $-u_0$ | $u_7$ | $u_6$ | $-u_5$ | $-u_4$ |
| $u_4$ | $u_4$ | $-u_5$ | $u_6$ | $-u_7$ | $-u_0$ | $u_1$ | $-u_2$ | $u_3$ |
| $u_5$ | $u_5$ | $u_4$ | $-u_7$ | $-u_6$ | $-u_1$ | $-u_0$ | $u_3$ | $u_2$ |
| $u_6$ | $u_6$ | $-u_7$ | $-u_4$ | $u_5$ | $u_2$ | $-u_3$ | $-u_0$ | $u_1$ |
| $u_7$ | $u_7$ | $u_6$ | $u_5$ | $u_4$ | $-u_3$ | $-u_2$ | $-u_1$ | $-u_0$ |

To express the above multiplication simply, we define $\varepsilon_{ij}$ $(\varepsilon_{ij} = \pm 1)$ and $\kappa_{ij}$ $(\kappa_{ij} = 0, 1, \ldots, 7)$ by

$$u_i u_j = \varepsilon_{ij} u_{\kappa_{ij}} \tag{3.1}$$

for $i, j = 0, 1, \ldots, 7$.

Let $M$ be a connected open submanifold of $\mathbf{C}ayP^2(\alpha)$ or $\mathbf{C}ayH^2(\alpha)$. The tangent space of $M$ can be identified with the set of ordered pair of Cayley numbers $\mathbf{C}ay \oplus \mathbf{C}ay$. The vector space $\mathbf{C}ay \oplus \mathbf{C}ay$ has a positive definite symmetric bilinear form $\langle\ ,\ \rangle$ given by $\langle (a,c),(b,d)\rangle = \langle a,b\rangle + \langle c,d\rangle$. We put $e_i = (u_i, 0)$, $e_{i^*} = (0, u_i)$ for $0 \le i \le 7$. Then the vectors $e_0, \ldots, e_7$, $e_{0^*}, \ldots, e_{7^*}$ form an orthonormal basis of the tangent space of $M$.

The curvature tensor $R$ of $M$ is given by

$$
\begin{aligned}
(3.2)\quad &\langle R((a,b),(c,d))(e,f),(g,h)\rangle \\
&= \alpha(\langle c,e\rangle\langle a,g\rangle - \langle a,e\rangle\langle c,g\rangle + \langle d,f\rangle\langle b,h\rangle - \langle b,f\rangle\langle d,h\rangle) \\
&\quad + \frac{\alpha}{4}(\langle ed,gb\rangle - \langle eb,gd\rangle + \langle cf,ah\rangle - \langle af,ch\rangle + \langle ad-cb, gf-eh\rangle)
\end{aligned}
$$

(for detail, see [4]).

*Remark* 1. From (3.2), we see the scalar curvature $\rho$ of $M$ is given by $\rho = 144\alpha$.

Now, we consider a $\lambda_x(>0)$-isotropic immersion $f$ of $M$ into a real space form $\tilde{M}^{16+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$. Using the equation of Gauss (2.1) and Lemma 2(3), we have

$$
\begin{aligned}
(3.3)\quad &3\langle \sigma(X,Y),\sigma(Z,W)\rangle \\
&= \langle R(Z,X)Y,W\rangle + \langle R(Z,Y)X,W\rangle + (\lambda_x^2 - 2\tilde{c})\langle X,Y\rangle\langle Z,W\rangle \\
&\quad + (\lambda_x^2 + \tilde{c})\{\langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle\}
\end{aligned}
$$

for all vectors $X, Y, Z, W \in T_xM$. For an orthonormal basis $\{e_0, \ldots, e_7, e_{0^*}, \ldots, e_{7^*}\}$ of the tangent space $T_xM$, we employ the simple notation $\sigma_{ij}$, $\sigma_{ij^*}$ and $\sigma_{i^*j^*}$ instead of $\sigma(e_i,e_j)$, $\sigma(e_i,e_{j^*})$ and $\sigma(e_{i^*},e_{j^*})$, respectively. Then, the equation (3.3), combined with (3.2), yields the following orthogonal relations:

$$
\begin{aligned}
(3.4)\quad &\langle \sigma_{ij},\sigma_{kl}\rangle = \langle \sigma_{i^*j^*},\sigma_{k^*l^*}\rangle \\
&= \frac{1}{3}\{\lambda_x^2 + 2(\alpha-\tilde{c})\}\delta_{ij}\delta_{kl} + \frac{1}{3}\{\lambda_x^2 - (\alpha-\tilde{c})\}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}),
\end{aligned}
$$

$$
(3.5)\qquad \langle \sigma_{ij},\sigma_{k^*l^*}\rangle = \frac{1}{3}\left(\lambda_x^2 + \frac{\alpha}{2} - 2\tilde{c}\right)\delta_{ij}\delta_{kl},
$$

$$
(3.6)\qquad \langle \sigma_{ij^*},\sigma_{kl^*}\rangle = \frac{\alpha}{4}\varepsilon_{ij}\varepsilon_{kl}\delta_{\kappa_{ij}\kappa_{kl}} + \frac{1}{3}\{\lambda_x^2 - (\alpha-\tilde{c})\}\delta_{ik}\delta_{jl},
$$

$$
(3.7)\qquad \langle \sigma_{ij},\sigma_{kl^*}\rangle = \langle \sigma_{ij^*},\sigma_{k^*l^*}\rangle = 0,
$$

where $\varepsilon_{ij}$ and $\kappa_{ij}$ are defined by (3.1).

A straightforward calculation shows the following lemmas.

LEMMA 4. *For a $\lambda_x(>0)$-isotropic immersion $f: M \to \tilde{M}^{16+p}(\tilde{c})$, we have*

$$\|\mathfrak{h}\|^2 = \frac{1}{2^3}(3\lambda_x^2 + 3\alpha - 5\tilde{c}), \tag{3.8}$$

*where $\mathfrak{h}$ is the mean curvature vector of $f$.*

LEMMA 5. *A $\lambda_x(>0)$-isotropic immersion $f: M \to \tilde{M}^{16+p}(\tilde{c})$ is pseudo umbilical.*

Consider the vector subspaces $S$, $T$, $T^*$, $U_m$ $(m = 0, \ldots, 7)$ of the first normal space $N_x^1(M)$ given by

$$S = Span_{\mathbf{R}}\{\sigma_{00}, \ldots, \sigma_{77}, \sigma_{0^*0^*}, \ldots, \sigma_{7^*7^*}\},$$
$$T = Span_{\mathbf{R}}\{\sigma_{ij}; 0 \le i < j \le 7\},$$
$$T^* = Span_{\mathbf{R}}\{\sigma_{i^*j^*}; 0 \le i < j \le 7\},$$
$$U_m = Span_{\mathbf{R}}\{\sigma_{ij^*}; \kappa_{ij} = m, 0 \le i \le 7, 0 \le j \le 7\}.$$

Then, the relations (3.4), ..., (3.7) tell us that the vector space $N_x^1(M)$ is decomposed into a direct sum of subspaces which are mutually orthogonal:

$$N_x^1(M) = S \oplus T \oplus T^* \oplus U_0 \oplus \cdots \oplus U_7.$$

Evaluating the dimension of each subspace, we obtain the following lemma which gives a necessary condition for an isometric immersion $f$ to be isotropic.

LEMMA 6. *Let $f$ be a $\lambda_x(>0)$-isotropic immersion of $M$ into a real space form $\tilde{M}^{16+p}(\tilde{c})$. Then the dimension of the first normal space $N_x^1(M)$ at the point $x$ of $M$ is equal to either* 9, 10, 126, 127, 135 *or* 136. *Moreover, we have*

(1) $\dim N_x^1(M) = 9 \Leftrightarrow \alpha > 0,\ \tilde{c} = \frac{3}{4}\alpha$ *and* $\lambda_x^2 = \frac{1}{4}\alpha$,
(2) $\dim N_x^1(M) = 10 \Leftrightarrow \alpha > 0,\ \tilde{c} < \frac{3}{4}\alpha$ *and* $\lambda_x^2 = \alpha - \tilde{c}$,
(3) $\dim N_x^1(M) = 126 \Leftrightarrow \alpha < 0,\ \tilde{c} = -\frac{3}{2}\alpha$ *and* $\lambda_x^2 = -\frac{7}{2}\alpha$,
(4) $\dim N_x^1(M) = 127 \Leftrightarrow \alpha < 0,\ \tilde{c} < -\frac{3}{2}\alpha$ *and* $\lambda_x^2 + 5\alpha + \tilde{c} = 0$,
(5) $\dim N_x^1(M) = 135 \Leftrightarrow -\frac{2}{3}\tilde{c} < \alpha < \frac{4}{3}\tilde{c}$ *and* $3\lambda_x^2 + 3\alpha - 5\tilde{c} = 0$,
(6) $\dim N_x^1(M) = 136 \Leftrightarrow \lambda_x^2 + 5\alpha + \tilde{c} > 0,\ 3\lambda_x^2 + 3\alpha - 5\tilde{c} > 0$ *and* $-\frac{5}{7}\lambda_x^2 < \alpha - \tilde{c} < \lambda_x^2$.

*Proof.* Denote by $K(X, Y)$ the sectional curvature of the plane spanned by vectors $X, Y \in T_xM = \mathbf{C}ay \oplus \mathbf{C}ay$. Then we see from (3.2) that

$$K((a,0),(b,0)) = \langle R((a,0),(b,0))(b,0),(a,0)\rangle / \|(a,0)\wedge(b,0)\|^2 = \alpha$$

for $(a,0),(b,0) \in T_xM$ with $(a,0)\wedge(b,0) \neq 0$. So the restriction $\Delta_x|_{\mathbf{C}ay\oplus\{0\}}$ of the discriminant to $\mathbf{C}ay \oplus \{0\}$ is constantly equal to $\alpha - \tilde{c}$ on the linear subspace $\mathbf{C}ay \oplus \{0\}$ of $T_xM$, and hence we can apply Lemma 3 to the subspace $\mathbf{C}ay \oplus \{0\}$. Our discussion is divided into the following three cases: (A) $\alpha - \tilde{c} = \lambda_x^2$, (B) $\alpha - \tilde{c} = -5\lambda_x^2/7$, (C) $-5\lambda_x^2/7 < \alpha - \tilde{c} < \lambda_x^2$.

First, we investigate the case (A). The relation (3.4) reduces to

$$\langle \sigma_{ij}, \sigma_{kl} \rangle = \langle \sigma_{i^*j^*}, \sigma_{k^*l^*} \rangle = \lambda_x^2 \delta_{ij} \delta_{kl}$$

so that we have $\sigma_{00} = \sigma_{11} = \cdots = \sigma_{77}$, $\sigma_{0^*0^*} = \sigma_{1^*1^*} = \cdots = \sigma_{7^*7^*}$ and $\sigma_{ij} = \sigma_{i^*j^*} = 0$ for $i < j$, that is, $\dim S = 1, 2$ and $\dim T = \dim T^* = 0$. The relation (3.5) becomes

$$\langle \sigma_{ij}, \sigma_{k^*l^*} \rangle = \left( \lambda_x^2 - \frac{\alpha}{2} \right) \delta_{ij} \delta_{kl}.$$

We here recall the Gram determinant $G(v_1, \ldots, v_n) (= G(\{v_i\}_{i=1,\ldots,n}))$ of a set of vectors $v_1, \ldots, v_n$ in a real metric vector space given by

$$G(v_1, \ldots, v_n) = \det \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_n \rangle \\ \vdots & \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \cdots & \langle v_n, v_n \rangle \end{pmatrix}.$$

It is well known that $G(v_1, \ldots, v_n) \geq 0$ and that the vectors $v_1, \ldots, v_n$ are linearly independent if and only if the Gram determinant $G(v_1, \ldots, v_n)$ is nonzero. For our vectors $\sigma_{00}, \sigma_{0^*0^*} \in S$, we have

$$G(\sigma_{00}, \sigma_{0^*0^*}) = \det \begin{pmatrix} \lambda_x^2 & \lambda_x^2 - \frac{\alpha}{2} \\ \lambda_x^2 - \frac{\alpha}{2} & \lambda_x^2 \end{pmatrix} = \alpha \left( \lambda_x^2 - \frac{\alpha}{4} \right).$$

This means that $0 < \alpha/4 \leq \lambda_x^2$ and the dimension of the subspace $S$ equals 1 (resp. 2) if and only if $\tilde{c} = 3\alpha/4$ (resp. $\tilde{c} < 3\alpha/4$), because $\alpha - \tilde{c} = \lambda_x^2$.

We see from (3.6) that

$$\|\sigma_{ij^*}\|^2 = \frac{\alpha}{4} \quad \text{and} \quad \langle \sigma_{ij^*}, \sigma_{kl^*} \rangle = \frac{\alpha}{4} \varepsilon_{ij} \varepsilon_{kl} \quad \text{for } \sigma_{ij^*}, \sigma_{kl^*} \in U_m.$$

Hence we have $\dim U_m = 1$ $(m = 0, \ldots, 7)$ so $\dim N_x^1(M) = 9$ or 10 according as $\tilde{c} = 3\alpha/4$ or $\tilde{c} < 3\alpha/4$. This proves the assertion (1), (2).

Next, we study the case (B). Lemma 3 says that $\sum_{i=0}^{7} \sigma_{ii} = 0$ and $\dim \mathit{Span}_{\mathbf{R}} \{\sigma_{ij}; 0 \leq i \leq 7, 0 \leq j \leq 7\} = 35$, so we have $\dim \mathit{Span}_{\mathbf{R}} \{\sigma_{00}, \ldots, \sigma_{66}\} = 7$ and $\dim T = 28$. On the other hand, for vectors $(0, a), (0, b) \in T_x M$ with $(0, a) \wedge (0, b) \neq 0$ we have $K((0, a), (0, b)) = \alpha$. Hence we can apply Lemma 3 to the linear subspace $\{0\} \oplus \mathbf{C}ay$ of $T_x M$. Since $\Delta_x|_{\{0\} \oplus \mathbf{C}ay} = \alpha - \tilde{c} = -5\lambda_x^2/7$, we see $\sum_{i=0}^{7} \sigma_{i^*i^*} = 0$ and $\dim \mathit{Span}_{\mathbf{R}} \{\sigma_{0^*0^*}, \ldots, \sigma_{6^*6^*}\} = 7$, $\dim T^* = 28$ as well. Thus, we find that our immersion $f$ is minimal: $\sum_{i=0}^{7} \sigma_{ii} + \sum_{i=0}^{7} \sigma_{i^*i^*} = 0$. Thanks to Lemma 4, we have $3\lambda_x^2 + 3\alpha - 5\tilde{c} = 0$. This, combined with the equality $\alpha - \tilde{c} = -5\lambda_x^2/7$, gives $\tilde{c} = -3\alpha/2$ and $\lambda_x^2 = -7\alpha/2$. Then, from the relation (3.5) we get

$$\langle \sigma_{ii}, \sigma_{k^*k^*} \rangle = 0,$$

which implies that vectors $\sigma_{00}, \ldots, \sigma_{66}$, $\sigma_{0^*0^*}, \ldots, \sigma_{6^*6^*}$ form an independent system and $\dim S = 14$.

We evaluate the dimension of $U_0 = Span_{\mathbf{R}}\{\sigma_{00^*}, \ldots, \sigma_{77^*}\}$. The relation (3.6) becomes

$$\langle \sigma_{ii^*}, \sigma_{kk^*} \rangle = \frac{\alpha}{4}\varepsilon_{ii}\varepsilon_{kk} - 2\alpha\delta_{ik},$$

so

$$G(\sigma_{00^*}, \ldots, \sigma_{ii^*}) = \det \left( \begin{array}{c|ccc} -\frac{7}{4}\alpha & -\frac{1}{4}\alpha & \cdots & -\frac{1}{4}\alpha \\ \hline -\frac{1}{4}\alpha & -\frac{7}{4}\alpha & & \frac{1}{4}\alpha \\ \vdots & & \ddots & \\ -\frac{1}{4}\alpha & \frac{1}{4}\alpha & & -\frac{7}{4}\alpha \end{array} \right)$$

for $1 \leq i \leq 7$. We see $G(\sigma_{00^*}, \ldots, \sigma_{77^*}) = 0$, $G(\sigma_{00^*}, \ldots, \sigma_{66^*}) > 0$ and we conclude $\dim U_0 = 7$.

For the other subspace $U_m$, we can also see $\dim U_m = 7$. Consequently, we get the assertion (3).

Lastly, we consider the case (C). In this case,

$$\dim Span_{\mathbf{R}}\{\sigma_{ij}; 0 \leq i \leq 7, 0 \leq j \leq 7\} = \dim Span_{\mathbf{R}}\{\sigma_{i^*j^*}; 0 \leq i \leq 7, 0 \leq j \leq 7\} = 36$$

so that $\dim Span_{\mathbf{R}}\{\sigma_{00}, \ldots, \sigma_{77}\} = \dim Span_{\mathbf{R}}\{\sigma_{0^*0^*}, \ldots, \sigma_{7^*7^*}\} = 8$ and $\dim T = \dim T^* = 28$. A computation gives

$$G(\sigma_{00}, \ldots, \sigma_{77}, \sigma_{0^*0^*}, \ldots, \sigma_{7^*7^*}) = \frac{2^{16}}{3^{15}}(\alpha - \tilde{c} - \lambda_x^2)^{14}(5\alpha + \tilde{c} + \lambda_x^2)(3\alpha - 5\tilde{c} + 3\lambda_x^2).$$

Suppose that $5\alpha + \tilde{c} + \lambda_x^2 = 0$. Since $-5\lambda_x^2/7 < \alpha - \tilde{c} < \lambda_x^2$, we have $-2\lambda_x^2/7 < \alpha < 0$. Then

$$G(\sigma_{00}, \ldots, \sigma_{77}, \sigma_{0^*0^*}, \ldots, \sigma_{6^*6^*}) = \frac{2^{14} \cdot 7^{13}}{3^{13}}\alpha^{14}(7\alpha + 2\lambda_x^2) > 0,$$

$$G(\sigma_{00^*}, \ldots, \sigma_{77^*}) = 0, \quad G(\sigma_{00^*}, \ldots, \sigma_{66^*}) = -2^4\alpha^7 > 0,$$

hence $\dim S = 15$ and $\dim U_0 = 7$. We see $\dim U_m = 7 (m = 1, \ldots, 7)$ as well as $U_0$. It follows that $\dim N_x^1(M) = 127$.

If $3\alpha - 5\tilde{c} + 3\lambda_x^2 = 0$, we have $-2\lambda_x^2/7 < \alpha < 4\lambda_x^2$, from which

$$G(\sigma_{00}, \ldots, \sigma_{77}, \sigma_{0^*0^*}, \ldots, \sigma_{6^*6^*}) = \frac{2^{15}}{3^{15} \cdot 5^3}(\alpha - 4\lambda_x^2)^2(7\alpha + 2\lambda_x^2) > 0,$$

$$G(\sigma_{00^*}, \ldots, \sigma_{77^*}) = G(\{\sigma_{ij^*}\}_{\kappa_{ij}=m}) = -\frac{2^9}{3^8 \cdot 5^8}(\alpha - 4\lambda_x^2)^7(7\alpha + 2\lambda_x^2) > 0,$$

and therefore $\dim N_x^1(M) = 135$.

Suppose $(5\alpha+\tilde{c}+\lambda_x^2)(3\alpha-5\tilde{c}+3\lambda_x^2)>0$. Then

$$G(\sigma_{00^*},\dots,\sigma_{77^*})=G(\{\sigma_{ij^*}\}_{\kappa_{ij}=m})=\frac{1}{3^8}(\lambda_x^2-\alpha+\tilde{c})^7(\lambda_x^2+5\alpha+\tilde{c})$$

does not vanish because $\lambda_x^2>\alpha-\tilde{c}$. Hence we have $\lambda_x^2+5\alpha+\tilde{c}>0$, $3\alpha-5\tilde{c}+3\lambda_x^2>0$ and $\dim S=16$, $\dim U_m=8$ $(m=0,\dots,7)$ so that $\dim N_x^1(M)=136$. This completes the proof. □

*Remark* 2. By Lemma 4 and Lemma 6 we find that if the immersion $f$ is minimal then $\tilde{c}>0$ and $\dim N_x^1(M)$ must be 9, 126 or 135.

## 4. The proof of Theorem 1

Our immersion $f$ is a full parallel isotropic immersion of $M$ into a real space form $\tilde{M}^{16+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$. Hence, the mean curvature vector $\mathfrak{h}_f$ of $f$ is parallel. Therefore, as an immediate consequence of Lemma 4 (3.8), we find that the immersion $f$ is constant isotropic. Moreover, we know from Lemma 5 that $f$ is pseudo umbilical. So, by using a similar method in [11] it can be shown that either $M$ is minimal in $\tilde{M}^{16+p}(\tilde{c})$ or $M$ is minimally immersed into a totally umbilical hypersurface $\overline{M}^{16+q}(\tilde{c})$ $(q=p-1)$ of $\tilde{M}^{16+p}(\tilde{c})$ which is orthogonal to the mean curvature vector $\mathfrak{h}_f$. Here note that our submanifold $M$ is not necessarily complete. The above fact holds without the hypothesis of completeness.

First, we consider the case that $M$ is minimal in $\tilde{M}^{16+p}(\tilde{c})$. From Remark 2 we have $\tilde{c}>0$, so that we can regard $M$ as a minimal submanifold of $S^{16+p}(\tilde{c})$ through a constant isotropic immersion $f$ whose isotropy constant is $\lambda$. Let $\iota$ be the natural embedding of $S^{16+p}(\tilde{c})$ into a Euclidean space $\mathbf{R}^{17+p}$. Then, we can see that the immersion $\iota$ is a constant $\sqrt{\tilde{c}}$-isotropic and $\|\mathfrak{h}_\iota\|=\sqrt{\tilde{c}}$. Thus, from Lemma 1, the composition $\iota\circ f: M\to\mathbf{R}^{17+p}$ is a parallel constant isotropic immersion and its isotropy constant is $\sqrt{\lambda^2+\tilde{c}}$. We denote by $D$ and $\nabla^\perp$ the covariant differentiation of $\mathbf{R}^{17+p}$ and that in the normal bundle of $M$ in $\mathbf{R}^{17+p}$, respectively. Let $\gamma=\gamma(s)$ be an arbitrary geodesic in $M$ parametrized by its arclength $s$. We have $\sqrt{\lambda^2+\tilde{c}}=\|\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma})\|$. Set $V=(1/\sqrt{\lambda^2+\tilde{c}})\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma})$. By the formula of Gauss we have

$$D_{\dot{\gamma}}\dot{\gamma}=\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma})=\sqrt{\lambda^2+\tilde{c}}V.$$

Moreover, by the formula of Weingarten

$$D_{\dot{\gamma}}V=(1/\sqrt{\lambda^2+\tilde{c}})\{-A_{\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma})}\dot{\gamma}+\nabla^\perp_{\dot{\gamma}}(\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma}))\}.$$

Since $\iota\circ f$ is isotropic, it follows that $A_{\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma})}\dot{\gamma}=(\lambda^2+\tilde{c})\dot{\gamma}$. In fact, we take a local field of orthonormal frames $\{e_1',e_2',\dots,e_{16}'\}$ around $\gamma(s)\in M$ in such a way that $e_1'=\dot{\gamma}(s)$. Then, owing to Lemma 2 (2), we see

$$\langle A_{\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma})}\dot{\gamma},e_i'\rangle=\langle\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma}),\sigma_{\iota\circ f}(\dot{\gamma},e_i')\rangle=\|\sigma_{\iota\circ f}(\dot{\gamma},\dot{\gamma})\|^2\delta_{1i},$$

which shows the equality. By the fact that $\iota \circ f$ is parallel, we have $\nabla^{\perp}_{\dot{\gamma}}(\sigma_{\iota\circ f}(\dot{\gamma}, \dot{\gamma})) = 0$. Hence we obtain

$$D_{\dot{\gamma}} V = -\sqrt{\lambda^2 + \tilde{c}}\dot{\gamma}.$$

Thus we see that every geodesic in $M$ is a circle in $\mathbf{R}^{17+p}$. In general, an isometric immersion of $M$ into $\tilde{M}$ is called a *planar geodesic immersion* if every geodesic in $M$ is mapped locally into a 2-dimensional totally geodesic submanifolds of $\tilde{M}$. Our immersion $\iota \circ f$ is planar geodesic.

Now, we have the following lemma due to S. L. Hong ([7]):

LEMMA 7. *Let $M$ be an $n$-dimensional $(n \geq 2)$ connected Riemannian manifold, $f : M \to \mathbf{R}^{n+p}$ a planar geodesic immersion. Then $f$ is a constant isotropic immersion. Denote by $\lambda$ its isotropic constant. If $\lambda > 0$, the maximal (resp. minimal) sectional curvature of $M$ is equal to $\lambda^2$ (resp. $\lambda^2/4$), that is,*

$$\frac{1}{4}\lambda^2 \leq K(X, Y) \leq \lambda^2$$

*for any orthonormal vectors $X, Y \in T_x M$.*

Applying this to our case, we can see that $\alpha > 0$ and $\alpha = \lambda^2 + \tilde{c}$. Therefore, by Lemma 6 and Remark 2, we conclude that $p = 9$.

Next, we investigate the case that $M$ is minimally immersed into a totally umbilical hypersurface $\overline{M}^{16+q}(\bar{c})$. Since the totally umbilical immersion of $\overline{M}^{16+q}(\bar{c})$ into $\tilde{M}^{16+p}(\tilde{c})$ is $\sqrt{\bar{c} - \tilde{c}}$-isotropic and its mean curvature vector is parallel, by virture of Lemma 1, the minimal immersion $f' : M \to \overline{M}^{16+q}(\bar{c})$ is a parallel constant isotropic immersion with its isotropy constant $\sqrt{\lambda^2 - (\bar{c} - \tilde{c})}$. So, by Remark 2, we have $\bar{c} > 0$. Along the same argument as above, we consider the composition of $\iota' \circ f'$ of a minimal immersion $f' : M \to S^{16+q}(\bar{c})$ and the natural embedding $\iota' : S^{16+q}(\bar{c}) \to \mathbf{R}^{17+q}$. Since, from Lemma 1 again, the immersion $\iota' \circ f'$ is a parallel constant isotropic immersion whose isotropy constant is equal to $\sqrt{\{\lambda^2 - (\bar{c} - \tilde{c})\} + \bar{c}} = \sqrt{\lambda^2 + \tilde{c}}$, it follows that $\alpha > 0$ and $q = 9$.

Y. Agaoka and E. Kaneda proved the following rigidity theorem ([3]):

THEOREM 3. *Let $f_0$ be the canonical isometric embedding of $\mathbf{C}ayP^2(\alpha)$ into the Euclidean space $\mathbf{R}^{26}$. Then, for any isometric immersion $f$ defined on a connected open set of $\mathbf{C}ayP^2(\alpha)$ into $\mathbf{R}^{26}$, there exists a Euclidean transformation $A$ of $\mathbf{R}^{26}$ satisfying $f = A \circ f_0$.*

Thanks to their result, we find that our immersion $f$ is locally congruent to either the first standard minimal immersion $\mathbf{C}ayP^2(\alpha) \to S^{25}(3\alpha/4)$ or the composition of the first standard minimal immersion and a totally umbilical immersion $\mathbf{C}ayP^2(\alpha) \to S^{25}(3\alpha/4) \to \tilde{M}^{26}(\tilde{c})$ $(3\alpha/4 \geq \tilde{c})$.

## 5. Cayley Frenet curves

We consider a Frenet curve $\gamma$ of proper order 2 in $\mathbf{C}ayP^2(\alpha)$ $(\alpha > 0)$ or $\mathbf{C}ayH^2(\alpha)$ $(\alpha < 0)$. We can see from (1.1) that the sectional curvature $K(\dot{\gamma}, V)$ given by the osculating plane spanned by $\dot{\gamma}$ and $V$ is constant along $\gamma$. Indeed, since $\nabla R \equiv 0$ we have

$$\begin{aligned}
&\nabla_{\dot{\gamma}}\langle R(\dot{\gamma}, V)V, \dot{\gamma}\rangle \\
&\quad = \langle R(\nabla_{\dot{\gamma}}\dot{\gamma}, V)V + R(\dot{\gamma}, \nabla_{\dot{\gamma}}V)V + R(\dot{\gamma}, V)\nabla_{\dot{\gamma}}V, \dot{\gamma}\rangle + \langle R(\dot{\gamma}, V)V, \nabla_{\dot{\gamma}}\dot{\gamma}\rangle \\
&\quad = \kappa\langle R(V, V)V, \dot{\gamma}\rangle - \kappa\langle R(\dot{\gamma}, \dot{\gamma})V, \dot{\gamma}\rangle - \kappa\langle R(\dot{\gamma}, V)\dot{\gamma}, \dot{\gamma}\rangle + \kappa\langle R(\dot{\gamma}, V)V, V\rangle \\
&\quad = 0.
\end{aligned}$$

A Frenet curve $\gamma$ of proper order 2 which satisfies $K(\dot{\gamma}, V) = \alpha$ is called a *Cayley Frenet curve*. If the curvature $\kappa$ of a Cayley Frenet curve $\gamma$ is constant, namely if $\gamma$ is a circle, we call $\gamma$ a *Cayley circle*. We regard a geodesic as a Cayley circle of null curvature.

A curve $\gamma$ in a Riemannian manifold $M$ is called a *plane curve* if the curve $\gamma$ is locally contained in some 2-dimensional totally geodesic submanifold of $M$. As a matter of course, every plane curve with positive curvature function is a Frenet curve of proper order 2. But in general, the converse does not hold. In the case that the space $M$ is a real space form $\tilde{M}^m(\tilde{c})$ of constant curvature $\tilde{c}$, it is easy to see that a curve $\gamma$ is a Frenet curve of proper order 2 if and only if the curve $\gamma$ is a plane curve with positive curvature.

Suppose that a connected open submanifold $M$ of $\mathbf{C}ayP^2(\alpha)$ or $\mathbf{C}ayH^2(\alpha)$ is isometrically immersed into a real space form $\tilde{M}^{16+p}(\tilde{c})$ through an immersion $f$. It is needless to say that the extrinsic shape $f \circ \gamma$ of a Frenet curve $\gamma$ of proper order 2 in $M$ is not always a plane curve in the ambient space $\tilde{M}^{16+p}(\tilde{c})$. However, we have the following ([2]):

PROPOSITION 1. *The immersions* (1), (2) *given in Theorem* 2 *map every Cayley circle in* $\mathbf{C}ayP^2(\alpha)$ *to a circle in the ambient space.*

## 6. The proof of Theorem 2

The proof is similar to that in [14]. But for readers we explain it in detail. Let $x$ be an arbitrary point of $M$ and $X \in T_xM$ an arbitrary unit vector. Let $\gamma = \gamma(s)$ be a Cayley Frenet curve in $M$ satisfying the equations (1.1) and the initial condition $\gamma(0) = x$, $\dot{\gamma}(0) = X$ and $K(X, V(0)) = \alpha$. Since the curve $f \circ \gamma$ is a plane curve in $\tilde{M}^{16+p}(\tilde{c})$ by assumption, there exist a (nonnegative) function $\tilde{\kappa} = \tilde{\kappa}(s)$ and a field of unit vectors $\tilde{V} = \tilde{V}(s)$ along $f \circ \gamma$ in $\tilde{M}^{16+p}(\tilde{c})$ which satisfy that

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \tilde{\kappa}\tilde{V}, \quad \tilde{\nabla}_{\dot{\gamma}}\tilde{V} = -\tilde{\kappa}\dot{\gamma}. \tag{6.1}$$

Then by the formula of Gauss, we have

$$\tilde{\kappa}\tilde{V} = \kappa V + \sigma(\dot{\gamma}, \dot{\gamma}), \tag{6.2}$$

hence

$$\tilde{\kappa}^2 = \kappa^2 + \|\sigma(\dot{\gamma},\dot{\gamma})\|^2. \tag{6.3}$$

The function $\tilde{\kappa}$ is positive because $\kappa > 0$.

For the left-hand side of (6.2), by using (6.1) and (6.2) again, we get

$$\tilde{\kappa}\tilde{\nabla}_{\dot{\gamma}}(\tilde{\kappa}\tilde{V}) = \tilde{\kappa}\dot{\tilde{\kappa}}\tilde{V} - \tilde{\kappa}^3\dot{\gamma} = \dot{\tilde{\kappa}}\{\kappa V + \sigma(\dot{\gamma},\dot{\gamma})\} - \tilde{\kappa}^3\dot{\gamma}. \tag{6.4}$$

And for the right-hand side of (6.2), by the formulae of Gauss and Weingarten we have

$$\begin{aligned}\tilde{\kappa}\tilde{\nabla}_{\dot{\gamma}}\{\kappa V + \sigma(\dot{\gamma},\dot{\gamma})\} &= \tilde{\kappa}\{\dot{\kappa}V + \kappa\tilde{\nabla}_{\dot{\gamma}}V - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + \nabla^{\perp}_{\dot{\gamma}}(\sigma(\dot{\gamma},\dot{\gamma}))\} \\ &= \tilde{\kappa}\{\dot{\kappa}V + \kappa(\nabla_{\dot{\gamma}}V + \sigma(\dot{\gamma},V)) - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + (\nabla'_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}) + 2\sigma(\nabla_{\dot{\gamma}}\dot{\gamma},\dot{\gamma})\} \\ &= \tilde{\kappa}\{\dot{\kappa}V - \kappa^2\dot{\gamma} + 3\kappa\sigma(\dot{\gamma},V) - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + (\nabla'_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\}.\end{aligned} \tag{6.5}$$

We compare the tangential components and the normal components for the submanifold $M$ in (6.4) and (6.5), respectively. Then we obtain the following equations:

$$\dot{\tilde{\kappa}}\kappa V - \tilde{\kappa}^3\dot{\gamma} = \tilde{\kappa}\{\dot{\kappa}V - \kappa^2\dot{\gamma} - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma}\}, \tag{6.6}$$

$$\dot{\tilde{\kappa}}\sigma(\dot{\gamma},\dot{\gamma}) = \tilde{\kappa}\{3\kappa\sigma(\dot{\gamma},V) + (\nabla'_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\}. \tag{6.7}$$

Differentiating the both sides of (6.3), we have

$$\tilde{\kappa}\dot{\tilde{\kappa}} = \kappa\dot{\kappa} + \langle(\nabla'_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle + 2\kappa\langle\sigma(V,\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle. \tag{6.8}$$

On the other hand, the equation (6.7) yields

$$\tilde{\kappa}\dot{\tilde{\kappa}}\sigma(\dot{\gamma},\dot{\gamma}) = \tilde{\kappa}^2\{3\kappa\sigma(\dot{\gamma},V) + (\nabla'_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\}. \tag{6.9}$$

Substitute (6.3) and (6.8) into (6.9) and set $s = 0$. Then we have

$$\begin{aligned}&\{\kappa(0)\dot{\kappa}(0) + \langle(\nabla'_X\sigma)(X,X),\sigma(X,X)\rangle \\ &\qquad + 2\kappa(0)\langle\sigma(X,X),\sigma(X,V(0))\rangle\}\sigma(X,X) \\ &\quad = \{\kappa(0)^2 + \|\sigma(X,X)\|^2\}\{3\kappa(0)\sigma(X,V(0)) + (\nabla'_X\sigma)(X,X)\}.\end{aligned} \tag{6.10}$$

We note that there exists another Cayley Frenet curve $\gamma_1 = \gamma_1(s)$ with the same curvature $\kappa$ in $M$ satisfying $\nabla_{\dot{\gamma}_1}\dot{\gamma}_1 = \kappa V_1$ and $\nabla_{\dot{\gamma}_1}V_1 = -\kappa\dot{\gamma}_1$ with the initial condition $\gamma_1(0) = x$, $\dot{\gamma}_1(0) = X$ and $V_1(0) = -V(0)$. Then the equality (6.10) for $\gamma_1$ turns to

$$\begin{aligned}&\{\kappa(0)\dot{\kappa}(0) + \langle(\nabla'_X\sigma)(X,X),\sigma(X,X)\rangle \\ &\qquad - 2\kappa(0)\langle\sigma(X,X),\sigma(X,V(0))\rangle\}\sigma(X,X) \\ &\quad = \{\kappa(0)^2 + \|\sigma(X,X)\|^2\}\{-3\kappa(0)\sigma(X,V(0)) + (\nabla'_X\sigma)(X,X)\}.\end{aligned} \tag{6.11}$$

Therefore, from (6.10) and (6.11) we obtain

$$2\langle\sigma(X,X),\sigma(X,V(0))\rangle\sigma(X,X)=3\{\kappa(0)^2+\|\sigma(X,X)\|^2\}\sigma(X,V(0)). \tag{6.12}$$

Taking the inner product of both sides of (6.12) with $\sigma(X,X)$, we get

$$\begin{aligned}&2\langle\sigma(X,X),\sigma(X,V(0))\rangle\|\sigma(X,X)\|^2\\&\quad=3\{\kappa(0)^2+\|\sigma(X,X)\|^2\}\langle\sigma(X,X),\sigma(X,V(0))\rangle\end{aligned}$$

and hence

$$\{3\kappa(0)^2+\|\sigma(X,X)\|^2\}\langle\sigma(X,X),\sigma(X,V(0))\rangle=0.$$

Since $3\kappa(0)^2+\|\sigma(X,X)\|^2>0$, we have $\langle\sigma(X,X),\sigma(X,V(0))\rangle=0$. Thus, again from (6.12), we see that $\sigma(X,V(0))=0$ holds for any $X\in T_xM$ and any $V(0)\in T_xM$ satisfying $K(X,V(0))=\alpha$ at an arbitrary point $x\in M$. It follows that

$$\sigma(\dot\gamma,V)=0 \quad \text{along } \gamma. \tag{6.13}$$

Taking the inner product of both sides of (6.6) with $V$, we have

$$\begin{aligned}\dot{\tilde\kappa}\kappa&=\tilde\kappa\dot\kappa-\tilde\kappa\langle A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma,V\rangle\\&=\tilde\kappa\dot\kappa-\tilde\kappa\langle\sigma(\dot\gamma,\dot\gamma),\sigma(\dot\gamma,V)\rangle.\end{aligned}$$

Owing to (6.13), the above equation becomes

$$\dot{\tilde\kappa}\kappa=\tilde\kappa\dot\kappa. \tag{6.14}$$

Then the equation (6.7), together with (6.13) and (6.14), yields

$$(\nabla'_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)=\frac{\dot{\tilde\kappa}}{\tilde\kappa}\sigma(\dot\gamma,\dot\gamma)=\frac{\dot\kappa}{\kappa}\sigma(\dot\gamma,\dot\gamma),$$

and

$$(\nabla'_X\sigma)(X,X)=\frac{\dot\kappa(0)}{\kappa(0)}\sigma(X,X).$$

Changing $X$ into $-X$, we get $(\nabla'_X\sigma)(X,X)=0$. Thanks to Codazzi's equation in a space of constant curvature $(\nabla'_X\sigma)(Y,Z)=(\nabla'_Y\sigma)(X,Z)$, the immersion $f$ is parallel.

Next, by (6.14) we see that the equation (6.6) reduces to $A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma=(\tilde\kappa^2-\kappa^2)\dot\gamma$. Therefore

$$\langle\sigma(X,X),\sigma(X,Y)\rangle=\langle A_{\sigma(X,X)}X,Y\rangle=0$$

for any orthonormal pair of vectors $X,Y\in T_xM$ at each point $x\in M$. Thus, by virtue of Lemma 2, the immersion $f$ is isotropic at each point $x\in M$. Hence, as mentioned in the proof of Theorem 1, $f$ is constant isotropic.

Now, since our immersion is parallel, we can see that the first normal space $N^1(M) = \bigcup N_x^1(M)$ is invariant under parallel translations with respect to the connection in the normal bundle and the dimension of $N_x^1(M)$ is constant on $M$. If the immersion $f$ is not full, thanks to a theorem of J. Erbacher [5], there exists a totally geodesic submanifold $\overline{M}^{16+q}(\tilde{c})$ of $\tilde{M}^{16+p}(\tilde{c})$ of dimension $16+q$ such that $f(M) \subset \overline{M}^{16+q}(\tilde{c})$, where $q = \dim N^1(M)$. Then the immersion $f : M \to \overline{M}^{16+q}(\tilde{c})$ is a full parallel isotropic immersion. Hence, by Theorem 1, our immersion $f$ is locally congruent to one of the examples (1), (2) in Theorem 2.

Finally, we shall show that the curve $\gamma$ satisfying the hypothesis of Theorem 2 is a Cayley circle. Assume that the curvature $\kappa$ is not constant. Then there exists some $s_0$ with $\dot{\kappa}(s_0) \neq 0$. Since $\kappa, \tilde{\kappa} > 0$, we find $\dot{\tilde{\kappa}}(s_0) \neq 0$ from (6.14). From the fact that $\nabla'\sigma = 0$ and (6.13) we can see that the equation (6.7) yields $\sigma(\dot{\gamma}(s_0), \dot{\gamma}(s_0)) = 0$. As we know that $f$ is constant isotropic, we conclude $\sigma(X, X) = 0$ for an arbitrary unit vector $X \in T_xM$ at each point $x \in M$. Hence the immersion $f : M \to \tilde{M}^{16+p}(\tilde{c})$ is totally geodesic. But it is known that the manifold $M$ cannot be immersed into a real space form as a totally geodesic submanifold. Thus we have a contradiction, so that the curve $\gamma$ is a Cayley circle. This completes the proof.

*Remark* 3. Recently, T. Adachi and T. Sugiyama characterized some isometric immersions from the view point of *curvature logarithmic derivatives* of curves. See for example [1].

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