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HOPF HYPERSURFACES OF LOW TYPE IN NON-FLAT COMPLEX SPACE FORMS

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Abstract

We classify Hopf hypersurfaces of non-flat complex space forms $\mathbf{C}P^m(4)$ and $\mathbf{C}H^m(-4)$, denoted jointly by $\mathbf{C}Q^m(4c)$, that are of 2-type in the sense of B. Y. Chen, via the embedding into a suitable (pseudo) Euclidean space of Hermitian matrices by projection operators. This complements and extends earlier classifications by Martinez and Ros (the minimal case) and Udagawa (the CMC case), who studied only hypersurfaces of $\mathbf{C}P^m$ and assumed them to have constant mean curvature instead of being Hopf. Moreover, we rectify some claims in Udagawa's paper to give a complete classification of constant-mean-curvature-hypersurfaces of 2-type. We also derive a certain characterization of CMC Hopf hypersurfaces which are of 3-type and mass-symmetric in a naturally-defined hyperquadric containing the image of $\mathbf{C}Q^m(4c)$ via these embeddings. The classification of such hypersurfaces is done in $\mathbf{C}Q^2(4c)$, under an additional assumption in the hyperbolic case that the mean curvature is not equal to $\pm 2/3$. In the process we show that every standard example of class B in $\mathbf{C}Q^m(4c)$ is mass-symmetric and we determine its Chen-type.

1. Introduction

The study of finite-type submanifolds of Euclidean and pseudo-Euclidean spaces has been an area of flourishing research initiated by B. Y. Chen in the 1980s [9]. Many geometers contributed to the theory and quite a number of important and interesting results coming from that study have been obtained on sharp eigenvalue estimates and characterizations of certain submanifolds by eigenvalue equalities [10]. A Riemannian n-manifold M^n isometrically immersed into a Euclidean or pseudo-Euclidean space by $x : M^n \to E^N_{(K)}$ is said to be of k-type (more precisely of Chen k-type) in $E^N_{(K)}$ if the position vector x can be decomposed, up to a translation by a constant vector x_0, into a sum of k

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nonconstant $E^N_{(K)}$-valued eigenfunctions of the Laplacian Δ_M from different eigenspaces, viz.

$$(1) \qquad x = x_0 + x_{t_1} + \cdots + x_{t_k}; \quad x_0 = \text{const}, \quad \Delta x_{t_i} = \lambda_{t_i} x_{t_i}, \quad i = 1, \dots, k,$$

where $x_{t_i} \neq$ const, $\lambda_{t_i} \in \mathbf{R}$ are all different, and the Laplacian acts on a vector-valued function componentwise. For a compact submanifold, the constant part x_0 is the center of mass and if x immerses M^n into a central hyperquadric of a Euclidean or pseudo-Euclidean space the immersion is said to be mass-symmetric in that hyperquadric if x_0 coincides with the center of the said hyperquadric. Moreover, decomposition (1) also makes sense for noncompact submanifolds, but x_0 may not be uniquely determined, namely when one of the eigenvalues λ_{t_i} above is zero. Such submanifolds are said to be of null k-type, and are, therefore, per definition mass-symmetric.

The study of finite-type submanifolds therefore treats an interesting question: To what extent is the geometric structure of a submanifold determined by a simple analytic information, that is, by the spectral resolution (1) of the immersion into finitely many terms? By placing a complex projective or a complex hyperbolic space into a suitable (pseudo) Euclidean space of Hermitian matrices using the embedding Φ by projectors in the standard way (cf. [31], [28], [29], [18], [15]), it is possible to study submanifolds, in particular hypersurfaces, of a complex space form in terms of finite-type property, where the immersion considered is the composite immersion with Φ. It is well-known that a 1-type submanifold is minimal in an appropriate hyperquadric of the ambient (pseudo) Euclidean space. 1-Type real hypersurfaces of a complex space form $\mathbf{C}Q^m$ were previously studied in [23], [18], and the present author subsequently classified 1-type submanifolds of these spaces of any dimension (see [14], [15]). In particular, 1-type hypersurface in $\mathbf{C}P^m(4)$ is a geodesic hypersphere of radius $r = \arctan\sqrt{2m+1}$, which has an interesting stability property [23], [15]. Type-2 (also called bi-order) hypersurfaces in the complex projective space were studied by Martinez and Ros [23] (the minimal case) and Udagawa [36], who classified them under the assumption that they have constant mean curvature. However, Udagawa's classification is incomplete and has some deficiencies which we rectify here. First, it was claimed without proof in [36, p. 194] that there are no 2-type hypersurfaces in $\mathbf{C}P^m$ among homogeneous examples of class B. We find a counterexample to this claim, producing two such hypersurfaces. Second, it was claimed in the same paper (pp. 192–193) that there are no geodesic hyperspheres (i.e. class-A_1 hypersurfaces) in $\mathbf{C}P^m$ which are mass-symmetric and of 2-type, whereas we prove that a geodesic hypersphere of radius $\cot^{-1}(1/\sqrt{m})$ exactly has these properties. Because of these erroneous claims, all three theorems of [36] are deficient in one way or another.

Kähler submanifolds of $\mathbf{C}P^m(4)$ of 2-type were successfully studied and classified in works of Ros [29] and Udagawa [35], whereas Shen [30] produced a classification of minimal surfaces (real dimension 2) in $\mathbf{C}P^m(4)$ of 2-type. On the other hand, there are only scant results so far on 3-type submanifolds of complex space forms (see [33], [34]) and their further study is warranted. An overview of the results on low-type submanifolds of projective and hyperbolic spaces via the immersion by projectors is presented in [16].

In this paper we further advance the study of hypersurfaces of non-flat complex space forms (that is, of both complex projective and complex hyperbolic space) which are of 2- or 3-type and produce some new classification results, with the starting (weaker) assumption that the hypersurfaces possess some simple compatibility property between the complex structure of the ambient space and the second fundamental form. One of the most studied kinds of hypersurfaces in complex space forms are the so-called Hopf hypersurfaces [3], [11], [25], defined by the property that the (almost contact) structure vector $U := -J\xi$, where ξ is the unit normal, is a principal curvature vector (i.e. proper for the shape operator). Equivalently, they are defined by integral curves of the structure vector field U being geodesics and in $\mathbf{C}P^m$ they are realized as tubes about complex submanifolds when the corresponding focal set has constant rank [8]. The above-mentioned examples of 2-type hypersurfaces studied in [23] and [36] are in fact certain homogeneous Hopf hypersurfaces. One of our results is that a 2-type Hopf hypersurface indeed has constant mean curvature, the key result towards their classification given in Theorems 1 and 2. Kähler submanifolds of 3-type in complex projective spaces are studied in [33], [34], where some examples are given, including compact irreducible Hermitian symmetric submanifolds of degree 3. In this paper we also undertake a study of 3-type Hopf hypersurfaces with constant mean curvature in non-Euclidean complex space forms, fulfilling the promise made in [13], based on the study of spherical hypersurfaces of constant mean curvature which are of 3-type via the second standard immersion of the unit sphere (see also [17]). Along the way we obtain a generalization of Nomizu-Smyth's formula for the trace Laplacian of the shape operator [26], and Simons'-type formula for the Laplacian of the squared norm of the second fundamental form, which may be useful in other contexts. For the background and additional clarification of the notation used in this article a reader should consult [15]. Excellent references on the geometry of hypersurfaces of complex space forms are [3], [4], [25], and a brief overview [5].

2. The basic background and relevant formulas

Let $\mathbf{C}Q^m(4c)$ denote m-dimensional non-flat model complex space form, that is either the complex projective space $\mathbf{C}P^m(4)$ or the complex hyperbolic space $\mathbf{C}H^m(-4)$ of constant holomorphic sectional curvature $4c$ $(c = \pm 1)$. By using a particular (pseudo) Riemannian submersion one can construct $\mathbf{C}Q^m$ and its embedding Φ into a certain (pseudo) Euclidean space of matrices. Consider first Hermitian form Ψ_c on $\mathbf{C}^{m+1}$ given by $\Psi_c(z,w) = c\bar{z}_0 w_0 + \sum_{j=1}^m \bar{z}_j w_j,\ z, w \in \mathbf{C}^{m+1}$ with the associated (pseudo) Riemannian metric $g_c = \operatorname{Re}\Psi_c$ and the quadric hypersurface $N^{2m+1} := \{z \in \mathbf{C}^{m+1} \mid \Psi_c(z,z) = c\}$. When $c = 1$, N^{2m+1} is the ordinary hypersphere S^{2m+1} of $\mathbf{C}^{m+1} = \mathbf{R}^{2m+2}$ and when $c = -1$, N^{2m+1} is the anti-de Sitter space H_1^{2m+1} in $\mathbf{C}_1^{m+1}$. The orbit space under the natural action of the circle group S^1 on N^{2m+1} defines $\mathbf{C}Q^m(4c)$, which is then the base space of a (pseudo) Riemannian submersion with totally geodesic fibers. The standard embedding Φ into the set of Ψ-Hermitian matrices $H^{(1)}(m+1)$ is achieved by identifying a point, that is a complex line (or a time-like complex line in the hyperbolic case) with the projection operator onto it. Then one gets the following matrix representation of Φ at a point $p = [z]$, where $z = (z_j) \in N^{2m+1} \subset \mathbf{C}^{m+1}_{(1)}$

$$\Phi([z]) = \begin{pmatrix} |z_0|^2 & cz_0\bar{z}_1 & \cdots & cz_0\bar{z}_m \\ z_1\bar{z}_0 & c|z_1|^2 & \cdots & cz_1\bar{z}_m \\ \vdots & \vdots & \ddots & \vdots \\ z_m\bar{z}_0 & cz_m\bar{z}_1 & \cdots & c|z_m|^2 \end{pmatrix}. \tag{2}$$

The second fundamental form σ of this embedding is parallel and the image $\Phi(\mathbf{C}Q^m)$ of the space form is contained in the hyperquadric of $H^{(1)}(m+1)$ centered at $I/(m+1)$ and defined by the equation

$$\langle P - I/(m+1), P - I/(m+1)\rangle = \frac{cm}{2(m+1)},$$

where I denotes the $(m+1)\times(m+1)$ identity matrix. For the fundamental properties of the embedding Φ see [31], [18], [28], [15].

If now $x : M^n \to \mathbf{C}Q^m(4c)$ is an isometric immersion of a Riemannian n-manifold as a real hypersurface of a complex space form $(n = 2m-1)$ then we have the associated composite immersion $\tilde{x} = \Phi \circ x$, which realizes M as a submanifold of the (pseudo) Euclidean space $E^N_{(K)} := H^{(1)}(m+1)$, equipped with the usual trace metric $\langle A, B\rangle = \frac{c}{2}\operatorname{tr}(AB)$. In this notation the subscripts and superscripts in parenthesis are present only in relation to $\mathbf{C}H^m$, so that the superscript 1 in $H^{(1)}(m+1)$ is optional and appears only in the hyperbolic case, since the construction of the embedding is based on the form Ψ in $\mathbf{C}^{m+1}_1$ of index 1.

Let ξ be a local unit vector field normal to M in $\mathbf{C}Q^m$, A the shape operator of the immersion x, and let $\alpha = (1/n)\operatorname{tr} A$ be the mean curvature of M in $\mathbf{C}Q^m$, so that the mean curvature vector H of the immersion equals $H = \alpha\xi$. Further, let $\overline{\nabla}$, $\bar{A}$, $\overline{D}$, denote respectively the Levi-Civita connection, the shape operator, and the metric connection in the normal bundle, related to $\mathbf{C}Q^m$ and the embedding Φ. Let the same letters without bar denote the respective objects for a submanifold M and the immersion x, whereas we use the same symbols with tilde to denote the corresponding objects related to the composite immersion $\tilde{x} := \Phi \circ x$ of M into the (pseudo) Euclidean space $H^{(1)}(m+1)$. As usual, we use σ for the second fundamental form of $\mathbf{C}Q^m$ in $E^N_{(K)}$ via Φ and h for the second fundamental form of a submanifold M in $\mathbf{C}Q^m$. An orthonormal basis of the tangent space T_pM at a general point will be denoted by $\{e_i\}$, $i = 1, 2, \ldots, n$. In general, indices i, j will range from 1 to n and Γ will denote the set of all (local) smooth sections of a bundle.

We give first some important formulas which will be repeatedly used throughout this paper. For a general submanifold M, local tangent fields $X, Y \in \Gamma(TM)$ and a local normal field $\xi \in \Gamma(T^{\perp}M)$, the formulas of Gauss and Weingarten are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y); \quad \bar{\nabla}_X \xi = -A_\xi X + D_X \xi. \tag{3}$$

In particular, for a hypersurface of a complex space form $\mathbf{C}Q^m$ with unit normal vector ξ and the corresponding shape operator A, they become

$$\bar{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle\xi; \quad \bar{\nabla}_X \xi = -AX \tag{4}$$

Let J be the Kähler almost complex structure of $\mathbf{C}Q^m$, and U be the distinguished tangent vector field $U := -J\xi$. Define an endomorphism S of the tangent space and a normal bundle valued 1-form F by

$$SX = (JX)_T, \quad FX = (JX)_N = \langle X, U\rangle\xi,$$

i.e for $X \in \Gamma(TM)$, $JX = SX + FX$ is the decomposition of JX into tangential and normal to submanifold parts. Then the following formulas are well known [3], [25]:

$$SU = 0, \quad SX = JX - \langle X, U\rangle\xi, \quad S^2X = -X + \langle X, U\rangle U \tag{5}$$

$$\nabla_X U = SAX, \quad (\nabla_X S)Y = \langle Y, U\rangle AX - \langle AX, Y\rangle U. \tag{6}$$

The curvature tensor of $\mathbf{C}Q^m(4c)$ is given by

$$\begin{aligned}\bar{R}(X, Y)Z = c[&\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX \\ &- \langle JX, Z\rangle JY - 2\langle JX, Y\rangle JZ],\end{aligned} \tag{7}$$

and the equations of Codazzi and Gauss for a hypersurface of $\mathbf{C}Q^m(4c)$ are respectively given by

$$(\nabla_X A)Y - (\nabla_Y A)X = c[\langle X, U\rangle SY - \langle Y, U\rangle SX - 2\langle SX, Y\rangle U], \tag{8}$$

$$\begin{aligned}R(X, Y)Z = c[&\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle SY, Z\rangle SX \\ &- \langle SX, Z\rangle SY - 2\langle SX, Y\rangle SZ] \\ &+ \langle AY, Z\rangle AX - \langle AX, Z\rangle AY.\end{aligned} \tag{9}$$

The following formulas of A. Ros for the shape operator of Φ in the direction of $\sigma(X, Y)$ are also well known, (see, for example, [28], [29] and [18])

$$\begin{aligned}\langle\sigma(X, Y), \sigma(V, W)\rangle = c[&2\langle X, Y\rangle\langle V, W\rangle + \langle X, V\rangle\langle Y, W\rangle \\ &+ \langle X, W\rangle\langle Y, V\rangle + \langle JX, V\rangle\langle JY, W\rangle \\ &+ \langle JX, W\rangle\langle JY, V\rangle],\end{aligned} \tag{10}$$

$$\begin{aligned}\bar{A}_{\sigma(X, Y)} V = c[&2\langle X, Y\rangle V + \langle X, V\rangle Y + \langle Y, V\rangle X \\ &+ \langle JX, V\rangle JY + \langle JY, V\rangle JX].\end{aligned} \tag{11}$$

One also verifies

(12) $\quad \sigma(JX, JY) = \sigma(X, Y), \quad \langle \sigma(X, Y), \tilde{x} \rangle = -\langle X, Y \rangle, \quad \langle \sigma(X, Y), I \rangle = 0.$

The gradient of a smooth function f is a vector field $\nabla f := \sum_i (e_i f) e_i$. The Hessian of f is a symmetric tensor field defined by

$$\mathrm{Hess}_f(X, Y) = \langle \nabla_X(\nabla f), Y \rangle = XYf - (\nabla_X Y) f,$$

and the Laplacian acting on smooth functions is defined as $\Delta f = -\mathrm{tr}\, \mathrm{Hess}_f$. The Laplace operator can be extended to act on a vector field V along $\tilde{x}(M)$ by

$$\Delta V = \sum_i [\tilde{\nabla}_{\nabla_{e_i} e_i} V - \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} V].$$

The product formula for the Laplacian, which will be often used in the ensuing computations, is

$$\Delta(fg) = (\Delta f) g + f(\Delta g) - 2 \sum_i (e_i f)(e_i g), \tag{13}$$

for smooth functions $f, g \in C^\infty(M)$, and it can then be extended to hold for the scalar product of vector valued functions, and thus also for product of matrices, in a natural way. We shall use the notation $f_k := \mathrm{tr}\, A^k$, and in particular $f := f_1 = \mathrm{tr}\, A$. For an endomorphism B of the tangent space of M we define $\mathrm{tr}(\nabla B) := \sum_{i=1}^n (\nabla_{e_i} B) e_i$. We shall assume all manifolds to be smooth and connected, but not necessarily compact.

3. Iterated Laplacians of a real hypersurface

Recall that

$$\Delta \tilde{x} = -n\tilde{H} = -f\xi - \sum_{i=1}^n \sigma(e_i, e_i), \tag{14}$$

where here, and in the following, we understand the Laplacian Δ of M to be applied to vector fields along M (viewed as $E^N_{(K)}$-valued functions, i.e. matrices) componentwise.

By the product formula above we have

$$\Delta^2 \tilde{x} := \Delta(\Delta \tilde{x}) = -(\Delta f)\xi - f(\Delta \xi) + 2\sigma(\nabla f, \xi) - 2A(\nabla f) - \sum_i \Delta(\sigma(e_i, e_i)). \tag{15}$$

Further,

$$\begin{aligned}
\Delta \xi &= \sum_i [\tilde{\nabla}_{\nabla_{e_i} e_i} \xi - \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \xi] \\
&= \sum_i [-A(\nabla_{e_i} e_i) + \sigma(\nabla_{e_i} e_i, \xi) + \overline{\nabla}_{e_i}(Ae_i) + \sigma(e_i, Ae_i) \\
&\qquad + \bar{A}_{\sigma(e_i, \xi)} e_i - \overline{D}_{e_i}(\sigma(e_i, \xi))].
\end{aligned}$$

Using (11), the parallelism of σ, and the fact that $\mathrm{tr}(\nabla A) = \nabla(\mathrm{tr}\, A) = \nabla f$ (by virtue of the Codazzi equation), we obtain

$$\Delta\xi = \nabla f + [f_2 + c(n-1)]\xi - f\sigma(\xi,\xi) + 2\sum_i \sigma(e_i, Ae_i). \tag{16}$$

One further computes

$$\begin{aligned}\sum_i \Delta(\sigma(e_i,e_i)) &= -4cJAU + 2c(n+3)f\xi + 2c(n+2)\sum_i \sigma(e_i,e_i)\\ &\quad + 2\sum_i \sigma(Ae_i,Ae_i) - 2\sigma(\xi,\nabla f) - 2(c+f_2)\sigma(\xi,\xi).\end{aligned} \tag{17}$$

Combining formulas (15)–(17) we finally obtain

$$\begin{aligned}\Delta^2\tilde{x} = &-[\Delta f + f(f_2 + c(3n+5)) - 4c\langle AU, U\rangle]\xi + 4cSAU\\ &- f\nabla f - 2A(\nabla f) + (2c + 2f_2 + f^2)\sigma(\xi,\xi) + 4\sigma(\nabla f,\xi)\\ &- 2c(n+2)\sum_i \sigma(e_i,e_i) - 2f\sum_i \sigma(e_i,Ae_i) - 2\sum_i \sigma(Ae_i,Ae_i).\end{aligned} \tag{18}$$

Compare this with formula (2.15) of [36], formula (2.9) of [18], and formula (2.8) of [13].

Let us now find $\Delta^3\tilde{x}$. The computation is long but straightforward, so we just outline the main steps. First we shall compute the trace-Laplacian of the shape operator defined as the endomorphism $\Delta A := \sum_i[\nabla_{\nabla_{e_i}e_i}A - \nabla_{e_i}(\nabla_{e_i}A)]$. This computation is modeled on the computation of Nomizu and Smyth [26] in the case of constant-mean-curvature-hypersurfaces of a real space form. However, here we do not assume the mean curvature to be constant and we are dealing with complex space forms. Let $K(X,Y) = \nabla_{\nabla_X Y}A - \nabla_X(\nabla_Y A)$. Then

$$K(X,Y) = K(Y,X) + [A, R(X,Y)], \tag{19}$$

where $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ is the curvature operator of the hypersurface and the bracketed expression on the right hand side denotes the commutator of the endomorphisms involved. Clearly, $\Delta A = \sum_i K(e_i,e_i)$. We compute

$$\begin{aligned}\sum_i K(X,e_i)e_i &= \sum_i[(\nabla_{\nabla_X e_i}A)e_i - (\nabla_X(\nabla_{e_i}A))e_i]\\ &= \sum_{i,j}\omega_i^j(X)[(\nabla_{e_j}A)e_i + (\nabla_{e_i}A)e_j] - \sum_i \nabla_X((\nabla_{e_i}A)e_i)\\ &= -\nabla_X(\nabla f),\end{aligned} \tag{20}$$

since the connection 1-forms ω_i^j are antisymmetric and the bracketed expression is symmetric in i, j. Since all the quantities involved are tensorial, to facilitate further computations let us assume that $\nabla_{e_i} e_j = 0$ for all $i, j = 1, \dots, n$ and moreover $\nabla_{e_i} X = 0$ at a point where the computations are being carried out. By the Codazzi equation we have

$$\begin{aligned}\sum_i K(e_i, X)e_i &= -\sum_i \nabla_{e_i}((\nabla_X A)e_i) \\ &= -\sum_i \nabla_{e_i}[(\nabla_{e_i} A)X + c(\langle X, U\rangle Se_i - \langle e_i, U\rangle SX - 2\langle SX, e_i\rangle U)] \\ &= \sum_i [K(e_i, e_i)X + 2c\nabla_{e_i}(\langle SX, e_i\rangle U) \\ &\qquad - c\nabla_{e_i}(\langle X, U\rangle Se_i - \langle e_i, U\rangle SX)].\end{aligned}$$

Using (5) and (6) we get

$$\begin{aligned}\sum_i \nabla_{e_i}(\langle SX, e_i\rangle U) &= \sum_i (\langle(\nabla_{e_i} S)X, e_i\rangle U + \langle SX, e_i\rangle \nabla_{e_i} U) \\ &= f\langle X, U\rangle U - \langle AX, U\rangle U + SASX,\end{aligned}$$

and in a similar fashion

$$\begin{aligned}\sum_i \nabla_{e_i}(\langle X, U\rangle Se_i - \langle e_i, U\rangle SX) &= \sum_i [\langle X, \nabla_{e_i} U\rangle Se_i + \langle X, U\rangle(\nabla_{e_i} S)e_i] \\ &\quad - \sum_i [\langle e_i, \nabla_{e_i} U\rangle SX + \langle e_i, U\rangle(\nabla_{e_i} S)X] \\ &= \langle AX, U\rangle U - f\langle X, U\rangle U - SASX.\end{aligned}$$

Combining these steps we get

$$\sum_i K(e_i, X)e_i = (\Delta A)X + 3cSASX - 3c\langle AX - fX, U\rangle U. \tag{21}$$

From (19) and (21) it follows

$$\begin{aligned}(\Delta A)X = &-3cSASX + 3c\langle AX - fX, U\rangle U \\ &+ \sum_i K(X, e_i)e_i + \sum_i [A, R(e_i, X)]e_i.\end{aligned}$$

By the Gauss equation (9) we have

$$\begin{aligned}\sum_i [A, R(e_i, X)]e_i &= \sum_i A(R(e_i, X)e_i) - \sum_i R(e_i, X)(Ae_i) \\ &= cfX + (f_2 - cn)AX - fA^2X + 3cAS^2X - 3cSASX.\end{aligned}$$

From (5) and the above we finally obtain the following extension of the Nomizu-Smyth formula [26] to hypersurfaces of non-Euclidean complex space forms:

$$(\Delta A)X = 3c[\langle AU - fU, X\rangle U + \langle U, X\rangle AU] - 6cSASX \\ + cfX + [f_2 - c(n+3)]AX - fA^2X - \nabla_X(\nabla f). \tag{22}$$

As in [26], we have $\Delta(\mathrm{tr}\, A^2) = 2\,\mathrm{tr}[(\Delta A)A] - 2\sum_i \mathrm{tr}(\nabla_{e_i}A)^2$, and thus we obtain the following Simons'-type formula for a hypersurface of $\mathbf{C}Q^m(4c)$:

$$\frac{1}{2}\Delta(\mathrm{tr}\, A^2) = 6c\langle AU, AU\rangle - 3cf\langle AU, U\rangle - 6c\,\mathrm{tr}(SA)^2 + cf^2 \\ + [f_2 - c(n+3)]f_2 - ff_3 - \|\nabla A\|^2 - \sum_i \mathrm{Hess}_f(Ae_i, e_i), \tag{23}$$

where $\|\nabla A\|^2 = \sum_{i,j}\langle(\nabla_{e_j}A)e_i, (\nabla_{e_j}A)e_i\rangle$. A similar, but rather long, calculation using (22) yields

$$\Delta(JAU) = [\mathrm{tr}(\nabla_U A^2)]U - fSA^2U + 2(c+f_2)SAU - 2\sum_i J(\nabla_{e_i}A)(SAe_i) \\ - J\nabla_U(\nabla f) - JAS(\nabla f) + \langle\nabla f, AU\rangle U \\ - [f\langle A^2U, U\rangle - 2(c+f_2)\langle AU, U\rangle + 2cf]\xi \\ + 2\sigma(A^2U, U) - f\sigma(\xi, JAU) - 2\sum_i \sigma(e_i, J\nabla_{e_i}(AU)). \tag{24}$$

Note that by the Codazzi equation and formulas (5), (6) we have

$$\nabla_{e_i}(AU) = ASAe_i + (\nabla_U A)e_i - cSe_i, \tag{25}$$

and also $\sigma(\xi, SX) = \sigma(U, X) - \langle X, U\rangle\sigma(\xi, \xi)$. Additional, somewhat involved, computations yield the following formulas:

$$\Delta(\sigma(\xi, \xi)) = 4cJAU + 2(c+f_2)\sigma(\xi, \xi) + 2\sigma(\nabla f, \xi) \\ + 2c\sum_i \sigma(e_i, e_i) - 2\sum_i \sigma(Ae_i, Ae_i), \tag{26}$$

$$\sum_i \Delta(\sigma(e_i, Ae_i)) = -2c\sum_i J(\nabla_{e_i}A)(Se_i) + 2cfJAU - 4cJA^2U + 8c\nabla f \\ + 2(cf^2 + 2cf_2 + n - 1)\xi - 4\sigma(\xi, \mathrm{tr}(\nabla A^2)) - 2f_3\sigma(\xi, \xi) \\ + 2\sigma(\xi, A(\nabla f)) + \sum_i[\sigma(e_i, (\Delta A)e_i) + 2\sigma(Ae_i, A^2e_i)] \\ + 2c\sum_i[f\sigma(e_i, e_i) + \sigma(e_i, Ae_i) - \sigma(e_i, JASe_i)], \tag{27}$$

and using

$$\sum_{i,j}\bar{A}_{\sigma((\nabla_{e_j}A)e_i, Ae_i)}e_j = c\,\mathrm{tr}(\nabla A^2) + c\nabla f_2 - c\sum_j J(\nabla_{e_j}A^2)(Se_j),$$

also

$$\sum_i \Delta(\sigma(Ae_i, Ae_i)) = -2\sum_{i,j} \sigma((\nabla_{e_j}A)e_i, (\nabla_{e_j}A)e_i) + 2\sum_i \sigma(Ae_i, (\Delta A)e_i) \tag{28}$$
$$+ 2\sum_i [cf_2\sigma(e_i, e_i) + c\sigma(e_i, A^2e_i) + \sigma(Ae_i, A^3e_i)]$$
$$- 2c\sum_i \sigma(e_i, JA^2Se_i) - 4\sigma(\xi, \mathrm{tr}(\nabla A^3)) + 2\sigma(\xi, A^2(\nabla f))$$
$$- 2f_4\sigma(\xi, \xi) + 4c\ \mathrm{tr}(\nabla A^2) + 4c\nabla f_2 + 2c(ff_2 + 2f_3)\xi$$
$$+ 2cfJA^2U - 4cJA^3U - 4c\sum_i J(\nabla_{e_i}A^2)(Se_i).$$

By a repeated use of the Codazzi equation we may deduce that

$$\mathrm{tr}(\nabla A) = \nabla(\mathrm{tr}\ A) = \nabla f, \tag{29}$$

$$\mathrm{tr}(\nabla A^2) = \frac{1}{2}\nabla f_2 + A(\nabla f) - 3cSAU, \tag{30}$$

$$\mathrm{tr}(\nabla A^3) = \frac{1}{3}\nabla f_3 + \frac{1}{2}A(\nabla f_2) + A^2(\nabla f) - 3cSA^2U - 3cASAU, \tag{31}$$

and in general, by induction,

$$\mathrm{tr}(\nabla A^k) = \sum_{r=1}^{k} \frac{1}{r}A^{k-r}(\nabla f_r) - 3c\sum_{r=1}^{k-1} A^{r-1}SA^{k-r}U.$$

Additionally, by using the Codazzi equation again one computes

$$\sum_i (\nabla_{e_i}A)(Se_i) = -c(n-1)U, \tag{32}$$

and for a symmetric endomorphism B one gets from (12)

$$\sum_i \sigma(e_i, JBe_i) = 0. \tag{33}$$

Although we can compute $\Delta(\sigma(\nabla f, \xi))$ in a similar fashion and obtain a formula for $\Delta^3\tilde{x}$ in general, we list this formula only in the special case when the mean curvature is (locally) constant. Thus assuming $f = \mathrm{const}$, from (18) and (22)–(33) we obtain

$$\Delta^3\tilde{x} = (\Delta^3\tilde{x})_T + (\Delta^3\tilde{x})_N,$$

where the component tangent to $\mathbf{C}Q^m$ equals

$$(34)\quad (\Delta^3\tilde{x})_T = 8c\sum_i J[(\nabla_{e_i}A^2)(Se_i) - (\nabla_{e_i}A)(SAe_i)] - 2fA(\nabla f_2) - 4c\nabla f_2$$
$$+ 12c\langle \nabla f_2, U\rangle U + 8cSA^3U + 8(2cf_2 + n + 7)SAU$$
$$+ [8c\langle A^3U, U\rangle + 8(2cf_2 + n + 4)\langle AU, U\rangle - f(\Delta f_2) - 8cf_3$$
$$- f(f_2^2 + 4c(n+4)f_2 + 4cf^2 + 7n^2 + 30n + 19)]\xi,$$

and the normal component is

$$(35)\quad (\Delta^3\tilde{x})_N = 4\sum_{i,j}\sigma((\nabla_{e_j}A)e_i, (\nabla_{e_j}A)e_i) + 6f\sigma(\xi, \nabla f_2) + 12\sigma(\xi, A(\nabla f_2))$$
$$+ \frac{8}{3}\sigma(\xi, \nabla f_3) - 16c\sigma(\xi, ASAU) - 32cf\sigma(AU, U)$$
$$- 32c\sigma(A^2U, U) - 12c\sigma(AU, AU) + 16cf\sum_i \sigma(e_i, SASe_i)$$
$$+ 16c\sum_i \sigma(e_i, SASAe_i) + 4c\sum_i \sigma(e_i, SA^2Se_i)$$
$$+ \{28c\langle A^2U, U\rangle + 28cf\langle AU, U\rangle + 2(\Delta f_2) + f^2[3f_2 + c(3n+13)]$$
$$+ 4f_2^2 + 4c(n+4)f_2 + 4ff_3 + 4f_4 + 4n + 20\}\sigma(\xi, \xi)$$
$$- 4(cf^2 + n^2 + 4n + 5)\sum_i \sigma(e_i, e_i)$$
$$- 4f[f_2 + c(n+3)]\sum_i \sigma(e_i, Ae_i)$$
$$- 4(c + 2f_2)\sum_i \sigma(Ae_i, Ae_i) - 4\sum_i \sigma(A^2e_i, A^2e_i).$$

Compare with formula (2.17) of [13] and a related expression in [17].

4. Hopf hypersurfaces of 2-type have constant principal curvatures

In this section we study hypersurfaces of complex space forms of Chen-type 2 and classify such hypersurfaces that are also assumed to be Hopf hypersurfaces, i.e. for which the structure vector field $U := -J\xi$ is principal. We denote by $\mathscr{D}^\perp$ the 1-dimensional distribution generated by U and by $\mathscr{D}$ the holomorphic distribution which is the orthogonal complement of $\mathscr{D}^\perp$ in TM at each point. By way of notation, V_μ will denote the eigenspace of the shape operator A for an eigenvalue (principal curvature) μ and $\mathfrak{s}(\mathscr{D})$, the spectrum of $A|_{\mathscr{D}}$, the set of all eigenvalues of A corresponding to eigenvectors belonging to $\mathscr{D}$ at a given point.

Let $M^n \subset \mathbf{C}Q^m$, $n = 2m-1$, be a 2-type hypersurface in $H^{(1)}(m+1)$, i.e. $\tilde{x} = \tilde{x}_0 + \tilde{x}_u + \tilde{x}_v$ where $\tilde{x}_0 = \text{const}$, $\Delta\tilde{x}_u = \lambda_u\tilde{x}_u$ and $\Delta\tilde{x}_v = \lambda_v\tilde{x}_v$, according to (1). Then

$$\Delta^2\tilde{x} - (\lambda_u + \lambda_v)\Delta\tilde{x} + \lambda_u\lambda_v\tilde{x} = \lambda_u\lambda_v\tilde{x}_0. \tag{36}$$

Let L be the vector field in $H^{(1)}(m+1)$ along M^n, represented by the left hand side of the equation (36) and let X be an arbitrary tangential vector field of M^n. Then

$$\begin{aligned}
0 = \langle\tilde{\nabla}_X L, \tilde{x}\rangle &= X\langle L, \tilde{x}\rangle - \langle L, X\rangle \\
&= X\left(-2c + f^2 + 2cn(n+2) - (\lambda_u + \lambda_v)n + \frac{c}{2}\lambda_u\lambda_v\right) \\
&\quad - 4c\langle SAU, X\rangle + \langle f\nabla f + 2A(\nabla f), X\rangle \\
&= \langle 2f\nabla f, X\rangle - 4c\langle SAU, X\rangle + \langle f\nabla f + 2A(\nabla f), X\rangle.
\end{aligned}$$

Therefore

$$2A(\nabla f) + 3f\nabla f - 4cSAU = 0. \tag{37}$$

Similarly, by considering the $\sigma(\xi,\xi)$-component, in combination with (37) we may obtain

$$\begin{aligned}
\nabla f_2 + A(\nabla f) - \frac{1}{2}f\nabla f - f(\nabla_U A)U - 2(\nabla_U A)(AU) \\
+ f\langle\nabla f, U\rangle U - 2\langle\nabla f, U\rangle AU - \langle AU, U\rangle\nabla f = 0,
\end{aligned} \tag{38}$$

and the other components are even more complicated. Although it is possible to characterize 2-type hypersurfaces of $\mathbf{C}Q^m$ by a set of equations involving the structure vector field U, the gradients of f and f_2, Δf, the shape operator, and various compositions of S and A, the equations involved are very complicated to enable the classification of such hypersurfaces without any extra conditions. At this point it seems beneficial to make some additional assumptions on a hypersurface in order to make the situation more tractable. The most facile assumption, which simplifies many terms, is that $f := \operatorname{tr} A = \text{const}$, immediately leading, by way of (37), to the conclusion that M is a Hopf hypersurface, since $SAU = 0$ is equivalent to $AU = \varkappa U$ for some function $\varkappa$. Moreover, it is known that in this case $\varkappa$ is (locally) constant [22], [25]. Using this, one can show that the hypersurface is homogeneous and has at most 5 distinct principal curvatures, all of which are constant. Using the complete list of such hypersurfaces available in [32], [24], [20], [3], [4], one obtains a classification of constant-mean-curvature (CMC) hypersurfaces whose Chen-type is 2. This has been already attempted by Udagawa [36] for hypersurfaces of $\mathbf{C}P^m(4)$, and for hypersurfaces of $\mathbf{C}H^m(-4)$ see below. Udagawa's classification in $\mathbf{C}P^m$, however, is incomplete (see below).

On the other hand, instead of assuming the mean curvature to be constant, it seems more challenging to make a weaker assumption that M is a 2-type Hopf hypersurface. In that case we have

$$AU = \varkappa U, \quad \varkappa = \text{const}, \quad \text{and} \tag{39}$$

$$A(\nabla f) = -\frac{3f}{2}\nabla f.$$

So we do not get $f = \text{const}$ immediately, although that will eventually turn out to be the case.

Let G be an open set defined by $G = \{p \in M \mid f(p) \cdot (\nabla f)(p) \neq 0\}$. The Hopf property implies $\langle U, \nabla f\rangle = Uf = 0$ [25, p. 253] and thus $\nabla f \in \mathscr{D} = (\mathbf{R}U)^{\perp}$. In addition, we have that $S(\nabla f)$ is also an eigenvector of A, see [4], [22]. Then, since the integral curves of U for a Hopf hypersurface are geodesics, (38) reduces to

$$\nabla f_2 = (2f + \varkappa)\nabla f. \tag{40}$$

Instead of showing more general formula (38), for our purposes it suffices to prove (40). By using (3) and parallelism of σ we have

$$\begin{aligned} 0 &= \langle \tilde{\nabla}_X L, \sigma(\xi,\xi)\rangle \\ &= X\langle L, \sigma(\xi,\xi)\rangle + \langle L, \bar{A}_{\sigma(\xi,\xi)}X\rangle - \langle L, \bar{D}_X(\sigma(\xi,\xi))\rangle \\ &= X\langle L, \sigma(\xi,\xi)\rangle + 2c\langle L, X + \langle X, U\rangle U\rangle + 2\langle L, \sigma(AX,\xi)\rangle. \end{aligned} \tag{41}$$

Using $AU = \varkappa U$ and $Uf = 0$ we obtain

$$\begin{aligned} \langle \Delta^2\tilde{x}, \sigma(\xi,\xi)\rangle &= 4c[f_2 - \varkappa f - \varkappa^2 - cn(n+3)], \\ \langle \Delta^2\tilde{x}, X + \langle X, U\rangle U\rangle &= -\langle f\nabla f + 2A(\nabla f), X\rangle, \\ \langle \Delta^2\tilde{x}, \sigma(AX,\xi)\rangle &= 4c\langle A(\nabla f), X\rangle. \end{aligned}$$

The metric products of $\Delta\tilde{x}$ and $\tilde{x}$ with these quantities are either zero or give constants which disappear after differentiation. Thus putting these together in (41) we obtain

$$4c\langle \nabla f_2 - \varkappa\nabla f, X\rangle - 2c\langle f\nabla f + 2A(\nabla f), X\rangle + 8c\langle A(\nabla f), X\rangle = 0,$$

so that (40) follows from this and (39).

We now show that $f = \text{const}$. On G let $e_1 := \nabla f/|\nabla f|$ be the unit vector of ∇f. Then

$$\begin{aligned} 0 &= \langle \tilde{\nabla}_{\nabla f} L, \sigma(e_1,e_1)\rangle \\ &= (\nabla f)\langle L, \sigma(e_1,e_1)\rangle + \langle L, \bar{A}_{\sigma(e_1,e_1)}\nabla f\rangle - \langle L, \bar{D}_{\nabla f}\sigma(e_1,e_1)\rangle \\ &= (\nabla f)\langle L, \sigma(e_1,e_1)\rangle + 4c\langle L, \nabla f\rangle - 2\langle L, \sigma(\nabla_{\nabla f} e_1, e_1)\rangle + 3f|\nabla f|\langle L, \sigma(\xi, e_1)\rangle. \end{aligned}$$

If $AX = \mu X$ for $X \in \mathscr{D}$ then, by the results of Maeda [22] (for the projective case) and Berndt [4] (for the hyperbolic case), also $A(SX) = \mu^*(SX)$, where μ^* is uniquely determined by the condition

$$(42)\quad (2\mu - \varkappa)(2\mu^* - \varkappa) = \varkappa^2 + 4c, \quad \text{i.e.} \quad \mu^* = \frac{\varkappa\mu + 2c}{2\mu - \varkappa}, \quad \text{and} \quad (\mu^*)^* = \mu.$$

Since $\langle U, \nabla f \rangle = 0$, then e_1, $Se_1 \in \mathscr{D}$. When $X = \nabla f$, we have $\mu = -3f/2$ and $\mu^* = \dfrac{3\varkappa f - 4c}{2(3f + \varkappa)}$. We may assume that $3f + \varkappa \neq 0$, for otherwise we may work on an open subset of G where $f \neq -\varkappa/3$ and invoke continuity of f. From here we compute using (11)

$$\langle L, \sigma(e_1, e_1) \rangle = -c(5f^2 + 4f\mu^* + 4\mu^{*2}) + \text{const},$$
$$\langle L, \nabla f \rangle = 2f|\nabla f|^2, \quad \langle L, \sigma(\nabla_{\nabla f} e_1, e_1) \rangle = 0, \quad \langle L, \sigma(\xi, e_1) \rangle = 4c|\nabla f|.$$

Substituting in the above equality we get

$$20f|\nabla f|^2 - (\nabla f)(5f^2 + 4f\mu^* + 4\mu^{*2}) = 0.$$

Since

$$(\nabla f)(\mu^*) = \frac{3(\varkappa^2 + 4c)}{2(3f + \varkappa)^2}|\nabla f|^2,$$

we see that $f = \operatorname{tr} A$ satisfies on G a polynomial equation of degree 4 with constant coefficients, viz.

$$135f^4 + 108\varkappa f^3 + 18\varkappa^2 f^2 - 2\varkappa(5\varkappa^2 + 12c)f + 16c\varkappa^2 + 48 = 0.$$

Consequently, f is (locally) constant since f is continuous and M is assumed connected. From (40) we also get $f_2 = \text{const}$.

Since $f = \text{const}$, (18) reduces to

$$(43)\quad \begin{aligned} \Delta^2 \tilde{x} = {} & [4c\varkappa - f(f_2 + c(3n+5))]\xi + (2c + 2f_2 + f^2)\sigma(\xi, \xi) \\ & - 2c(n+2)\sum_i \sigma(e_i, e_i) - 2f\sum_i \sigma(Ae_i, e_i) - 2\sum_i \sigma(Ae_i, Ae_i). \end{aligned}$$

Differentiating (36) with respect to an arbitrary tangent field $X \in \Gamma(TM)$ we have

$$(44)\quad \tilde{\nabla}_X(\Delta^2 \tilde{x}) - p\tilde{\nabla}_X(\Delta \tilde{x}) + qX = 0,$$

where $p := \lambda_u + \lambda_v$ and $q := \lambda_u \lambda_v$. Conversely, if (44) holds then $\tilde{x}$ satisfies the polynomial equation (36) in the Laplacian of the form $P(\Delta)(\tilde{x} - \tilde{x}_0) = 0$, with $P(t) = t^2 - pt + q$. According to a result of Chen and Petrovic [12] if such polynomial has simple real roots the submanifold is of 2-type (if not already of 1-type). Let $V_\mu \subset \mathscr{D}$ be an eigenspace of an eigenvalue $\mu \in \mathfrak{s}(\mathscr{D})$ at each point and let $X \in V_\mu$ be a unit vector. Taking the metric product of (44) with X and observing that $\langle \Delta \tilde{x}, X \rangle = \langle \Delta^2 \tilde{x}, X \rangle = 0$ for any tangent vector X, we have

$$\begin{aligned} 0 &= \langle \tilde{\nabla}_X(\Delta^2 \tilde{x}), X\rangle - p\langle \tilde{\nabla}_X(\Delta \tilde{x}), X\rangle + q \\ &= X\langle \Delta^2 \tilde{x}, X\rangle - \langle \Delta^2 \tilde{x}, \overline{\nabla}_X X + \sigma(X,X)\rangle + p\langle \Delta \tilde{x}, \overline{\nabla}_X X + \sigma(X,X)\rangle + q \\ &= -\mu\langle \Delta^2 \tilde{x}, \xi\rangle - \langle \Delta^2 \tilde{x}, \sigma(X,X)\rangle + p\mu\langle \Delta \tilde{x}, \xi\rangle + p\langle \Delta \tilde{x}, \sigma(X,X)\rangle + q. \end{aligned}$$

Using (10) and the above-mentioned results of Maeda and Berndt that $AX = \mu X$ implies $A(SX) = \mu^*(SX)$ where μ^* is given by (42) we get from here

$$\begin{aligned} 0 = q &+ [f(f_2 + 3c(n+3)) - pf - 4c\varkappa]\mu + 2cf^2 - 2pc(n+2) \\ &+ 4(n+1)(n+3) + 4c\mu^2 + 4cf\mu^* + 4c\mu^{*2}. \end{aligned} \tag{45}$$

Substituting the value of μ^* from (42) and clearing of denominators we get a fourth degree polynomial equation in μ with constant coefficients (since f, f_2, and $\varkappa$ are all constant). We conclude that $A|_{\mathscr{D}}$ has at most four eigenvalues, i.e. the hypersurface has at most five distinct principal curvatures, all of them constant.

Hopf hypersurfaces of $\mathbf{C}P^m(4)$ and $\mathbf{C}H^m(-4)$ for $m \geq 2$ with constant principal curvatures are homogeneous and they are known. By a result of Takagi [32] (see also [19], [20]) there are six types or six classes of Hopf hypersurfaces with constant principal curvatures in $\mathbf{C}P^m(4)$, given as (possibly open portions of) the model hypersurfaces in the following list (the so-called Takagi's list):

(A_1) A geodesic hypersphere of radius $r \in \left(0, \frac{\pi}{2}\right)$;

(A_2) A tube of any radius $r \in \left(0, \frac{\pi}{2}\right)$ around a canonically embedded (totally geodesic) $\mathbf{C}P^k$ for some $k \in \{1, \dots, m-2\}$;

(B) A tube of any radius $r \in \left(0, \frac{\pi}{4}\right)$ around a canonically embedded complex quadric $Q^{m-1} = SO(m+1)/SO(2) \times SO(m-1)$;

(C) A tube of radius $r \in \left(0, \frac{\pi}{4}\right)$ around the Segre embedding of $\mathbf{C}P^1 \times \mathbf{C}P^k$ in $\mathbf{C}P^m, m = 2k+1$;

(D) A tube of radius $r \in \left(0, \frac{\pi}{4}\right)$ of dimension 17 in $\mathbf{C}P^9$ around the Plücker embedding of the complex Grassmannian of 2-planes $G_2(\mathbf{C}^5)$;

(E) A tube of radius $r \in \left(0, \frac{\pi}{4}\right)$ of dimension 29 in $\mathbf{C}P^{15}$ around the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$.

We call these the standard examples or the model hypersurfaces in $\mathbf{C}P^m$. To avoid confusion with the notion of Chen-type, these hypersurfaces will be referred to as being of class A (with subclasses A_1, A_2), B, C, D, E, rather than being of type A, B, C, D, E, as is customary in the literature. For the example B we note that a tube of radius $r \in \left(0, \frac{\pi}{4}\right)$ around Q^{m-1} in $\mathbf{C}P^m$ can be regarded also as the tube of radius $\frac{\pi}{4} - r$ around the canonically embedded (totally geodesic) $\mathbf{R}P^m$ in $\mathbf{C}P^m$, which is the other focal submanifold of that hypersurface [5], [8].

These model hypersurfaces have two, three, or five principal curvatures given by

$$\varkappa = 2\cot(2r), \quad \text{and} \quad \mu_i = \cot\left(r + (i-1)\frac{\pi}{4}\right), \quad i = 1, 2, 3, 4,$$

where r is the radius of the tube involved and $\varkappa$ the principal curvature of U. The table of principal curvatures and their multiplicities for these hypersurfaces is compiled by Takagi [32] and reads as follows (see also [3], [25]):

Table 1. *Principal curvatures of the standard examples in* $\mathbf{C}P^m$ *and their multiplicities*

	$2\cot(2r)$	$\cot r$	$\cot\left(r+\frac{\pi}{4}\right)$	$\cot\left(r+\frac{\pi}{2}\right)$	$\cot\left(r+\frac{3\pi}{4}\right)$
A_1	1	$2(m-1)$	—	—	—
A_2	1	$2(m-k-1)$	—	$2k$	—
B	1	—	$m-1$	—	$m-1$
C	1	$m-3$	2	$m-3$	2
D	1	4	4	4	4
E	1	8	6	8	6

It is known that the almost complex structure J leaves eigenspaces V_{μ_1} and V_{μ_3} invariant and interchanges eigenspaces V_{μ_2} and V_{μ_4}.

In the complex hyperbolic space the number of principal curvatures is two or three. The list (the so-called Montiel's list after [24], completed by Berndt [3], [4], see also [25]) of Hopf hypersurfaces with constant principal curvatures in $\mathbf{C}H^m(-4)$ consists of (open portions of) the following:

(A_0) A horosphere in $\mathbf{C}H^m$;

(A_1') A geodesic hypersphere of any radius $r \in \mathbf{R}_+$;

(A_1'') A tube of any radius $r \in \mathbf{R}_+$ over a totally geodesic complex hyperbolic hyperplane $\mathbf{C}H^{m-1}$;

(A_2) A tube of any radius $r \in \mathbf{R}_+$ about the canonically embedded $\mathbf{C}H^k$ in $\mathbf{C}H^m$ for $k = 1, \dots, m-2$;

(B) A tube of any radius $r \in \mathbf{R}_+$ about the canonically embedded (totally geodesic, totally real) $\mathbf{R}H^m$ in $\mathbf{C}H^m$.

We note that a canonically embedded $\mathbf{R}H^m \subset \mathbf{C}H^m$ is of 1-type in $H^1(m+1)$, [15]. In a recent work Berndt and Díaz-Ramos [6], [7] classified hypersurfaces of $\mathbf{C}H^m$ with three constant principal curvatures, without assuming them to be Hopf.

The table of principal curvatures $\varkappa$, μ, ν and their multiplicities $m_\varkappa$, m_μ, m_ν is as follows [3], [4], [25]:

Table 2. *Principal curvatures of the standard examples in* $\mathbf{C}H^m$ *and their multiplicities*

	$\varkappa$	μ	ν	$m_\varkappa$	m_μ	m_ν
A_0	2	—	1	1	—	$2m-2$
A_1'	$2\coth(2r)$	$\coth r$	—	1	$2(m-1)$	—
A_1''	$2\coth(2r)$	—	$\tanh r$	1	—	$2(m-1)$
A_2	$2\coth(2r)$	$\coth r$	$\tanh r$	1	$2(m-k-1)$	$2k$
B	$2\tanh(2r)$	$\coth r$	$\tanh r$	1	$m-1$	$m-1$

It is known that the eigenspaces V_μ and V_ν are interchanged by the action of J for a class-B hypersurface and they are J-invariant (holomorphic) for any of the class-A hypersurfaces. A hypersurface of class A_2 has three principal curvatures and so does a hypersurface of class B, except in one case, namely when the radius of the tube is $r = \frac{1}{2}\ln(2+\sqrt{3})$ and then $\mu = \varkappa = \sqrt{3}$.

In both settings, $\varkappa$ is the principal curvature corresponding to $U := -J\xi$. These classifications enable us to prove our results for 2-type Hopf hypersurfaces. In the subsequent investigation of 2-type Hopf hypersurfaces of $\mathbf{C}Q^m(4c)$ we may assume we are dealing with Hopf hypersurfaces with constant principal curvatures and therefore with one from the Takagi's list in the projective space or one from the Montiel's list in the hyperbolic space.

5. The classification of 2-type Hopf hypersurfaces of $\mathbf{C}Q^m(4c)$

We begin by analyzing various components of equation (44). Let $X \in \Gamma(TM)$. Then using the Gauss and Weingarten formulas (3) and the fact that σ is parallel, we get from (14)

$$\tilde{\nabla}_X(\Delta\tilde{x}) = 2c(n+2)X + fAX - 2c\langle X, U\rangle U - f\sigma(X,\xi) - 2\sigma(AX,\xi) \tag{46}$$

and from (43)

$$\begin{aligned}
\tilde{\nabla}_X(\Delta^2\tilde{x}) = {} & [2cf^2 + 4(n+1)(n+3)]X - [4c\varkappa - f(f_2 + 3c(n+3))]AX \\
& + 4cA^2X - 2c[2f_2 + f^2 + 2c(n+3)]\langle X, U\rangle U - 4cfJASX \\
& - 4cJA^2SX + [4c\varkappa - f(f_2 + c(3n+5))]\sigma(X,\xi) \\
& - 2[2f_2 + f^2 + 2c(n+3)]\sigma(AX,\xi) - 4f\sigma(A^2X,\xi) \\
& - 4\sigma(A^3X,\xi) - 2f\sum_i \sigma((\nabla_X A)e_i, e_i) - 4\sum_i \sigma((\nabla_X A)e_i, Ae_i).
\end{aligned}$$

Therefore, separating the part of equation (44) that is tangent to $\mathbf{C}Q^m$ we get

$$
\begin{aligned}
0 = &-2c[2f_2 + f^2 + 2c(n+3) - p]\langle X, U\rangle U - 4cfJASX \\
&- 4cJA^2SX + [4(n+1)(n+3) + 2cf^2 - 2pc(n+2) + q]X \\
&- [4c\varkappa - f(f_2 + 3c(n+3) - p)]AX + 4cA^2X
\end{aligned} \tag{47}
$$

and the part normal to $\mathbf{C}Q^m$ yields

$$
\begin{aligned}
&[4c\varkappa - f(f_2 + c(3n+5) - p)]\sigma(X,\xi) - 4\sigma(A^3X,\xi) \\
&\quad - 4f\sigma(A^2X,\xi) - 2[2f_2 + f^2 + 2c(n+3) - p]\sigma(AX,\xi) \\
&\quad - 2f\sum_i \sigma((\nabla_X A)e_i, e_i) - 4\sum_i \sigma((\nabla_X A)e_i, Ae_i) = 0.
\end{aligned} \tag{48}
$$

These expressions are linear in X. Further separation of parts relative to the splitting $\mathscr{D} \oplus \mathbf{R}U \oplus \mathbf{R}\xi$ of the tangent space of $\mathbf{C}Q^m$ yields the following

LEMMA 1. *Let M^n be a Hopf hypersurface (not necessarily compact) of $\mathbf{C}Q^m(4c)$ ($m \geq 2$, $n = 2m-1$). If M is of 2-type via $\tilde{x}$ satisfying 2-type condition (44) then M has at most five distinct principal curvatures, all of which are constant, and the following relations hold:*

(E_1) $[2c(n+1) + \varkappa f]p = q + \varkappa f[f_2 + 3c(n+3)] - 4cf_2 + 4n(n+3)$;

(E_2)

$$
\begin{aligned}
[2c(n+2) + \mu f]p = q &+ 2cf^2 + 4cf\mu^* + 4c\mu^{*2} + 4c\mu^2 \\
&+ [f(f_2 + 3c(n+3)) - 4c\varkappa]\mu + 4(n+1)(n+3),
\end{aligned}
$$

for any principal curvature $\mu \in \mathfrak{s}(\mathscr{D})$;

(E_3)

$$
\begin{aligned}
(f + 2\mu)p = &-4c\varkappa + 4\mu(f\mu + \mu^2 + f\mu^* + \mu^{*2}) \\
&+ 2\mu[2f_2 + f^2 + 2c(n+3) - 2f\varkappa - 2\varkappa^2] + f[f_2 + c(3n+5)],
\end{aligned}
$$

for any $\mu \in \mathfrak{s}(\mathscr{D})$;

(E_4)

$$
\begin{aligned}
&f\langle(\nabla_X A)Y, Z\rangle + f\langle(\nabla_X A)(SY), SZ\rangle \\
&\quad + \langle(\nabla_X A^2)Y, Z\rangle + \langle(\nabla_X A^2)(SY), SZ\rangle = 0,
\end{aligned}
$$

for every $X, Y, Z \in \Gamma(\mathscr{D})$.

Conversely, if (E_1)–(E_4) hold for a Hopf hypersurface with constant principal curvatures, where p and q are constants and $\mu \in \mathfrak{s}(\mathscr{D})$ is an arbitrary principal curvature on $\mathscr{D}$, then the formula (44) holds and the submanifold is of type ≤ 2 if the corresponding monic polynomial $P(t) = t^2 - pt + q$ has two distinct real roots.

Proof. From the above discussion it follows that M has constant principal curvatures. (E_1) follows from (47) when $X = U$. (E_2) is the formula (45), and it follows from (47) when $X \in V_\mu \subset \mathscr{D}$ is chosen to be a principal direction of a principal curvature $\mu \in \mathfrak{s}(\mathscr{D})$. Note that (47) is linear in X, so it suffices to consider X to be one of the principal directions.

(E_3) and (E_4) follow from the normal part (48). Recall that the normal space of $\mathbf{C}Q^m$ in $H^{(1)}(m+1)$ is spanned by $\tilde{x}$ and the values of σ on various pairs of tangent vectors to $\mathbf{C}Q^m$, namely $\sigma(\xi,\xi)$, $\sigma(\xi, X)$, $\sigma(X, Y)$ for $X, Y \in \Gamma(\mathscr{D})$, see e.g. [15]. Note that by (12) $\sigma(\xi, U) = 0$, $\sigma(U, U) = \sigma(\xi,\xi)$ and $\sigma(U, X) = \sigma(\xi, JX)$ for $X \in \mathscr{D}$. By (12) and the constancy of f and f_2, from (48) we conclude that the equation (44) has no $\tilde{x}$-component.

Let L denote the left-hand side of (48). Then $\bar{A}_L = 0$. Conversely, if $\bar{A}_L = 0$, for some $L \in T^\perp \mathbf{C}Q^m$ then $L = kI$ is a multiple of the identity, but since L is a linear combination of terms of the form $\sigma(V, W)$ then by (12) $L = 0$. Consider first $\bar{A}_L\xi = 0$. The condition $\langle \bar{A}_L\xi, \xi\rangle = 0$ gives no information since by (10) $\langle L, \sigma(\xi,\xi)\rangle = 0$ is trivially satisfied and the same holds for $\langle \bar{A}_L\xi, U\rangle = 0$. Now take $Y \in \mathscr{D}$ and consider $\langle \bar{A}_L\xi, Y\rangle = \langle L, \sigma(\xi, Y)\rangle = 0$. Using

$$\bar{A}_{\sigma(X,\xi)}\xi = c(X - \langle X, U\rangle U), \quad \sum_i \bar{A}_{\sigma((\nabla_X A)e_i, e_i)}\xi = 2c\varkappa JSAX - 2cJASAX,$$

$$\text{and} \quad \sum_i \bar{A}_{\sigma((\nabla_X A)e_i, Ae_i)}\xi = c\varkappa^2 JSAX - cJA^2SAX,$$

from (48) it follows

$$\begin{aligned} (49) \qquad & 4\langle SA^2SAX, Y\rangle + 4f\langle SASAX, Y\rangle - 4\langle A^3X, Y\rangle - 4f\langle A^2X, Y\rangle \\ & - 2[2f_2 + f^2 + 2c(n+3) - p - 2f\varkappa - 2\varkappa^2]\langle AX, Y\rangle \\ & + [4c\varkappa + fp - f(f_2 + c(3n+5))]\langle X, Y\rangle = 0. \end{aligned}$$

Since $A\mathscr{D} \subset \mathscr{D}$, $S\mathscr{D} = \mathscr{D}$, and the expression is linear in $X, Y \in \mathscr{D}$ we can drop Y and take $X \in V_\mu \subset \mathscr{D}$ to get (E_3). Considering $\bar{A}_L U = 0$ gives no additional information beyond (E_3) by virtue of $\langle \bar{A}_L U, Y\rangle = \langle \bar{A}_L\xi, JY\rangle$, returning it to the case above. Next we exploit the condition $\bar{A}_L Y = 0$ for $Y \in \mathscr{D}$. By (11) we have

$$\sum_i \bar{A}_{\sigma((\nabla_X A)e_i, e_i)} Y = 2c(\nabla_X A)Y - 2cJ(\nabla_X A)(SY)$$

$$\sum_i \bar{A}_{\sigma((\nabla_X A)e_i, Ae_i)} Y = c(\nabla_X A^2)Y - cJ(\nabla_X A^2)(SY).$$

In particular, when $X = U$ by the Codazzi equation and formula (6) we have

$$\sum_i \bar{A}_{\sigma((\nabla_U A)e_i, e_i)} Y = 4SY + 2c[\varkappa SAY + \varkappa ASY - ASAY + (SA)^2SY]$$

and a similar, somewhat longer, expression is obtained for $\sum_i \bar{A}_{\sigma((\nabla_U A)e_i, Ae_i)} Y$. Then taking $Y \in V_\mu$ and using $2\mu\mu^* = 2c + \varkappa(\mu + \mu^*)$, by way of (42), we see that $\bar{A}_L Y = 0$ reduces to a trivial identity when $X = U$. Thus consider as the last condition to check $\langle \bar{A}_L Y, Z \rangle = 0$. Choosing $Z = \xi$ or $Z = U$ gives back (E_3) and when $X, Y, Z \in \mathscr{D}$ from $\langle L, \sigma(Y, Z) \rangle = 0$ we get (E_4). Conversely, since we considered all possible components, the conditions (E_1)–(E_4) are equivalent to (47) and (48) by linearity and thus we get (44), from which it follows that a hypersurface is of type ≤ 2, provided that the corresponding polynomial has two distinct real roots.

Note that by a result of Niebergall and Ryan [25, p. 264] any of the class-A hypersurfaces in $\mathbf{C}Q^m$ from either list is characterized by

$$(\nabla_X A)Y = -c[\langle SX, Y \rangle U + \langle U, Y \rangle SX],$$

so that the condition (E_4) is trivially satisfied for those hypersurfaces. Further, by eliminating q from (E_1) and (E_2) we get

$$\begin{aligned} (50) \qquad [2c + f(\mu - \varkappa)]p = {} & 4c(\mu^2 + \mu^{*2}) + 4cf\mu^* - 4c\varkappa\mu + 4(n+3) \\ & + 4cf_2 + 2cf^2 + f(\mu - \varkappa)[f_2 + 3c(n+3)], \end{aligned}$$

and if p can be uniquely determined from this condition (regardless of the choice of μ and consistent with (E_3)) then q is uniquely determined from (E_1).

We now examine which of the Hopf hypersurfaces with constant principal curvatures are of 2-type. This has been already considered by Udagawa for hypersurfaces of $\mathbf{C}P^m$ [36]. Although our argument is different from Udagawa's and relies on the analysis of the conditions (E_1)–(E_4), rather than on the matrix representation of the immersion in $H^{(1)}(m+1)$, it partly overlaps Udagawa's investigation and reaches the same classification for 2-type CMC real hypersurfaces in $\mathbf{C}P^m$ of class A. However, Udagawa's paper contains errors regarding mass-symmetric hypersurfaces and in particular hypersurfaces of class B, as a result of which the three theorems in that work contain inaccuracies and incomplete classifications. Moreover, our more detailed analysis clearly exhibits the manner of 2-type decompositions involved. Also, the benefit of our uniform approach is that it produces results for hypersurfaces of $\mathbf{C}H^m$ at the same time, the case which is not treated in earlier papers, and the same technique will be used to study 3-type submanifolds.

First we note that a horosphere in $\mathbf{C}H^m$ is not of any finite type since, as shown in [18], it satisfies $\Delta^2\tilde{x} = \text{const} \neq 0$ and therefore cannot satisfy equation (1), for otherwise equation (44) would hold for some constants p and q, which would force p and q, and thus also $\Delta^2\tilde{x}$, to be zero or $p\tilde{\nabla}_X(\Delta\tilde{x})$ to be a multiple of X, contradicting (46). For hypersurfaces of class A_1 (geodesic spheres, equidistant hypersurfaces) we have

Lemma 2. (i) *A geodesic hypersphere in* $\mathbf{C}P^m(4)$ *of any radius* $r \in (0, \pi/2)$, $r \neq \cot^{-1}\sqrt{1/(2m+1)}$ *is of 2-type in* $H(m+1)$. *A geodesic hypersphere in*

$\mathbf{C}H^m(-4)$ *of arbitrary radius* $r > 0$ *is of* 2*-type in* $H^1(m+1)$ *via* $\tilde{x}$ *and the same holds true for a tube of an arbitrary radius* $r > 0$ *about a totally geodesic complex hyperbolic hyperplane* $\mathbf{C}H^{m-1}(-4) \subset \mathbf{C}H^m(-4)$. *These statements are also valid for any open portion of the respective submanifolds.*

(ii) *The only complete mass-symmetric hypersurfaces of class* A_1 *are geodesic hyperspheres of radius* $r = \cot^{-1}\sqrt{1/m}$ *in* $\mathbf{C}P^m(4)$.

Proof. (i) For a geodesic sphere (class A_1 in $\mathbf{C}P^m$ and A_1' in $\mathbf{C}H^m$) define

$$\cot_c(r) = \begin{cases} \cot r, & \text{when } c = 1 \text{ (projective case)} \\ \coth r, & \text{when } c = -1 \text{ (hyperbolic case)} \end{cases}$$

and let $\mu = \cot_c(r)$ be the principal curvature of multiplicity $2(m-1) = n-1$ and $\varkappa = 2\cot_c(2r)$ the principal curvature (of U) of multiplicity 1, whereas $\mu = \tanh r$, $\varkappa = 2\coth(2r)$ for a tube about a complex hyperbolic hyperplane $\mathbf{C}H^{m-1}(-4)$ of class A_1''. Then

$$\mu^* = \mu, \quad \varkappa = \mu - \frac{c}{\mu}, \quad f = n\mu - \frac{c}{\mu}, \quad f_2 = n\mu^2 + \frac{1}{\mu^2} - 2c. \tag{51}$$

From (50) we get

$$[(n+2)c - \mu^{-2}]p = c(3n+2)(n+2)\mu^2 + (3n^2+6n+4) - \frac{(2n+1)c}{\mu^2} - \frac{1}{\mu^4}. \tag{52}$$

We may assume that $(n+2)c \neq 1/\mu^2$, certainly true when $c = -1$, and when $c = 1$ the equality would lead to $\mu = \sqrt{1/(n+2)}$ i.e. to $r = \cot^{-1}\sqrt{1/(2m+1)}$. However, the geodesic hypersphere of this radius in $\mathbf{C}P^m(4)$ is of 1-type (see e.g. [23], [15]). Thus dividing (52) by $(n+2)c - \mu^{-2}$ we get

$$p = (3n+2)\mu^2 + 3c(n+1) + \frac{1}{\mu^2} = (\mu^2 + c)\left(3n + 2 + \frac{c}{\mu^2}\right). \tag{53}$$

Then from (E_1) we find

$$q = 2(n+1)\left[n\mu^4 + c(2n+1)\mu^2 + \frac{c}{\mu^2} + (n+2)\right]. \tag{54}$$

Solving (E_3) for p gives the same value as in (53), so the conditions (E_1)–(E_3) are consistent and satisfied by the above values of p and q, the condition (E_4) being trivially satisfied. According to Lemma 1, the equation (44) then holds, hence also (36). Moreover, the polynomial $P(\lambda) = \lambda^2 - p\lambda + q$ has two distinct real roots $\lambda_u = 2(n+1)(\mu^2 + c)$ and $\lambda_v = \frac{1}{\mu^2}(\mu^2 + c)(n\mu^2 + c)$, which are the two eigenvalues of the Laplacian from the 2-type decomposition of $\tilde{x}$.

(ii) Let $\mathscr{D}$ be the holomorphic distribution in TM as before and choose an orthonormal basis $\{e_i\}$ of the tangent space so that $e_n = U$ and $e_i \in \mathscr{D}$ for

$i = 1, 2, \dots, n-1$. To see which hypersurfaces of class A_1 are mass-symmetric first we find from (14), (18), and (43)

$$\Delta\tilde{x} = -\left(n\mu - \frac{c}{\mu}\right)\xi - \sigma(\xi,\xi) - \sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i), \quad \text{and}$$

$$\begin{aligned}\Delta^2\tilde{x} = &-\left[n^2\mu^3 + c(3n^2 + 2n - 4)\mu - \frac{2n-1}{\mu} - \frac{c}{\mu^3}\right]\xi \\ &+ \left[(n^2 - 2)\mu^2 - 2cn - \frac{1}{\mu^2}\right]\sigma(\xi,\xi) - 2(n+1)(\mu^2 + c)\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i)\end{aligned}$$

and then compute, using $\lambda_u - \lambda_v = (1/\mu^2)(\mu^2 + c)[(n+2)\mu^2 - c]$, that

$$\begin{aligned}(55) \qquad \tilde{x}_u &= \frac{1}{\lambda_u(\lambda_u - \lambda_v)}(\Delta^2\tilde{x} - \lambda_v\Delta\tilde{x}) \\ &= \frac{m-1}{4m(\mu^2 + c)^2}\left\{-4c\mu\xi + 2\mu^2\sigma(\xi,\xi) - \frac{\mu^2 + c}{m-1}\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i)\right\}\end{aligned}$$

$$(56) \qquad \tilde{x}_v = \frac{1}{\lambda_v(\lambda_v - \lambda_u)}(\Delta^2\tilde{x} - \lambda_u\Delta\tilde{x}) = -\frac{\mu}{(\mu^2 + c)^2}[(\mu^2 - c)\xi + \mu\sigma(\xi,\xi)].$$

From Lemma 1 of [15] we have

$$(57) \qquad \tilde{x} = \frac{I}{m+1} - \frac{c}{2(m+1)}\sigma(\xi,\xi) - \frac{c}{4(m+1)}\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i).$$

Now the center of mass can be found as $\tilde{x}_0 = \tilde{x} - \tilde{x}_u - \tilde{x}_v$ to yield

$$(58) \qquad \tilde{x}_0 = \frac{I}{m+1} + \frac{m\mu^2 - c}{m(\mu^2 + c)^2}\left[\mu\xi + \frac{1}{2}\sigma(\xi,\xi) + (\mu^2 + c)\left(\tilde{x} - \frac{I}{m+1}\right)\right]$$

We observe that the same formula applies also for the center of mass of a 1-type hypersphere in $\mathbf{C}P(4)$ for an appropriate value of μ. Note that by our definition of mass-symmetry, any null 2-type hypersurface is per force mass-symmetric, since the constant part $\tilde{x}_0$ can be manipulated and changed to be equal to $I/(m+1)$, and the existing constant $\tilde{x}_0$ moved to be a part of the 0-eigenfunction. However, in our case, for A_1 hypersurface both λ_u and λ_v as given above are nonzero, because (in the hyperbolic case) $\mu = \coth r > 1$. Since the ξ-component of the right hand side of (58) is the only part tangent to $\mathbf{C}Q^m$ and $\mu = \cot_c r \neq 0$, a class-A_1 hypersurface is mass-symmetric, i.e. $\tilde{x}_0 = I/(m+1)$ if and only if $m\mu^2 = c$. This is possible only when $c = 1$ and $\mu = \cot r = \sqrt{1/m}$. Thus a geodesic hypersphere in $\mathbf{C}P^m(4)$ of radius $r = \cot^{-1}\sqrt{1/m}$ is the only complete mass-symmetric hypersurface of class A_1. We observe that this hypersphere with the given radius does satisfy the equation (3.14) of [36], but it is completely overlooked in that paper.

LEMMA 3. (i) *There are no* 2*-type hypersurfaces in* $\mathbf{C}H^m(-4)$ *of class* A_2, *i.e. no* 2*-type tubes about canonically embedded* $\mathbf{C}H^k \subset \mathbf{C}H^m$, $1 \le k \le m-2$. *A hypersurface of class* A_2 *in* $\mathbf{C}P^m(4)$ *is of* 2*-type if and only if it is an open portion of either* (a) *the tube of radius* $r = \cot^{-1}\sqrt{\dfrac{k+1}{m-k}}$ *or* (b) *the tube of radius* $r = \cot^{-1}\sqrt{\dfrac{2k+1}{2(m-k)+1}}$, *about a canonically embedded, totally geodesic* $\mathbf{C}P^k(4) \subset \mathbf{C}P^m(4)$, *for any* $k = 1, 2, \ldots, m-2$.

(ii) *The only complete mass-symmetric* 2*-type hypersurfaces of class* A_2 *are those in the first series of tubes* (a) *above.*

Proof. (i) Let $\mu_1 = \cot r$, $\mu_3 = \cot\left(r + \dfrac{\pi}{2}\right) = -\dfrac{1}{\mu_1}$ for model hypersurface of class A_2 in $\mathbf{C}P^m$ and $\mu_1 = \mu = \coth r$, $\mu_3 = \nu = \tanh r = \dfrac{1}{\mu_1}$ for model hypersurface of class A_2 in $\mathbf{C}H^m$. Then μ_1, μ_3 have respective multiplicities $2l$ and $2k$ for some positive integers k, l with $l = m-k-1$ i.e. $n = 2l + 2k + 1$. Moreover,

$$\mu_1^* = \mu_1, \quad \mu_3^* = \mu_3, \quad \mu_1\mu_3 = -c, \quad \varkappa = \mu_1 - \frac{c}{\mu_1} = \mu_1 + \mu_3 \tag{59}$$

$$f = L\mu_1 + K\mu_3, \quad f^2 = L^2\mu_1^2 + K^2\mu_3^2 - 2cKL, \quad f_2 = L\mu_1^2 + K\mu_3^2 - 2c, \tag{60}$$

where $K := 2k+1$ and $L := 2l+1$. Our goal is to examine when the equations (E_1)–(E_3) are consistent and when constants p and q can be found to satisfy them (Once again, the condition (E_4) is satisfied by every class-A_2 hypersurface). That comes down to the pair of equations consisting of (E_3) and (50), having the same solution for p for either value of $\mu \in \{\mu_1, \mu_3\}$. Consider the equation (50) in which $\mu = \mu_1$, multiplied by $[2c + f(\mu_3 - \varkappa)] = (2c - f\mu_1)$ and the same equation with $\mu = \mu_3$ multiplied by $(2c - f\mu_3)$. Subtract the two multiplied equations to eliminate p. We get

$$f(f_2 + f^2) + 2\varkappa f(f + \varkappa) - c(n+3)f - 4c\varkappa = 0. \tag{61}$$

This is a necessary and sufficient condition for p to have the same value from (50), regardless of the choice of μ. On the other hand subtracting the two equations obtained from (50) for $\mu = \mu_1, \mu_2$, gives

$$pf = ff_2 + c(3n+13)f + 4c\varkappa. \tag{62}$$

Similarly, from the two equations contained in (E_3) for $\mu = \mu_1, \mu_3$ by subtracting we get

$$p = 2f_2 + f^2 + 2\varkappa(f + \varkappa) + 2c(n+5), \tag{63}$$

and by eliminating p from these two equations we get exactly the same condition (61) as before. Moreover, assuming (61), we check that (62) and (63) are consistent, so there is only one condition, namely (61), to be satisfied in order to make (E_1)–(E_3) consistent, regardless of the choice of μ, and enable us to solve for p and q. Replacing the values from (59) and (60) into (61), using $\varkappa = \mu_1 + \mu_3$ we get

$$\begin{aligned} 0 = {} & L(L+1)(L+2)\mu_1^3 + K(K+1)(K+2)\mu_3^3 \\ & - c\mu_1(3L^2K + 3L^2 + 6LK + 8L + 2K + 4) \\ & - c\mu_3(3LK^2 + 3K^2 + 6LK + 8K + 2L + 4), \end{aligned}$$

or

$$\begin{aligned} (64) \qquad & [(L+1)\mu_1^2 - c(K+1)] \\ & \times [L(L+2)\mu_1^4 - 2c(LK + K + L + 2)\mu_1^2 + K(K+2)] = 0, \end{aligned}$$

which has the following three solutions

$$\text{(a)}\ \mu_1^2 = \frac{(K+1)c}{L+1} \quad \text{(b)}\ \mu_1^2 = \frac{Kc}{L+2} \quad \text{(c)}\ \mu_1^2 = \frac{(K+2)c}{L}.$$

Clearly, when $c = -1$ none of them is possible, so there are no 2-type hypersurfaces of $\mathbf{C}H^m(-4)$ among A_2-hypersurfaces. When $c = 1$ the last two possibilities generate the same set of examples. From (63) we find

$$p = (L^2 + 4L + 2)\mu_1^2 + (K^2 + 4K + 2)\mu_3^2 - 2LK$$

and we can also compute q from (E_1) in terms of μ_1, μ_3. Then using these we find the two eigenvalues of the Laplacian from the 2-type decomposition to be

$$\begin{aligned} (65) \qquad & \lambda_u = (L+1)(L+2)\mu_1^2 + (K+1)(K+2)\mu_3^2 - (L + K + 2LK), \\ & \lambda_v = L\mu_1^2 + K\mu_3^2 + L + K, \quad \mu_1 = \cot r,\ \mu_3 = -\tan r. \end{aligned}$$

In the case (a), we get $\lambda_u = 2(n+3)$, $\lambda_v = 2(n+1) - \dfrac{(l-k)^2}{(l+1)(k+1)} = \dfrac{4(m+1)(LK+m)}{(L+1)(K+1)}$, $\lambda_u > \lambda_v$, so the hypersurface is of 2-type. Since $\mu_1^2 = \cot^2 r = \dfrac{K+1}{L+1}$, from Takagi's list it follows that the hypersurface is an open portion of the tube of radius $r = \cot^{-1}\sqrt{\dfrac{K+1}{L+1}} = \cot^{-1}\sqrt{\dfrac{k+1}{m-k}}$ about a canonically embedded $\mathbf{C}P^k(4) \subset \mathbf{C}P^m(4)$, for any $k = 1, \ldots, m-2$; see also [8], [25]. For case (b), (65) yields

$$\lambda_u = \frac{4(k+1)(n+3)}{2k+1} = 4(m+1)\frac{K+1}{K}, \quad \lambda_v = \frac{4(l+1)(n+3)}{2l+3} = 4(m+1)\frac{L+1}{L+2},$$

$\lambda_u > \lambda_v$. Since $\mu_1^2 = \cot^2 r = \dfrac{K}{L+2}$, we identify such hypersurface as an open portion of the tube of radius $r = \cot^{-1}\sqrt{\dfrac{K}{L+2}} = \cot^{-1}\sqrt{\dfrac{2k+1}{2(m-k)+1}}$ about a canonically embedded $\mathbf{C}P^k(4) \subset \mathbf{C}P^m(4)$, for any $k = 1, \ldots, m-2$.

(ii) For an A_2-hypersurface we have from (65)

$$\lambda_u - \lambda_v = (L^2 + 2L + 2)\mu_1^2 + (K^2 + 2K + 2)\mu_3^2 - 2(L + K + LK). \tag{66}$$

Note that from Tables 1 and 2 and the accompanying discussion, in addition to principal curvature $\varkappa = 2\cot_c(2r)$ an A_2-hypersurface has also two more principal curvatures $\mu_1 = \cot_c r$ and $\mu_3 = -c\tan_c r$, with corresponding principal subspaces $V_1 := V_{\mu_1}$ and $V_3 := V_{\mu_3}$, being J-invariant and $\mathscr{D} = V_1 \oplus V_3$. Then from (14) and (43) for a basis of principal directions $\{e_i\}$ in $\mathscr{D}$ we get

$$\Delta\tilde{x} = -(L\mu_1 + K\mu_3)\xi - \sigma(\xi,\xi) - \sum_{e_i \in V_1}\sigma(e_i,e_i) - \sum_{e_j \in V_3}\sigma(e_j,e_j),$$

$$\begin{aligned}\Delta^2\tilde{x} = &-[L^2\mu_1^3 + K^2\mu_3^3 + (L^2 + 4mL - 4)\mu_1 + (K^2 + 4mK - 4)\mu_3]\xi \\ &+ [(L^2 - 2)\mu_1^2 + (K^2 - 2)\mu_3^2 - 2LK]\sigma(\xi,\xi) \\ &- 2(L+1)(\mu_1^2 + 1)\sum_{e_i \in V_1}\sigma(e_i,e_i) - 2(K+1)(\mu_3^2 + 1)\sum_{e_j \in V_3}\sigma(e_j,e_j).\end{aligned}$$

Then $\tilde{x}_u$ and $\tilde{x}_v$ can be computed as in (55)–(56). Since the hypersurface of $\mathbf{C}Q^m$ is mass-symmetric via $\tilde{x}$ we have $\tilde{x}_0 = \tilde{x} - (\tilde{x}_u + \tilde{x}_v) = I/(m+1)$. Because I and $\tilde{x}$ are normal to $\tilde{x}(\mathbf{C}Q^m)$, a necessary condition for mass-symmetry in $H^{(1)}(m+1)$ is that the ξ-component of $\tilde{x}_u + \tilde{x}_v$ be zero. The ξ-component of $\tilde{x}_u$ equals

$$\frac{-4}{\lambda_u(\lambda_u - \lambda_v)}[(mL - 1)\mu_1 + (mK - 1)\mu_3]$$

and the ξ-component of $\tilde{x}_v$ is

$$\begin{aligned}\frac{1}{\lambda_v(\lambda_v - \lambda_u)}\{&L(L^2 + 2L + 2)\mu_1^3 - [8mL + LK(3L + 2) + 2K - 4]\mu_1 \\ &+ K(K^2 + 2K + 2)\mu_3^3 - [8mK + LK(3K + 2) + 2L - 4]\mu_3\}.\end{aligned}$$

Observing the corresponding values of λ_u, λ_v in each of the cases we see that the ξ-component of $\tilde{x}_u + \tilde{x}_v$ for hypersurfaces in (b) is never zero, whereas for hypersurfaces of case (a) this component is identically equal to zero. An additional computation verifies that for any hypersurface of case (a) other components $\sigma(\xi,\xi)$, $\sum_{e_i \in V}\sigma(e_i,e_i)$ on both sides of mass-symmetric 2-type decomposition are matched.

The two families of tubes referred to in Lemma 3 have also another representation. Let

$$M_{2k+1,2l+1}(r) := S^{2k+1}(\cos r) \times S^{2l+1}(\sin r), \quad 0 < r < \pi/2,$$

be the family of generalized Clifford tori in an odd-dimensional sphere $S^{n+2} \subset \mathbf{C}^{m+1}$, $n = 2m - 1$. By choosing the two spheres (with the indicated radii) in the above product to lie in complex subspaces we get the fibration $S^1 \to M_{2k+1,2l+1}(r) \to M_{k,l}^{\mathbf{C}}(r) := \pi(M_{2k+1,2l+1}(r))$ compatible with the Hopf fibration

$\pi : S^{n+2} \to \mathbf{C}P^m(4)$, which submerses $M_{2k+1,2l+1}(r)$ onto $M^{\mathbf{C}}_{k,l}(r)$ [21]. Cecil and Ryan have shown [8] that $M^{\mathbf{C}}_{k,l}(r)$ is a tube of radius r about totally geodesic $\mathbf{C}P^k(4)$ with principal curvatures $\cot r$, $-\tan r$, $2\cot(2r)$ of respective multiplicities $2l$, $2k$, and 1. Accordingly, the family of hypersurfaces corresponding to the case (a) is given as open portions of

$$M^{\mathbf{C}}_{k,l}(r) = \pi\left(S^K\left(\sqrt{\frac{K+1}{n+3}}\right) \times S^L\left(\sqrt{\frac{L+1}{n+3}}\right)\right), \quad \cot^2 r = \frac{K+1}{L+1},$$

and the family of hypersurfaces corresponding to the case (b) is

$$M^{\mathbf{C}}_{k,l}(r) = \pi\left(S^K\left(\sqrt{\frac{K}{n+3}}\right) \times S^L\left(\sqrt{\frac{L+2}{n+3}}\right)\right), \quad \cot^2 r = \frac{K}{L+2},$$

where for both families $n+3 = 2(m+1)$ and $K = 2k+1$ and $L = 2l+1$ are odd positive integers with $K + L = 2m$. It is in exactly this form that they appear in Udagawa's paper. The family of hypersurfaces corresponding to the case (c) is the same family as in (b), with the roles of K and L interchanged and the factors reversed. Hypersurfaces of case (c) can be also described as tubes over $\mathbf{C}P^k(4)$ of radius $\rho = \cot^{-1}\sqrt{\dfrac{2k+3}{2(m-k)-1}}$, for $k = 1, 2, \ldots, m-2$, but are not listed as a separate case since they constitute the same family as the one under case (b). Namely, the tube about $\mathbf{C}P^k(4)$ of this radius ρ is the same as the tube over the other focal variety $\mathbf{C}P^l(4)$ of radius $\dfrac{\pi}{2} - \rho = \cot^{-1}\sqrt{\dfrac{2l+1}{2(m-l)+1}}$, which appears within family (b).

Remark. Note that according to a result of Barbosa et al. [1], tubes over $\mathbf{C}P^k(4)$ of radius r satisfying $\cot^{-1}\sqrt{\dfrac{2k+3}{2(m-k)-1}} \le r \le \cot^{-1}\sqrt{\dfrac{2k+1}{2(m-k)+1}}$ are stable with respect to normal variations preserving the enclosed volume. Hence the 2-type tubes over $\mathbf{C}P^k(4)$ of radii $\cot^{-1}\sqrt{\dfrac{2k+1}{2(m-k)+1}}$ and $\cot^{-1}\sqrt{\dfrac{2k+3}{2(m-k)-1}}$ are distinguished by being maximal, respectively minimal, stable tubes over $\mathbf{C}P^k$, for each $k = 1, 2, \ldots, m-2$, i.e. the values of radii in cases (b) and (c) are precisely the endpoints of the stability interval for r.

Lemma 4. *There are no 2-type hypersurfaces in $\mathbf{C}H^m(-4)$ among hypersurfaces of class B. A class-B hypersurface of $\mathbf{C}P^m(4)$ is of Chen 2-type if and only if it is an open portion of either the tube of radius $r_1 = \cot^{-1}(\sqrt{m} + \sqrt{m+1})$ or the tube of radius $r_2 = \cot^{-1}\sqrt{\sqrt{2m^2-1} + \sqrt{2m^2-2}}$, $r_1 < r_2$, about a complex quadric $Q^{m-1} \subset \mathbf{C}P^m(4)$. In both instances, these tubes are also mass-symmetric in the hypersphere $S^{N-1}_{I/(m+1)}\left(\sqrt{\dfrac{m}{2(m+1)}}\right)$ of $E^N = H(m+1)$ that contains them.*

Proof. Let $\mu_2 = \cot\left(r + \frac{\pi}{4}\right)$ and $\mu_4 = \cot\left(r + \frac{3\pi}{4}\right)$, $\varkappa = 2\cot(2r)$ for the standard examples B through E in $\mathbf{C}P^m(4)$ and $\mu_2 = \coth r$, $\mu_4 = \tanh r$, $\varkappa = 2\tanh(2r)$ for an example of class B in $\mathbf{C}H^m(-4)$. For all of these hypersurfaces we have

$$\mu_2\mu_4 = -c, \quad \mu_2 + \mu_4 = -\frac{4c}{\varkappa}, \quad \mu_2^* = \mu_4, \quad \mu_4^* = \mu_2. \tag{67}$$

Setting $\mu = \mu_2, \mu_4$ in (50) produces two equations, from which by eliminating p we get

$$2\left[\varkappa^2 + c(m-3) - \frac{16}{\varkappa^2}\right]f + 2\left(\varkappa + \frac{4c}{\varkappa}\right)f^2 - f(f^2 + f_2) - 4c\varkappa = 0. \tag{68}$$

The same condition is obtained from (E_3) by setting $\mu = \mu_2, \mu_4$ and eliminating p and is also a necessary condition for the values of p obtained from (50) and (E_3) to be equal for any hypersurface of class B. For class-B hypersurface in either setting the common multiplicity of μ_2, μ_4 is $m-1$ and we have

$$f = \varkappa - \frac{4c(m-1)}{\varkappa}, \quad f_2 = \varkappa^2 + \frac{16(m-1)}{\varkappa^2} + 2c(m-1). \tag{69}$$

From (68) and (69) we get

$$\varkappa^6 - 4c(m-1)\varkappa^4 - 8(m^2 + 2m - 1)\varkappa^2 + 32cm(m^2 - 1) = 0. \tag{70}$$

Thus, for hypersurfaces of class B, (70) represents a necessary and sufficient condition for p to have the same value from (50) and (E_3), regardless of the choice of $\mu = \mu_2, \mu_4$, and also for p and q to be uniquely determined from the conditions (E_1)–(E_3) in Lemma 1. One needs to check also condition (E_4) by computing the connection coefficients of the hypersurface considered or by invoking the η-parallelism of the shape operator for hypersurfaces of class B, [20], [25]. We shall work instead with condition (44), which is a necessary and sufficient condition for type ≤ 2, provided that the roots of the corresponding quadratic equation are real and distinct, and obtain expressions for $\Delta\tilde{x}$ and $\Delta^2\tilde{x}$ that will enable us to find the explicit 2-type decomposition of $\tilde{x}$ for certain hypersurfaces of class B. Let M be such hypersurface in either $\mathbf{C}P^m(4)$ or $\mathbf{C}H^m(-4)$. With μ_2, μ_4 as above, for the corresponding eigenspaces $V := V_{\mu_2}$ and V_{μ_4} we have $V_{\mu_4} = JV$ and $\mathscr{D} = V \oplus SV$. In the case of tube of radius $r = \frac{1}{2}\ln(2 + \sqrt{3})$ in $\mathbf{C}H^m(-4)$ which has only two constant principal curvatures (since $\varkappa = \mu_2$), we consider V_{μ_2} to consists of eigenvectors of μ_2 belonging to $\mathscr{D}$ only, thus not including U. Note that $\sigma(U, U) = \sigma(\xi, \xi)$ and $\sigma(Je_i, Je_i) = \sigma(e_i, e_i)$ by (12).

The fact that J interchanges V_{μ_2} and V_{μ_4} for every hypersurface of class B is crucial here and will enable us to find suitable expressions for $\Delta\tilde{x}$, $\Delta^2\tilde{x}$ and later $\Delta^3\tilde{x}$. Let $\{e_i\} = \{e_j, Se_j\}$ be a J-basis of the holomorphic distribution $\mathscr{D}$ (where

$e_j \in V,\ j = 1, 2, \ldots, m-1$), which is the basis of principal directions of $A|_{\mathscr{D}}$ with $Ae_j = \mu_j e_j$ and $A(Se_j) = \mu_j^* Se_j$, where $\mu_j,\ \mu_j^*$ satisfy relation (42), or equivalently

$$2c + \varkappa(\mu_j + \mu_j^*) = 2\mu_j\mu_j^* = -2c. \tag{71}$$

Then we have

$$\sum_{e_i \in \mathscr{D}} \sigma(e_i, Ae_i) = \sum_{e_i \in \mathscr{D}} \mu_i \sigma(e_i, e_i) = \sum_{e_j \in V} [\mu_j \sigma(e_j, e_j) + \mu_j^* \sigma(Se_j, Se_j)]$$
$$= \sum_{e_j \in V} (\mu_j + \mu_j^*)\sigma(e_j, e_j) = -\frac{2c}{\varkappa} \sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i), \quad \text{and} \tag{72}$$

$$\sum_{e_i \in \mathscr{D}} \sigma(Ae_i, Ae_i) = \sum_{e_i \in \mathscr{D}} \mu_i^2 \sigma(e_i, e_i) = \sum_{e_j \in V} (\mu_j^2 + \mu_j^{*2})\sigma(e_j, e_j)$$
$$= \left(\frac{8}{\varkappa^2} + c\right) \sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i). \tag{73}$$

Then using (67) and (69), formulas (14) and (18) become respectively

$$\Delta\tilde{x} = \left[\frac{2c(n-1)}{\varkappa} - \varkappa\right]\xi - \sigma(\xi, \xi) - \sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i), \tag{74}$$

$$\Delta^2\tilde{x} = \left[\frac{16c(n-1)^2}{\varkappa^3} + \frac{8n(n-1)}{\varkappa} - 2c(n+1)\varkappa - \varkappa^3\right]\xi$$
$$+ \left[\frac{4(n-1)(n+3)}{\varkappa^2} - 4c - \varkappa^2\right]\sigma(\xi, \xi) - 2(n+1)\left(\frac{4}{\varkappa^2} + c\right) \sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i). \tag{75}$$

For $X \in \Gamma(TM)$ we get the following using (3) and (11):

$$\tilde{\nabla}_X \xi = -AX + \sigma(X, \xi), \quad \tilde{\nabla}_X(\sigma(\xi, \xi)) = -2cX - 2c\langle X, U\rangle U - 2\sigma(AX, \xi), \tag{76}$$

$$\tilde{\nabla}_X\left(\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i)\right) = -2c(n+1)X + 4c\langle X, U\rangle U + 4\sigma(AX, \xi). \tag{77}$$

Therefore, differentiating $\Delta\tilde{x}$ and $\Delta^2\tilde{x}$ with respect to X we substitute in (44) using (72)–(77) to get

$$\left\{\frac{16c(n-1)^2}{\varkappa^3} + \frac{2(n-1)(4n - cp)}{\varkappa} + [p - 2c(n+1)]\varkappa - \varkappa^3\right\}[-AX + \sigma(X, \xi)]$$
$$+ \left[\frac{8c(n^2 + 2n + 5)}{\varkappa^2} + 4(n^2 + 2n + 3) + 2c\varkappa^2 + q - 2c(n+2)p\right]X$$
$$+ 2\left[p + \varkappa^2 - 4cn - \frac{4(n^2 + 6n + 1)}{\varkappa^2}\right][c\langle X, U\rangle U + \sigma(AX, \xi)] = 0. \tag{78}$$

Equate with zero the normal to $\mathbf{C}Q^m$ component of (78), which is a linear combination of $\sigma(X,\xi)$ and $\sigma(AX,\xi)$, and take respectively $X \in V_{\mu_2}$ and $X \in V_{\mu_4}$ to get two equations, from which by subtracting and solving for p we get

$$p = \frac{4(n^2+6n+1)}{\varkappa^2} + 4cn - \varkappa^2. \tag{79}$$

Thus the last line of equation (78) drops out and the coefficient of $\sigma(X,\xi)$ on the top line would have to be zero. With the value of p from (79), equating that coefficient with 0 yields

$$\varkappa^6 - 2c(n-1)\varkappa^4 - 2(n^2+6n+1)\varkappa^2 + 4c(n-1)(n+1)(n+3) = 0. \tag{80}$$

Under this condition the AX-component is also zero and then the X-component in the middle line of (78) must be zero, which gives the following value of q:

$$q = 2c(n+3)\left[\frac{4(n^2+4n-1)}{\varkappa^2} + 2c(n-1) - \varkappa^2\right]. \tag{81}$$

Thus under the condition (80) it is possible to satisfy equation (78), that is the equation (44), for the values of p and q as in (79) and (81). This means that a class-B hypersurface satisfying (80) is of 2-type. The equation (80) is, not surprisingly, the compatibility condition (70), which we now see is also a sufficient condition for a hypersurface of class B to be of 2-type. Moreover, that condition is equivalent to the equation

$$[\varkappa^2 - 2c(n+1)][\varkappa^4 + 4c\varkappa^2 - 2(n-1)(n+3)] = 0, \tag{82}$$

which has three roots $\varkappa^2 = 2c(n+1)$ and $\varkappa^2 = -2c \pm c\sqrt{2(n^2+2n-1)}$. When $c = -1$, none of them is possible since $0 < \varkappa^2 < 4$. For $c = 1$ (the case of a hypersurface of class B in $\mathbf{C}P^m(4)$) we have the following two possibilities:

$$\text{(a) } \varkappa^2 = 2(n+1) \quad \text{and} \quad \text{(b) } \varkappa^2 = \sqrt{2(n^2+2n-1)} - 2.$$

In case (a) we find $p = \lambda_u + \lambda_v$, $q = \lambda_u\lambda_v$ from (79) and (81) and the two eigenvalues $\lambda_u < \lambda_v$ to be

$$p = \frac{4n(n+3)}{n+1}, \quad q = \frac{4(n-1)(n+3)^2}{n+1} \tag{83}$$

$$\lambda_u = \frac{2(n-1)(n+3)}{n+1} = 4(m-1/m), \quad \lambda_v = 2(n+3) = 4(m+1). \tag{84}$$

The corresponding hypersurface is a tube of radius r about the complex quadric Q^{m-1}, where $\cot r - \tan r = \varkappa$, i.e. $\cot r = \dfrac{\varkappa + \sqrt{\varkappa^2+4}}{2} = \sqrt{m} + \sqrt{m+1}$.

In case (b) we find

$$p = \frac{1}{(n-1)(n+3)}[2(2n^3+7n^2+8n-1) + (n^2+10n+5)\sqrt{2(n^2+2n-1)}], \tag{85}$$

$$q = \frac{2}{n-1}[2(n^3+4n^2+5n-2) + (n^2+6n+1)\sqrt{2(n^2+2n-1)}], \tag{86}$$

$$\lambda_u = \frac{2}{n+3}[2\sqrt{2(n^2+2n-1)} + (n+1)^2], \tag{87}$$
$$\lambda_v = \frac{n+3}{n-1}[\sqrt{2(n^2+2n-1)} + 2n].$$

The corresponding hypersurface is a tube about Q^{m-1} of radius r, with $\cot r = \sqrt{\sqrt{2m^2-1} + \sqrt{2m^2-2}}$. These two tubes are therefore of 2-type in $H(m+1)$. Moreover, they are also mass-symmetric in the hypersphere containing $\Phi(\mathbf{C}P^m)$, which means that the center of mass is $\tilde{x}_0 = I/(m+1) = 2I/(n+3)$. Indeed by a Lemma of [15] we have the expression

$$I = (m+1)\tilde{x} + \frac{c}{2}\sigma(\xi,\xi) + \frac{c}{4}\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i). \tag{88}$$

Then it is a straightforward verification using (74), (75), (79), (81) and (88) that any class-B hypersurface satisfying condition (80) is of 2-type since it satisfies the equation

$$\Delta^2\tilde{x} - p\Delta\tilde{x} + q\left(\tilde{x} - \frac{I}{m+1}\right) = 0, \tag{89}$$

and it is, obviously, not of 1-type. Specifically, for the two tubes about Q^{m-1} discussed above, (89) holds for the indicated values of p and q from (83), respectively (85)–(86). The corresponding vector-eigenfunctions $\tilde{x}_u$ and $\tilde{x}_v$ of λ_u and λ_v in 2-type decomposition of $\tilde{x}$ can be found from

$$\tilde{x}_u = \frac{1}{\lambda_u - \lambda_v}\left[\Delta\tilde{x} - \lambda_v\left(\tilde{x} - \frac{I}{m+1}\right)\right], \quad \tilde{x}_v = \frac{1}{\lambda_v - \lambda_u}\left[\Delta\tilde{x} - \lambda_u\left(\tilde{x} - \frac{I}{m+1}\right)\right]. \tag{90}$$

For example, for the tube of radius $r_1 = \cot^{-1}(\sqrt{m} + \sqrt{m+1})$ we get

$$\tilde{x}_u = \frac{\sqrt{2(n+1)}}{2(n+3)}\xi - \frac{n+1}{4(n+3)}\sigma(\xi,\xi), \quad \text{and}$$
$$\tilde{x}_v = -\frac{\sqrt{2(n+1)}}{2(n+3)}\xi + \frac{n-3}{4(n+3)}\sigma(\xi,\xi) - \frac{1}{2(n+3)}\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i).$$

It can be also directly verified, using (16), (26) and (17), that $\tilde{x}_u$, $\tilde{x}_v$ are indeed eigenfunctions of Δ for the indicated eigenvalues. Incidental to this finding, we obtain two simple eigenvalue estimates for the first two non-zero eigenvalues λ_1, λ_2 for the hypersurface for which (a) holds: $\lambda_1 \le 4(m - 1/m)$ and $\lambda_2 \le 4(m+1)$.

LEMMA 5. *There are no* 2*-type hypersurfaces in* $\mathbf{C}P^m(4)$ *among any of the standard examples of class* C, D, *or* E.

Proof. This was shown in [36]. For the sake of completeness we include a different proof here using our approach. In addition to principal curvatures μ_2, μ_4 and formulas (67)–(68), we have also principal curvatures μ_1, μ_3, for which the relations (59) hold ($c = 1$ throughout). If we substitute $\mu = \mu_1$, $\mu = \mu_3$ in (E_3) and subtract the two resulting equations we get

$$p = 2f_2 + f^2 + 2\varkappa f + 2\varkappa^2 + 2(n+5). \tag{91}$$

The same manipulation with $\mu = \mu_2, \mu_4$ yields

$$p = 2f_2 + f^2 - 2\left(\varkappa + \frac{4}{\varkappa}\right)f + 2(n+5) + \frac{32}{\varkappa^2} - 2\varkappa^2. \tag{92}$$

On the other hand, substituting $\mu = \mu_1, \mu_3$ into (50) and subtracting we get

$$fp = ff_2 + (3n+13)f + 4\varkappa, \tag{93}$$

and the same procedure using $\mu = \mu_2, \mu_4$ leads to

$$fp = ff_2 + (3n+5)f - 4\varkappa. \tag{94}$$

Combining (93) and (94) we get $f = -\varkappa$ and subtracting (91) and (92) leads to $(\varkappa + 2/\varkappa)f + \varkappa^2 - 8/\varkappa^2 = 0$, which is incompatible with $f = -\varkappa$.

Now we can formulate our main classification results for 2-type Hopf hypersurfaces of $\mathbf{C}Q^m$. In the complex projective space we have

Theorem 1. *Let M^{2m-1} be a Hopf hypersurface of $\mathbf{C}P^m(4)$, $(m \geq 2)$. Then M^{2m-1} is of 2-type in $H(m+1)$ via Φ if and only if it is an open portion of one of the following*

(i) *A geodesic hypersphere of any radius $r \in \left(0, \frac{\pi}{2}\right)$, except for $r = \cot^{-1}\sqrt{\frac{1}{2m+1}}$;*

(ii) *The tube of radius $r = \cot^{-1}\sqrt{\frac{k+1}{m-k}}$ about a canonically embedded totally geodesic $\mathbf{C}P^k(4) \subset \mathbf{C}P^m(4)$, for any $k = 1, \ldots, m-2$;*

(iii) *The tube of radius $r = \cot^{-1}\sqrt{\frac{2k+1}{2(m-k)+1}}$ about a canonically embedded $\mathbf{C}P^k(4) \subset \mathbf{C}P^m(4)$, for any $k = 1, \ldots, m-2$.*

(iv) *The tube of radius $r = \cot^{-1}(\sqrt{m} + \sqrt{m+1})$ about a complex quadric $Q^{m-1} \subset \mathbf{C}P^m(4)$.*

(v) *The tube of radius $r = \cot^{-1}\sqrt{\sqrt{2m^2-1} + \sqrt{2m^2-2}}$ about a complex quadric $Q^{m-1} \subset \mathbf{C}P^m(4)$.*

Proof. As shown before, a 2-type Hopf hypersurface must have constant principal curvatures and therefore must be one from the Takagi's list in $\mathbf{C}P^m(4)$. The rest follows from Lemmas 1–5.

As commented before, the same classification holds when M is assumed to have constant mean curvature (CMC) instead of being Hopf. In that regard Theorems 1 and 2 in [36] are deficient and incomplete, since Udagawa's list contains examples (i)–(iii) only. The list of items (i)–(v) is the correct and complete classification of CMC hypersurfaces of 2-type in $\mathbf{C}P^m(4)$. Likewise, a previous announcement of our theorem in [16] is incomplete, since it was anticipated based on Udagawa's classification.

In the same manner, since being a Hopf hypersurface and having constant mean curvature imply each other for hypersurfaces of 2-type, Lemmas 1–5 yield

Theorem 2. *Let M^{2m-1} be a real hypersurface of $\mathbf{C}H^m(-4)$, $(m \geq 2)$ for which we assume that it is a Hopf hypersurface or has constant mean curvature. Then M^{2m-1} is of 2-type in $H^1(m+1)$ via Φ if and only if it is (an open portion of) either a geodesic hypersphere of arbitrary radius $r > 0$ or a tube of arbitrary radius $r > 0$ about a canonically embedded totally geodesic complex hyperbolic hyperplane $\mathbf{C}H^{m-1}(-4)$.*

Regarding mass-symmetric hypersurfaces, from the analysis above we have

Corollary 1. *A complete Hopf (or CMC) hypersurface of $\mathbf{C}P^m(4)$ is of 2-type and mass-symmetric in the hypersphere of $H(m+1)$ containing $\Phi(\mathbf{C}P^m)$ if and only if it is one of the hypersurfaces (tubes) in* (ii), (iv) *and* (v) *or the geodesic hypersphere of radius* $\cot^{-1}(1/\sqrt{m})$. *There exists no 2-type mass-symmetric (in particular, no null 2-type) hypersurface of $\mathbf{C}H^m(-4)$.*

This rectifies the claim made in Theorem 2 of [36].

6. CMC Hopf hypersurfaces of 3-type

It is not difficult to see that the hypersurfaces of class A_2 are, generally speaking, of 3-type (except for those two families of tubes in $\mathbf{C}P^m$ given in Theorem 1 (ii), (iii), which are or 2-type). Consider $p \in M \subset \mathbf{C}Q^m(4c)$ where $p = [\zeta]$ is represented by a column vector

$$\zeta \in \pi^{-1}(p) \subset N^{2m+1} \subset \mathbf{C}^{m+1}_{(1)} = \mathbf{C}^{k+1}_{(1)} \oplus \mathbf{C}^{l+1}.$$

Let $z = (z_i) = (\zeta_0, \dots, \zeta_k)^T$ and $w = (w_\alpha) = (\zeta_{k+1}, \dots, \zeta_m)^T$ and consider in $\mathbf{C}^{k+1}_{(1)}$ the quadric $N^{2k+1}(r_1)$ (the sphere or anti-de Sitter space of radius r_1) and in $\mathbf{C}^{l+1}$ the sphere $S^{2l+1}(r_2)$ so that $r_1^2 + cr_2^2 = 1$. In the projective case we have $c = 1$ and we set $r_1 = \cos r$, $r_2 = \sin r$, whereas in the hyperbolic case $c = -1$, $r_1 = \cosh r$, $r_2 = \sinh r$. The corresponding class-A_2 hypersurfaces which are the tubes of radius r about totally geodesic $\mathbf{C}Q^k(4c)$ are obtained as the Hopf projections, defining the submersion: $\pi(S^{2k+1}(\cos r) \times S^{2l+1}(\sin r))$ in $\mathbf{C}P^m(4)$ and $\pi(H_1^{2k+1}(\cosh r) \times S^{2l+1}(\sinh r))$ in $\mathbf{C}H^m(-4)$, $k + l = m - 1$. According to (2), the coordinate representation of $\tilde{x}(p)$ in $H^{(1)}(m+1)$ has the matrix block form

$$\tilde{x} = \begin{pmatrix} a_{ij} & b_{i\beta} \\ c_{\alpha j} & d_{\alpha\beta} \end{pmatrix}, \quad 0 \leq i, j \leq k, \ k+1 \leq \alpha, \beta \leq m,$$

where, for example, $d_{\alpha\beta} = cw\overline{w}^T$, $b_{i\beta} = cz\overline{w}^T$, and $a_{ij} = (\pm z_i \bar{z}_j)$ is formed by the signed products, plus in the first column minus otherwise in $\mathbf{C}H^m$-case, all plus in $\mathbf{C}P^m$-case. Then using the fact that π is a (pseudo) Riemannian submersion with totally geodesic fibers [2], one can compute the iterated Laplacians of $\pi(N^{2k+1}(r_1) \times S^{2l+1}(r_2))$ as follows, see [23], [36], [18]:

$$\Delta\tilde{x} = \begin{pmatrix} \dfrac{2c(K+1)}{r_1^2} a_{ij} - 4cI_{k+1} & \left(\dfrac{cK}{r_1^2} + \dfrac{L}{r_2^2}\right) b_{i\beta} \\ \left(\dfrac{cK}{r_1^2} + \dfrac{L}{r_2^2}\right) c_{\alpha j} & \dfrac{2(L+1)}{r_2^2} d_{\alpha\beta} - 4cI_{l+1} \end{pmatrix},$$

and in general for an integer $s \geq 1$

$$\Delta^s\tilde{x} = \begin{pmatrix} \dfrac{2^s c^s (K+1)^s}{r_1^{2s}} a_{ij} - \dfrac{2^{s+1} c^s (K+1)^{s-1}}{r_1^{2(s-1)}} I_{k+1} & \left(\dfrac{cK}{r_1^2} + \dfrac{L}{r_2^2}\right)^s b_{i\beta} \\ \left(\dfrac{cK}{r_1^2} + \dfrac{L}{r_2^2}\right)^s c_{\alpha j} & \dfrac{2^s (L+1)^s}{r_2^{2s}} d_{\alpha\beta} - \dfrac{2^{s+1} c (L+1)^{s-1}}{r_2^{2(s-1)}} I_{l+1} \end{pmatrix}.$$

Then one checks that the following equation is satisfied

$$\Delta^3\tilde{x} + p\Delta^2\tilde{x} + q\Delta\tilde{x} + r(\tilde{x} - \tilde{x}_0) = 0 \tag{95}$$

for

$$p = -\left[\frac{c(3K+2)}{r_1^2} + \frac{3L+2}{r_2^2}\right], \quad r = -\frac{4c(K+1)(L+1)}{r_1^2 r_2^2}\left(\frac{cK}{r_1^2} + \frac{L}{r_2^2}\right),$$

$$q = 2\left[\frac{K(K+1)}{r_1^4} + \frac{L(L+1)}{r_2^4} + \frac{c(4KL + 3K + 3L + 2)}{r_1^2 r_2^2}\right],$$

and

$$\tilde{x}_0 = \begin{pmatrix} \dfrac{2r_1^2}{K+1} I_{k+1} & O \\ O & \dfrac{2cr_2^2}{L+1} I_{l+1} \end{pmatrix}, \quad k + l = m - 1. \tag{96}$$

This means that any A_2-hypersurface is of 3-type if the polynomial $\lambda^3 + p\lambda^2 + q\lambda + r$ has simple real roots (and the hypersurface is not already of lower type). Those roots are found to be

$$\lambda_u = \frac{cK}{r_1^2} + \frac{L}{r_2^2}, \quad \lambda_v = \frac{2c(K+1)}{r_1^2}, \quad \lambda_w = \frac{2(L+1)}{r_2^2}.$$

When $c = 1$, the equality of any two among these three roots leads to 2-type examples (ii) and (iii) in Theorem 1. If we look for mass-symmetric examples of class A_2 then $\tilde{x}_0 = I/(m+1)$, which gives $\cot^2 r = \dfrac{K+1}{L+1}$, thus again leading to the example (ii), which is of 2-type. So there are no mass-symmetric 3-type examples among A_2-hypersurfaces in $\mathbf{C}P^m$. On the other hand, when $c = -1$ no equality between the roots λ_u, λ_v, λ_w is possible and we know from Lemma 3 that no example of class A_2 in $\mathbf{C}H^m(-4)$ is of 2-type, they are all, therefore, of 3-type. Since the constant part $\tilde{x}_0$ in 3-type decomposition has the form given in (96) and cannot clearly equal $I/(m+1)$, the only way such hypersurface can be mass-symmetric, according to our definition, is that the hypersurface is of null 3-type, i.e. the eigenvalue $\lambda_u = 0$, in which case $\tilde{x}_0$ can be changed to equal $I/(m+1)$. This gives the condition $\coth^2 r = K/L$, i.e. the radius of the tube about $\mathbf{C}H^k(-4)$ is $r = \coth^{-1}\sqrt{\frac{2k+1}{2l+1}}$, $1 \le l < k \le m-2$, $k + l = m - 1$. In that case we get a mass-symmetric null 3-type hypersurface in $\mathbf{C}H^m(-4)$:

$$\pi(H_1^K(\cosh r) \times S^L(\sinh r)) = \pi\left(H_1^{2k+1}\left(\sqrt{\frac{2k+1}{2(k-l)}}\right) \times S^{2l+1}\left(\sqrt{\frac{2l+1}{2(k-l)}}\right)\right).$$

Additional examples of mass-symmetric 3-type hypersurfaces have to be searched for among classes B, C, D, and E. We derive next certain necessary conditions for hypersurface with $\operatorname{tr} A = \text{const}$ to be mass-symmetric and of 3-type.

Let M^n be a CMC Hopf hypersurface of $\mathbf{C}Q^m(4c)$, $(n = 2m-1)$ which is of 3-type via $\tilde{x}$ and mass-symmetric in the hyperquadric centered at $I/(m+1)$ containing $\Phi(\mathbf{C}Q^m)$ and defined by $\left\langle P - \frac{I}{m+1}, P - \frac{I}{m+1}\right\rangle = \frac{cm}{2(m+1)}$. Then

$$\Delta^3\tilde{x} + p\Delta^2\tilde{x} + q\Delta\tilde{x} + r(\tilde{x} - I/(m+1)) = 0, \tag{97}$$

where p, q, r are the (signed) elementary symmetric functions of the eigenvalues λ_u, λ_v, λ_w associated with a 3-type decomposition of $\tilde{x}$. We will consider various components of this equation to derive a set of necessary conditions for a Hopf hypersurface with constant $\operatorname{tr} A$ to be mass-symmetric and of 3-type. Those will include the conditions $\operatorname{tr} A^k = \text{const}$, $1 \le k \le 4$. Recall that the normal space $T_P^\perp \mathbf{C}Q^m$ in $H^{(1)}(m+1)$ is spanned by the position vector P and vectors of the form $\sigma(Z, W)$, $Z, W \in T_P\mathbf{C}Q^m$ [15]. Using (12) and (23) we get from (14), (18), and (34) respectively

$$\langle \Delta\tilde{x}, \tilde{x}\rangle = n, \quad \langle \Delta^2\tilde{x}, \tilde{x}\rangle = f^2 + 2c(n^2 + 2n - 1), \tag{98}$$

$$\begin{aligned}\langle \Delta^3\tilde{x}, \tilde{x}\rangle = {} & f^2[f_2 + 5c(n-1)] - 8c\varkappa^2 + 4n(n+2)^2 - 20 \\ & + 16c\varkappa f - 16cf \operatorname{tr}(SAS) - 4c \operatorname{tr}(SA^2S) + 8c \operatorname{tr}(SA)^2.\end{aligned} \tag{99}$$

Further, choosing a J-basis $\{e_i, Se_i\}$ of $\mathscr{D}$ and using (42) in the form $\mu_i\mu_i^* = c + \frac{\varkappa}{2}(\mu_i + \mu_i^*)$, we compute

$$\operatorname{tr}(SAS) = \varkappa - f, \quad \operatorname{tr}(SA^2S) = \varkappa^2 - f_2, \quad \operatorname{tr}(SA)^2 = \varkappa^2 - \varkappa f - (n-1)c.$$

Substituting in (99) we obtain $\langle \Delta^3 \tilde{x}, \tilde{x} \rangle$ as a sum of several terms, one of which is $(f^2 + 4c)f_2$ and the others are constants depending only on $\varkappa$, f, c, n. Therefore taking the metric product of (97) with $\tilde{x}$ and using the above information, we see that if $f^2 + 4c \neq 0$ (this condition is always satisfied in the projective case) it follows that $f_2 = \operatorname{tr} A^2$ is constant. Thus we will subsequently assume that $f^2 \neq 4$ in the hyperbolic case to ensure the constancy of f_2.

Next, we look at the ξ-component of (97). From (5) we compute

$$\sum_i \langle J[(\nabla_{e_i} A)^2 (Se_i) - (\nabla_{e_i} A)(SAe_i)], \xi \rangle = \varkappa^2 f - \varkappa f_2,$$

so that the ξ-component of $\Delta^3 \tilde{x}$ equals

$$\begin{aligned} \text{(100)} \quad \langle \Delta^3 \tilde{x}, \xi \rangle = {} & 8c(\varkappa^2 f - \varkappa f_2) + 8c\varkappa^3 + 8(2cf_2 + n + 4)\varkappa \\ & - 8cf_3 - f[f_2^2 + 4c(n+4)f_2 + 4cf^2 + 7n^2 + 30n + 19]. \end{aligned}$$

We also have that $\langle \Delta \tilde{x}, \xi \rangle = -f$ and $\langle \Delta^2 \tilde{x}, \xi \rangle = 4c\varkappa - f[f_2 + c(3n+5)]$ are constant. Thus taking the metric product of (97) with ξ we get $f_3 = \text{const.}$ In finding the $\sigma(\xi, \xi)$-component of (97) note that

$$\langle \sigma(\xi, \xi), \sigma(\xi, \xi) \rangle = 4c, \quad \sum_i \langle \sigma(e_i, SASe_i), \sigma(\xi, \xi) \rangle = -2c(f - \varkappa),$$

$$\sum_i \langle \sigma(e_i, SA^2Se_i), \sigma(\xi, \xi) \rangle = -2c(f_2 - \varkappa^2),$$

$$\sum_i \langle \sigma(e_i, (SA)^2 e_i), \sigma(\xi, \xi) \rangle = -2(n-1) - 2c\varkappa(f - \varkappa),$$

and

$$\begin{aligned} & \sum_{i,j} \langle \sigma((\nabla_{e_j} A)e_i, (\nabla_{e_j} A)e_i), \sigma(\xi, \xi) \rangle \\ & \quad = 2c\|\nabla A\|^2 + 2c \sum_j \langle (\nabla_{e_j} A)U, (\nabla_{e_j} A)U \rangle \\ & \quad = 2c\|\nabla A\|^2 - 2c\varkappa^2 \operatorname{tr}(AS^2A) + 4c\varkappa \operatorname{tr}(ASASA) - 2c \operatorname{tr}(ASA)^2 \\ & \quad = 2c[f_2 - c(n+3)]f_2 - 2cff_3 + 14(n-1)c \\ & \qquad + 2f^2 + 6\varkappa f + c\varkappa^2 f_2 - (n-1)\varkappa^2 - c\varkappa^3 f, \end{aligned}$$

by way of (23) and (71). Now, when the inner product of (97) with $\sigma(\xi, \xi)$ is taken we get a sum of several terms equal to zero. The only term in this sum containing f_4 is $8cf_4$ and the other terms depend only on $\varkappa$, f, f_2, f_3, p, q, r, c, n, and are thus constant. It follows, therefore, that $f_4 = \text{const.}$ We note that $\sigma(U, \xi) = 0$ by (12) and considering $\sigma(X, \xi)$-component of (97) for $X \in \Gamma(\mathscr{D})$ we compute using the Codazzi equation that $\mathrm{tr}((\nabla_X A) \circ [A, S]) = 0$. For the part of (97) normal to $\mathbf{C}Q^m(4c)$ it remains to consider $\sigma(X, Y)$-component for $X \in \Gamma(\mathscr{D})$ and $Y \in \Gamma(TM)$. We compute

$$\langle \sigma(\xi, \xi), \sigma(X, Y) \rangle = 2c\langle X, Y \rangle + 2c\langle X, U \rangle \langle Y, U \rangle,$$

$$\begin{aligned}\sum_i &\langle \sigma(A^k e_i, A^l e_i), \sigma(X, Y) \rangle \\ &= 2c[\mathrm{tr}(A^{k+l})\langle X, Y \rangle + \langle A^{k+l} X, Y \rangle - \langle SA^{k+l} SX, Y \rangle],\end{aligned}$$

with k, l integers ≥ 0, and $A^0 = I$. Further,

$$\sum_i \langle \sigma(e_i, SASe_i), \sigma(X, Y) \rangle = 2c(\varkappa - f)\langle X, Y \rangle - 2c\langle AX, Y \rangle + 2c\langle SASX, Y \rangle,$$

$$\begin{aligned}\sum_i &\langle \sigma(e_i, SA^2 Se_i), \sigma(X, Y) \rangle \\ &= 2c(\varkappa^2 - f_2)\langle X, Y \rangle - 2c\langle A^2 X, Y \rangle + 2c\langle SA^2 SX, Y \rangle,\end{aligned}$$

$$\begin{aligned}\sum_i \langle \sigma(Ae_i, SASe_i), \sigma(X, Y) \rangle &= \sum_i \langle \sigma(e_i, SASAe_i), \sigma(X, Y) \rangle \\ &= 2c[\varkappa^2 - \varkappa f - (n-1)c]\langle X, Y \rangle \\ &\quad + 2c\langle [(SA)^2 + (AS)^2]X, Y \rangle,\end{aligned}$$

$$\begin{aligned}\sum_{i,j} &\langle \sigma((\nabla_{e_j} A)e_i, (\nabla_{e_j} A)e_i), \sigma(X, Y) \rangle \\ &= 2c\|\nabla A\|^2 \langle X, Y \rangle + 2c\langle BX, Y \rangle + 2c\langle B(SX), SY \rangle,\end{aligned}$$

where $B := \sum_j (\nabla_{e_j} A)^2$ is a well-defined endomorphism of TM, independent of the choice of the basis $\{e_i\}$. Next, we compute

$$\langle \Delta \tilde{x}, \sigma(X, Y) \rangle = -2c(n+2)\langle X, Y \rangle,$$

$$\begin{aligned}\langle \Delta^2 \tilde{x}, \sigma(X, Y) \rangle &= -4c\langle ASX, ASY \rangle - 4cf\langle ASX, SY \rangle - 4c\langle AX, AY \rangle \\ &\quad - 4cf\langle AX, Y \rangle - [4(n+1)(n+3) + 2cf^2]\langle X, Y \rangle,\end{aligned}$$

$$\begin{aligned}\langle \Delta^3 \tilde{x}, \sigma(X, Y) \rangle &= 8c\langle BX, Y \rangle + 8c\langle BSX, SY \rangle + 32\langle [(SA)^2 + (AS)^2]X, Y \rangle \\ &\quad + 8f(cf_2 + n + 7)\langle SASX, Y \rangle + 16(1 + cf_2)\langle SA^2 SX, Y \rangle \\ &\quad + 8c\langle SA^4 SX, Y \rangle - 8c\langle A^4 X, Y \rangle - 16(1 + cf_2)\langle A^2 X, Y \rangle \\ &\quad - 8f(cf_2 + n + 7)\langle AX, Y \rangle + [8\varkappa^2 + 16\varkappa f - 8f_2 - 2cf^2 f_2 \\ &\quad - 2(5n+19)f^2 - 8c(n^3 + 6n^2 + 10n + 7)]\langle X, Y \rangle.\end{aligned}$$

Thus taking the inner product of (97) with $\sigma(X, Y)$ and dropping Y we get

$$(101)\qquad BX - SBSX = A^4X - SA^4SX + a(A^2X - SA^2SX) + b(AX - SASX) - 4c[(SA)^2 + (AS)^2]X + dX,$$

where $X \in \Gamma(\mathscr{D})$, $a = (p/2) + 2(c + f_2)$, $b = (pf/2) + cf(cf_2 + n + 7)$, and

$$d = n^3 + 6n^2 + 10n + 7 + \frac{c}{4}(5n + 19)f^2 + \frac{1}{4}f^2 f_2 + cf_2 - 2c\varkappa f - c\varkappa^2 + \frac{p}{4}[2c(n + 1)(n + 3) + f^2] + \frac{q}{4}(n + 2) + \frac{cr}{8}.$$

There remains the part of (97) tangent to M to be considered. Relation (97) has no U-component and for $X \in \mathscr{D}$ we have by the Codazzi equation

$$\sum_i \langle J[(\nabla_{e_i}A^2)(Se_i) - (\nabla_{e_i}A)(SAe_i)], X\rangle = \mathrm{tr}((\nabla_{SX}A) \circ [S, A]).$$

The right-hand side of this is also the result of the metric product of the left-hand side of (97) with X and therefore must be equal to zero, which is the same piece of information contained in the $\sigma(X, \xi)$-component. Hence we have the following

Lemma 6. *Let $M^n \subset \mathbf{C}Q^m(4c)$ be a Hopf hypersurface $(n = 2m - 1)$ with constant mean curvature (in the hyperbolic case we assume, additionally, that $(\mathrm{tr}\, A)^2 \neq 4$). If M^n is mass-symmetric and of 3-type in $H^{(1)}(m + 1)$ then we have*

(i) $\mathrm{tr}\, A^k = \mathrm{const}$, *for* $k = 1, 2, 3, 4$;
(ii) $\mathrm{tr}((\nabla_X A) \circ [A, S]) = 0$, *for every* $X \in \Gamma(\mathscr{D})$;
(iii)

$$BX - SBSX = A^4X - SA^4SX + a(A^2X - SA^2SX) + b(AX - SASX) - 4c[(SA)^2 + (AS)^2]X + dX,$$

where $B := \sum_j (\nabla_{e_j}A)^2$, $X \in \Gamma(\mathscr{D})$, $c = \pm 1$, *and* a, b, d *are constants.*

Essentially, the conditions (i)–(iii) are also sufficient conditions for such M to be mass-symmetric and of type ≤ 3 since we obtained these conditions by considering all of the components of equation (97), provided that the constants a, b, c, d and $\mathrm{tr}\, A^k$ are such to enable p, q, r to be real and the polynomial equation $t^3 + pt^2 + qt + r = 0$ to have simple roots [12]. Note that the condition (ii) is automatically satisfied when M is a Hopf hypersurface whose induced Hopf foliation is a Riemannian foliation [3, p. 64] since then U is a Killing vector field and $AS = SA$. See also [27], characterizing class-A hypersurfaces by the condition $AS = SA$.

Corollary 2. *Let M be a CMC Hopf hypersurface of $\mathbf{C}Q^m$ satisfying $(\operatorname{tr} A)^2 \neq -4c$ and having at most four distinct principal curvatures at each point. If M is mass-symmetric and of 3-type via $\tilde{x}$ then M has constant principal curvatures.*

We show next that every hypersurface of class B is mass-symmetric in the corresponding hyperquadric and, apart from those two tubes in Lemma 4, of 3-type. Using the information from Lemma 4, we get respectively from (16), (26), (17) and (67) the following

$$\Delta\xi = [f_2 + c(n-1)]\xi + (2\varkappa - f)\sigma(\xi,\xi) - \frac{4c}{\varkappa}\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i), \tag{102}$$

$$\Delta(\sigma(\xi,\xi)) = 4c\varkappa\xi + 2\left[\frac{8(n-1)}{\varkappa^2} + c(n+1)\right]\sigma(\xi,\xi) - \frac{16}{\varkappa^2}\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i), \tag{103}$$

$$\begin{aligned} \sum_{e_i \in \mathscr{D}} \Delta(\sigma(e_i, e_i)) = {} & 2\left[\frac{16}{\varkappa^2} + c(n+3)\right]\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i) \\ & - 2\left[\frac{16(n-1)}{\varkappa^2} + c(n-1)\right]\sigma(\xi,\xi) \\ & + 2c\left[(n-1)\varkappa - \frac{2c(n-1)(n+3)}{\varkappa}\right]\xi. \end{aligned} \tag{104}$$

The third iterated Laplacian $\Delta^3\tilde{x}$ for a hypersurface of class B can be computed from (34)–(35) but that would require finding the connection coefficients of the hypersurface. It seems easier to find $\Delta^3\tilde{x}$ directly by applying the Laplacian to (75) and using (102)–(104). We get

$$\begin{aligned} \Delta^3\tilde{x} = {} & \left[\frac{128c(n-1)^3}{\varkappa^5} + \frac{128(n-1)(n^2+1)}{\varkappa^3} + \frac{8c(n-1)(3n^2+2n+3)}{\varkappa}\right. \\ & \left. - 16n\varkappa - 4c(n+1)\varkappa^3 - \varkappa^5\right]\xi \\ & + \left[\frac{32(n-1)(n+3)(3n+1)}{\varkappa^4} + \frac{8c(n-1)(3n^2+14n+3)}{\varkappa^2}\right. \\ & \left. + 8(n^2-4n+1) - 2c(3n+1)\varkappa^2 - \varkappa^4\right]\sigma(\xi,\xi) \\ & - \left[\frac{128(n+1)^2}{\varkappa^4} + \frac{48c(n+1)^2}{\varkappa^2} + 4(n-1)(n+3) - 4c\varkappa^2\right]\sum_{e_i \in \mathscr{D}} \sigma(e_i, e_i). \end{aligned} \tag{105}$$

Then using (74), (75), and (105) we have

Lemma 7. *Every class-B Hopf hypersurface with constant principal curvatures in* $\mathbf{C}H^m(-4)$ *is mass-symmetric and of* 3*-type via* $\tilde{x}$. *The same is true for hypersurfaces of class B in* $\mathbf{C}P^m(4)$, *with the exception of those two tubes about* Q^{m-1} *referred to in Lemma* 4, *which are mass-symmetric and of* 2*-type.*

Proof. These hypersurfaces are not of 1-type and the only 2-type examples are given in Lemma 4. We show that they satisfy the 3-type equation (97) and that the polynomial $P(t) = t^3 + pt^2 + qt + r$ has three distinct real roots except for one value of $\varkappa$, so that for other values of $\varkappa$ the result of [12] proves it then to be mass-symmetric and of type ≤ 3. Moreover, they will be exactly of 3-type if $P(t)$ is the minimal polynomial of the immersion $\tilde{x} - \tilde{x}_0$, i.e. the hypersurface does not satisfy a lower-degree polynomial in the Laplacian. Indeed, using the Gauss elimination with

$$p = -\frac{1}{\varkappa^2}(\varkappa^2 + 4c)[\varkappa^2 + 2c(3n+1)],$$

$$q = \frac{4}{\varkappa^4}(\varkappa^2 + 4c)[c(n+1)\varkappa^4 + (3n^2 + 6n - 1)\varkappa^2 + 8c(n^2 - 1)], \quad \text{and}$$

$$r = -\frac{4(n-1)(n+3)}{\varkappa^4}(\varkappa^2 + 4c)^2[\varkappa^2 + 2c(n+1)]$$

we can verify that (97) holds by equating all components with zero. Note that the normal space to $\mathbf{C}Q^m$ in $H^{(1)}(m+1)$ at a point $P \in M^n$ is spanned by vectors of the form $\tilde{x}$, $\sigma(X, Y), \sigma(X, \xi), \sigma(\xi, \xi)$, for $X, Y \in \mathscr{D}$. Moreover, the roots of the cubic equation $t^3 + pt^2 + qt + r = 0$ are real and they are found to be

$$\lambda_u = \frac{2c(n-1)(\varkappa^2 + 4c)}{\varkappa^2},$$

$$\lambda_v, \lambda_w = \frac{(\varkappa^2 + 4c)[\varkappa^2 + 4c(n+1)] \pm \sqrt{(\varkappa^2 + 4c)[\varkappa^6 - 12c\varkappa^4 + 64c(n+1)^2]}}{2\varkappa^2}.$$

Note that $\varkappa^2 + 4c \neq 0$ for a class-B hypersurface. Equality of any two roots is possible only when $c = 1$ and $\lambda_u = \lambda_w$ (with minus sign at the radical), where $\varkappa^2 = 2(\sqrt{2m^2 - 1} - 1)$, identifying it as the 2-type example of Theorem 1 (v). For the example given in Theorem 1 (iv) we have $\varkappa^2 = 4m$ and

$$P(t) = \left[t^2 - \frac{4(m+1)(2m-1)}{m}t + 16\frac{(m-1)(m+1)^2}{m}\right][t - 8(m+1)],$$

where the quadratic trinomial in the first pair of brackets is the minimal polynomial of that 2-type hypersurface according to (83), (84).

Now we are in a position to prove our classification result for 3-dimensional Hopf hypersurfaces of $\mathbf{C}Q^2(4c)$:

Theorem 3. *Let M^3 be a Hopf hypersurface of $\mathbf{C}P^2(4)$ with constant mean curvature. Then M^3 is mass-symmetric and of* 3*-type in* $H(3)$ *if and only if M^3 is a class-B hypersurface, that is, an open portion of a tube of any radius* $r \in (0, \pi/4)$ *about the complex quadric* Q^1 (*equivalently the tube of radius* $\dfrac{\pi}{4} - r$ *about a canonically embedded* $\mathbf{R}P^2$), *except when* $\cot r = \sqrt{2} + \sqrt{3}$ *and* $\cot r = \sqrt{\sqrt{6} + \sqrt{7}}$.

Proof. According to Corollary 2 such hypersurface has constant principal curvatures and thus it is one from the Takagi's list in $\mathbf{C}P^2$. Earlier analysis shows that it cannot be a hypersurface of class A_2, which is of 2-type when mass-symmetric nor any of geodesic spheres, which are of 1- and 2-type. Standard examples of class C, D, and E need not be considered because of dimension restriction. Then Lemma 7 proves that class-B examples are in fact the only ones, excluding the two 2-type tubes referred to in Lemma 4.

Theorem 4. *Let M^3 be a Hopf hypersurface of $\mathbf{C}H^2(-4)$ with constant mean curvature and* $(\mathrm{tr}\, A)^2 \neq 4$. *Then M^3 is mass-symmetric and of* 3*-type in* $H^1(3)$ *if and only if M^3 is a class-B hypersurface, that is, an open portion of a tube of any radius* $r > 0$ *about a canonically embedded, totally real, totally geodesic* $\mathbf{R}H^2 \subset \mathbf{C}H^2(4)$.

Proof. We know by Corollary 1 that the principal curvatures are constant and therefore examples are to be found among the standard ones from the Montiel's list. Every example of class B is mass-symmetric and of 3-type by Lemma 7. Moreover, by (67), (69), with $c = -1$, $m = 2$, we see that $\mathrm{tr}\, A \neq 2$ for these hypersurfaces. A class-A_0 hypersurface in $\mathbf{C}H^m$ (a horosphere) is not of finite type since it satisfies $\Delta^2 \tilde{x} = \mathrm{const} \neq 0$ [18]. Class-A_1 hypersurfaces (geodesic spheres and tubes about the complex hyperbolic hyperplane) are of 2-type. Class-A_2 hypersurfaces i.e. tubes about totally geodesic $\mathbf{C}H^k(-4)$, $1 \leq k \leq m-2$, are of 3-type as shown before and among them there are some mass-symmetric ones. But because $m = 2$ here, these examples are not possible. Note that an A_2-hypersurface degenerates into an A_1-hypersurface when $l = 0$ or $k = 0$. $\square$

The case of 3-type hypersurfaces of $\mathbf{C}H^2(-4)$ with $(\mathrm{tr}\, A)^2 = 4$ is also interesting, but a different analysis is needed to study them. Because of this property, they are akin to the so-called Bryant surfaces in $\mathbf{R}H^3$, see [17]. Also it would be interesting to determine the Chen-type of the standard examples of class C, D and E in $\mathbf{C}P^m(4)$ and, generally, study CMC Hopf hypersurfaces of 3-type in $\mathbf{C}Q^m(4c)$ when $m \geq 3$. The techniques developed here can be modified to study curvature-adapted hypersurfaces of low type in quaternionic space forms and partly also in octonion planes, the topic that will be treated in our subsequent papers.

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