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PSEUDO-ANOSOV ELEMENTS OF MAPPING CLASS GROUPS OF HEEGAARD SURFACES OF THE 3-SPHERE

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Abstract

An infinite family of pseudo-Anosov diffeomorphisms over Heegaard surface of the 3-sphere is constructed, when genus is at least 3.

1. Introduction

By Thurston [14], isotopy classes of diffeomorphisms over closed oriented surface are classified into periodic, reducible or pseudo-Anosov, according to their dynamical properties. In this paper, we concretely construct an infinite family of pseudo-Anosov diffeomorphisms which satisfy a certain condition.

A genus g *handlebody* H_g is an oriented 3-dimensional manifold constructed from a 3-ball by attaching g 1-handles. Then $\partial H_g = \Sigma_g$, a closed oriented surface of genus g. We embed H_g in S^3 such that $H_g^* = \overline{S^3 \backslash H_g}$ is a genus g handlebody. The triple (S^3, H_g, H_g^*) is called a *Heegaard splitting* of S^3, and $\partial H_g \subset S^3$ is called a *Heegaard surface* of S^3. Pseudo-Anosov diffeomorphisms over Σ_g which are restrictions of diffeomorphisms over H_g are constructed by Fathi and Laudenbach [4], and pseudo-Anosov diffeomorphisms over Heegaard surfaces of S^3 which are restrictions of diffeomorphisms over S^3 are constructed by Kobayashi [9], Johnson and Rubinstein [8]. In this paper, we introduce other family of pseudo-Anosov diffeomorphisms which satisfy the same conditions as above. Our construction is simple, easy to visualize, and easy to be generalized to construct infinitely many pseudo-Anosov diffeomorphisms over Heegaard surfaces of S^3. When the genus g is at least 3, these diffeomorphisms are constructed as follows (for the definition of diffeomorphisms explained in the later sentence, see the next section). Let ρ be a rotation of H_g, ω_1 be a twist of the 1-st knob of H_g, and $\eta_{1,j}$ be a sliding of the 1-st handle over the j-th handle $(2 \leq j \leq g)$. Let $a_k = \dfrac{g!}{k!(g-k)!}$ for $1 \leq k \leq g-2$ and $a_{g-1} = g+2$. Then $\omega_1 {\eta_{1,2}}^{a_1} {\eta_{1,3}}^{a_2} \cdots {\eta_{1,g}}^{a_{g-1}} \rho$ is pseudo-Anosov.

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2. Elements of Goeritz group

The mapping class group $\mathcal{M}_g$ of Σ_g is the group of isotopy classes of orientation preserving diffeomorphisms over Σ_g. For a simple closed curve a on Σ_g, a Dehn twist τ_a about this circle a is the diffeomorphism over Σ_g which affects an arc crossing a by causing it to turn right as it approaches a, run once around a, and then progress on as before. For any elements ϕ_1, ϕ_2 in $\mathcal{M}_g$, $\phi_1\phi_2$ means apply ϕ_2 first, then apply ϕ_1. Let H_g and (S^3, H_g, H_g^*) be as explained in §1. Let $\mathcal{H}_g$ be the subgroup of $\mathcal{M}_g$ defined by

$$\mathcal{H}_g = \left\{ \phi \in \mathcal{M}_g \,\middle|\, \begin{array}{l} \text{there is an orientation preserving diffeomorphism } \Phi \\ \text{over } S^3 \text{ such that } \Phi|_{\partial H_g} = \phi \end{array} \right\}.$$

This group $\mathcal{H}_g$ is called *Goeritz group*. When the genus $g = 2$, $\mathcal{H}_2$ is finitely generated by Goeritz [5], and by Scharlemann [12] with a modern proof. Akbas [1] and Cho [3] obtained finite presentations of $\mathcal{H}_2$. In [11], Powell claimed that $\mathcal{H}_g$ are finitely generated for the general g, but Scharlemann [12] pointed out a gap in its proof. It is still an open question, for the general g, whether $\mathcal{H}_g$ is finitely generated or not (see also Remark 1).

In order to introduce some elements of $\mathcal{H}_g$, we settle some notations. Let P_g be a disk D_0 removed the interior of g disks $D_1, \ldots, D_g$, and $\alpha_1, \ldots, \alpha_g$ be the arcs properly embedded in P_g such that α_i connects ∂D_0 and ∂D_i, and o be the center of D_0 (see the left of Figure 1). We embed P_g into the equatorial sphere S^2 in S^3, and add thickness to this P_g, then we get an embedding of $H_g = P_g \times [0,1]$ into S^3. The closure of $S^3 \backslash H_g$ is homeomorphic to the genus g 3-dimensional handlebody H_g^*. Let N_i be a regular neightborhood of α_i, then $h_i = N_i \times [0,1]$ is 1-handle attached to the 3-ball $\overline{(P_g \backslash (N_1 \cup \cdots \cup N_g))} \times [0,1]$. We call h_i the i-th handle. Let $x_1 = \partial(\alpha_1 \times [0,1]), \ldots, x_g = \partial(\alpha_g \times [0,1])$, $y_1 =$

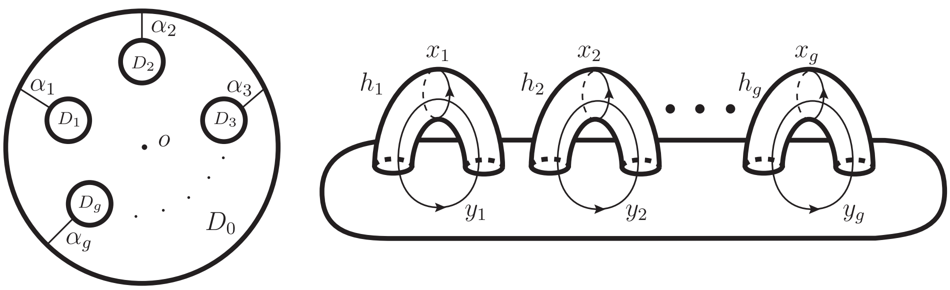

FIGURE 1. P_g and oriented curves and handles of H_g

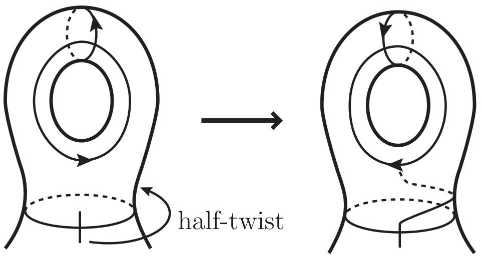

FIGURE 2. Twisting the knob K_1

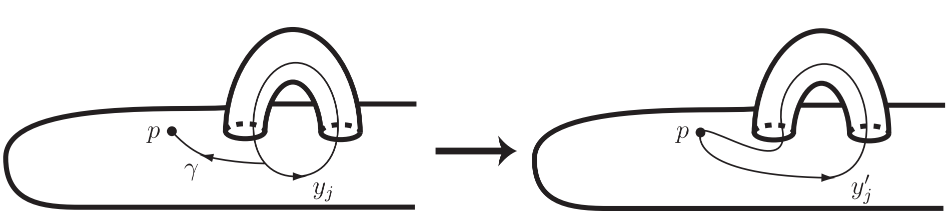

FIGURE 3. Deformation of y_j

$\partial D_1 \times \{1/2\}, \dots, y_g = \partial D_g \times \{1/2\}$, and be oriented as in the right of Figure 1. We define elements of $\mathscr{H}_g$ as follows.

ρ: *a cyclic translation of handles* (this is an extension to S^3 of ρ defined in [13]).

ρ is a rotation of H_g about the vertical axis $o \times [0,1]$ through $2\pi/g$ radius in the clockwise orientation.

ω_1: *a twisting a knob* (this is an extension to S^3 of ω_1 defined in [13]).

Let K_1 be a regular neighborhood of $h_1 \cup y_1$ in H_g. We twist K_1 as indicated in Figure 2.

$\eta_{1,j}$ $(2 \le j \le g)$: *slidings the* 1*-st handle* (this is an extension to S^3 of $\theta_{1,j}\tau_1^{-1}$ in [13]).

The 1-st handle h_1 is attached to the 3-ball $\overline{(P_g \backslash (N_1 \cup \cdots \cup N_g))} \times [0,1]$ along the two disks B_1 and B_2, where B_1 is the left foot of h_1 and B_2 is the right foot of h_1. Let p be the center of B_2, and γ be the arc from a point on y_j to p as shown in Figure 3. We deform y_j along this arc γ as indicated in Figure 3, and the resulting circle was denoted by y'_j. Let Δ be a disk properly embedded in $\overline{S^3 \backslash (H_g \backslash h_1)}$ such that $\partial\Delta = y'_j$. We take a regular neighborhood $N(\Delta)$ of Δ in $\overline{S^3 \backslash (H_g \backslash h_1)}$ such that $N(\Delta) \cap h_1$ is as illustrated in the left of Figure 4. The map indicated in Figure 4 is $\eta_{1,j}$. The restriction of this map $\eta_{1,j}$ to the boundary ∂H_g is equal to $\tau_a \tau_b^{-1} \tau_c$, where a, b and c are illustrated in Figure 5.

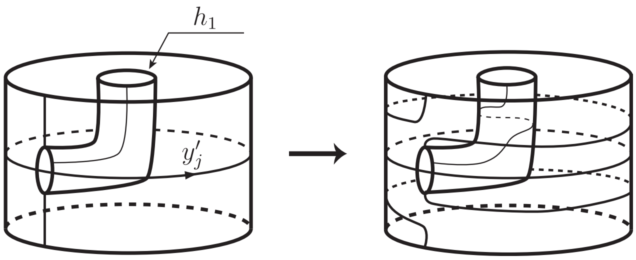

FIGURE 4. Sliding the 1-st handle over the j-th handle

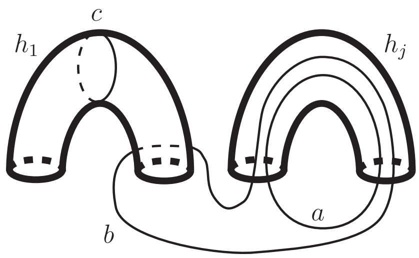

FIGURE 5. Dehn twists presentation of $\eta_{1,j}$

Remark 1. In [6], the author claimed that $\mathscr{H}_g$ is generated by ρ, ω_1 $\eta_{1,2}$ and one more element, but there is a serious gap in a proof of Lemma 5 in that paper.

Let N_x (resp. N_y) be the $\mathbf{Z}$-submodule of $H_1(\Sigma_g, \mathbf{Z})$ generated by $\{x_1, \dots, x_g\}$ (resp. $\{y_1, \dots, y_g\}$). If $\phi \in \mathscr{H}_g$, then $\phi_* : H_1(\Sigma_g, \mathbf{Z}) \to H_1(\Sigma_g, \mathbf{Z})$ preserves N_x and N_y as sets. For each element $\phi \in \mathscr{M}_g$, we define a $2g \times 2g$ matrix M_ϕ by

$$(\phi_*(x_1), \dots, \phi_*(x_g), \phi_*(y_1), \dots, \phi_*(y_g)) = (x_1, \dots, x_g, y_1, \dots, y_g)M_\phi.$$

If $\phi \in \mathscr{H}_g$, then $M_\phi = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$, where A is $g \times g$ matrix, 0 is a $g \times g$ zero matrix, and A^t is the transpose of A. Let U_1 and U_3 be the $g \times g$ matrix given by

$$U_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and $U_{1,j}$ $(2 \le j \le g)$ be the $g \times g$ matrix whose diagonal entries and the $(1, j)$-entry are 1 and other entries are 0. In the above notations, we dropped U_2, in order to follow the notations in [7]. For the elements of $\mathscr{H}_g$ introduced above, $M_\rho = \begin{pmatrix} U_1 & 0 \\ 0 & (U_1^t)^{-1} \end{pmatrix}$, $M_{\omega_1} = \begin{pmatrix} U_3 & 0 \\ 0 & (U_3^t)^{-1} \end{pmatrix}$, and $M_{\eta_{1,j}} = \begin{pmatrix} U_{1,j} & 0 \\ 0 & (U_{1,j}^t)^{-1} \end{pmatrix}$.

3. Margalit-Spallone condition

For $\phi \in \mathscr{M}_g$, let M_ϕ be the matrix presentation of the homomorphism $\phi_* : H_1(\Sigma_g, \mathbf{Z}) \to H_1(\Sigma_g, \mathbf{Z})$ introduced in the previous section, and $p_\phi(x)$ be the characteristic polynomial of M_ϕ. In [2, Lemma 5.1], Casson and Bleiler proved the following fact.

Theorem 1. *Let $\phi \in \mathscr{M}_g$. If $p_\phi(x)$ is irreducible over $\mathbf{Z}$, has no roots of unity as zeros, and is not a polynomial in x^k for $k > 1$, then ϕ is pseudo-Anosov.*

If $\phi \in \mathscr{H}_g$, then $p_\phi(x)$ is reducible polynomial. Hence, we can not apply the above criterion for elements of $\mathscr{H}_g$. In [10], Margalit and Spallone introduced a more subtle criterion. Let *Sym* be the map from $\mathbf{Z}[x]$ to itself defined by

$$Sym(q(x)) = x^{\deg(q)} \cdot q\left(x + \frac{1}{x}\right).$$

This map *Sym* is multiplicative and injective. In [10, Proposition 2 and 6], Margalit and Spallone proved the following fact.

Theorem 2. *Let $\phi \in \mathscr{M}_g$, and $q_\phi(x) \in \mathbf{Z}[x]$ such that $Sym(q_\phi(x)) = p_\phi(x)$. If $q_\phi(x) = x^g + a_{g-1}x^{g-1} + \cdots + a_1 x + a_0$ satisfies the following condition*
$(*)$ $q_\phi(x)$ is irreducible, and $|a_{g-1}| > 2g$,
then ϕ is pseudo-Anosov.

We call the above condition $(*)$ for $\phi \in \mathscr{M}_g$ the *Margalit-Spallone condition*.

4. Pseudo-Anosov elements in $\mathscr{H}_g$

Let *Rev P* be the map from $\mathbf{Z}[x]$ to itself defined by

$$Rev\ P(q(x)) = q(x) \cdot x^{\deg(q)} \cdot q\left(\frac{1}{x}\right).$$

Let $\phi \in \mathscr{H}_g$, $M_\phi = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$, and $p_A(x)$ be the characteristic polynomial of A. Then $p_\phi(x) = Rev\ P(p_A(x))$. The following two lemmas are pointed out by Takashi Ichikawa. The proof of the following two lemmas due to him is given in Appendix below.

Lemma 3 (Ichikawa). *If* $q(x) = x^g + a_1x^{g-1} + \cdots + a_{g-1}x + 1 \in \mathbf{Z}[x]$ *is irreducible and satisfies the following condition,*

$(**)$ $|a_1 + a_{g-1}| > 2g$, *and* $q(x) \notin \mathrm{Im}(Sym)$ (*e.g.* $a_1 \neq a_{g-1}$),

then $Sym^{-1}(Rev\ P(q(x))) = x^g + b_{g-1}x^{g-1} + \cdots + b_1x + b_0$ *is irreducible and* $|b_{g-1}| > 2g$.

Lemma 4 (Ichikawa). *If* p *is a prime number and* n *is an integer which is positive or less than* $-4g/p$, *and satisfies* $\gcd\{p, n\} = 1$, *then the polynomial* $(x+1)^g + np(x+1) - np$ *is irreducible and satisfies the condition* $(**)$ *in Lemma* 3.

By using Lemma 3, we show

Theorem 5. *If* $q(x) = x^g + a_1x^{g-1} + \cdots + a_{g-1}x + 1$ *is an irreducible polynomial and satisfies* $|a_1 + a_{g-1}| > 2g$ *and* $a_1 \neq a_{g-1}$, *then* $\phi_{a_1,\dots,a_{g-1}} = \omega_1\eta_{1,2}{}^{a_1}\eta_{1,3}{}^{a_2} \cdots \eta_{1,g}{}^{a_{g-1}}\rho \in \mathscr{H}_g$ *is pseudo-Anosov.*

Proof. We set

$$A = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{g-1} & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

then $M_{\phi_{a_1,\dots,a_{g-1}}} = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$, hence $p_{\phi_{a_1,\dots,a_{g-1}}}(x) = Rev\ P(q(x))$. By Lemma 3, $\phi_{a_1,\dots,a_{g-1}}$ satisfies the Margalit-Spallone condition. Therefore, $\phi_{a_1,\dots,a_{g-1}}$ is pseudo-Anosov. □

Example 1. By using the above theorem and Lemma 4, we construct infinitely many pseudo-Anosov elements of $\mathscr{H}_g$ when $g \geq 3$. Let p be a prime number, and n be a positive integer such that $\gcd(p, n) = 1$. Let $a_k = \dfrac{g!}{k!(g-k)!}$ for $1 \leq k \leq g-2$ and $a_{g-1} = g + np$ then, by Lemma 4, $x^g + a_1x^{g-1} + a_2x^{g-2} + \cdots + a_{g-1}x + 1$ is irreducible and satisfies the condition in Theorem 5. Therefore, $\omega_1\eta_{1,2}{}^{a_1}\eta_{1,3}{}^{a_2} \cdots \eta_{1,g}{}^{a_{g-1}}\rho$ is pseudo-Anosov. Since there are infinitely many such pairs (p, n) as above, this construction gives us infinitely many pseudo-Anosov elements in $\mathscr{H}_g$. In §1, we explained the case where $p = 2$ and $n = 1$.

Remark 2. Let ϕ be any element of $\mathscr{M}_2$ which is a restriction of an orientation preserving diffeomorphism over H_2. Then ϕ does not satisfy the Margalit-Spallone condition.

Appendix. Proof of Lemmas 3 and 4 (by Takashi Ichikawa)

Proof of Lemma 3. Put $f(x) = Sym^{-1}(Rev\ P(q(x)))$. Then

$$x^g \cdot f\left(x + \frac{1}{x}\right) = q(x) \cdot x^g \cdot q\left(\frac{1}{x}\right),$$

and hence by comparing these terms of degree $2g - 1$, we have

$$|b_{g-1}| = |a_1 + a_{g-1}| > 2g.$$

If $f(x)$ were not irreducible, then by Gauss's lemma, there are nonconstant monic polynomials $f_1(x), f_2(x) \in \mathbf{Z}[x]$ such that $f(x) = f_1(x) \cdot f_2(x)$. Since $x^g \cdot q(1/x)$ is irreducible and $\mathbf{Z}[x]$ is a UFD (unique factorization domain), $q(x)$ is either $Sym(f_1(x))$ or $Sym(f_2(x))$ which contradicts that $q(x) \notin \mathrm{Im}(Sym)$. Therefore, $f(x)$ is irreducible.

Proof of Lemma 4. By Eisenstein's criterion, $(x+1)^g + np(x+1) - np \in \mathbf{Z}[x]$ is irreducible as a polynomial of $x + 1$, and hence is so as a polynomial of x. Since

$$(x+1)^g + np(x+1) - np = x^g + gx^{g-1} + \cdots + (g + np)x + 1,$$

the condition $(**)$ is equivalent to that $n > 0$ or $n < -4g/p$.

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