H.-W. XU, F. HUANG AND F. XIANG
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AN EXTRINSIC RIGIDITY THEOREM FOR SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE IN A SPHERE

Hong-Wei Xu, Fei Huang and Fei Xiang

Abstract

Let M be an n-dimensional closed submanifold with parallel mean curvature in S^{n+p}, $\tilde{h}$ the trace free part of the second fundamental form, and $\tilde{\sigma}(u) = \|\tilde{h}(u,u)\|^2$ for any unit vector $u \in TM$. We prove that there exists a positive constant $C(n,p,H)(\geq \frac{1}{3})$ such that if $\tilde{\sigma}(u) \leq C(n,p,H)$, then either $\tilde{\sigma}(u) \equiv 0$ and M is a totally umbilical sphere, or $\tilde{\sigma}(u) \equiv C(n,p,H)$. A geometrical classification of closed submanifolds with parallel mean curvature satisfying $\tilde{\sigma}(u) \equiv C(n,p,H)$ is also given. Our main result is an extension of the Gauchman theorem [4].

1. Introduction

Let M be an n-dimensional closed Riemannian manifold isometrically immersed in a unit sphere S^{n+p}. Simons [10], Lawson [5], Chern, do Carmo and Kobayashi [3] proved a rigidity theorem for closed minimal manifolds in a sphere with bounded second fundamental form. Later, A. M. Li and J. M. Li [6] improved Simons' pinching constant to $\max\{n/(2-1/p), 2n/3\}$.

It was extended to submanifolds with parallel mean curvature in a sphere, first by Okumura [7] and Yau [16, 17], then by Xu [11], and finally by Cheng and Nakagawa [2] in codimension 1, and by Xu [12, 13] in codimension p independently. Later, Shiohama and Xu [9, 14] generalized the rigidity theorem to the case where the ambient space is a pinched Riemannian manifold.

Set $\sigma(u) = \|h(u,u)\|^2$ for any $u \in UM$, where h is the second fundamental form of the immersion, UM is the unit tangent bundle over M. Gauchman [4] proved that if M is an n-dimensional closed minimal submanifold in S^{n+p}, and if $\sigma(u) \leq \frac{1}{3}$ for any unit vector $u \in TM$, then either $\sigma(u) \equiv 0$ or $\sigma(u) \equiv \frac{1}{3}$. Moreover he gave a geometrical classification of closed minimal submanifolds satisfying $\sigma(u) = \frac{1}{3}$.

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We give the following

Definition 1.1. Choose a local orthonormal frame field $\{e_1,\dots,e_{n+p}\}$ on S^{n+p} such that $e_1,\dots,e_n$ are tangent to M and let $\{\omega_1,\dots,\omega_n\}$ be the dual frame field of $\{e_1,\dots,e_n\}$. Denote by $h=\sum_{\alpha,i,j} h_{ij}^{\alpha}\omega_i\otimes\omega_j\otimes e_\alpha$ the second fundamental form of M. We define the trace free second fundamental form of M by

$$\tilde{h} := \sum_{\substack{1\le i,j\le n\\ n+1\le\alpha\le n+p}} \tilde{h}_{ij}^{\alpha}\omega_i\otimes\omega_j\otimes e_\alpha,$$

where $\tilde{h}_{ij}^{\alpha}=h_{ij}^{\alpha}-\left(\frac{1}{n}\sum_i h_{ii}^{\alpha}\right)\delta_{ij}$.

From the above definition it is easy to see that $\mathrm{tr}(\tilde{h}_{ij}^{\alpha})=\sum_i \tilde{h}_{ii}^{\alpha}=0$ for any α and that $\tilde{h}$ is a symmetric bilinear mapping $T_xM\times T_xM\to T_x^{\perp}M$ for $x\in M$, where T_xM is the tangent space of M at x and $T_x^{\perp}M$ is the normal space to M at x. Define the squared norm of $\tilde{h}(u,u)$ by

$$\tilde{\sigma}(u) := \|\tilde{h}(u,u)\|^2.$$

If $\tilde{\sigma}(u)\equiv 0$ for any $u\in UM$, then $S\equiv nH^2$, and M^n is a totally umbilical sphere in S^{n+p}.

The following examples will help us to state our result precisely.

Example 1.2. Let $S^m(r)$ be an m-dimensional sphere of radius r in $\mathbf{R}^{m+1}$. We imbed $S^m(a)\times S^m(b)$ in S^{2m+1} as follows. Let $u\in S^m(a)$ and $v\in S^m(b)$ be the vectors of length a and b in $\mathbf{R}^{m+1}$. If $a^2+b^2=1$, then we can consider (u,v) as a unit vector in $\mathbf{R}^{2m+2}=\mathbf{R}^{m+1}\times\mathbf{R}^{m+1}$. It is easy to see that $S^m(a)\times S^m(b)$ is a submanifold of $S^{2m+1}(1)$ with parallel mean curvature $H=\left|\frac{b^2-a^2}{2ab}\right|$. In particular,

$$S^m\left(\frac{1}{\sqrt{2(1+H^2+H\sqrt{1+H^2})}}\right)\times S^m\left(\frac{1}{\sqrt{2(1+H^2-H\sqrt{1+H^2})}}\right)$$

is an isoparametric hypersurface of S^{2m+1} with parallel mean curvature H and $\max_{u\in UM}\tilde{\sigma}(u)=1+H^2$.

Example 1.3 (see [4, 8]). Denote by RP^2, CP^2, QP^2 and $Cay\,P^2$ the projective plane over the real numbers, complex numbers, quaternions and octonions, $\psi_1: RP^2\to S^4(1)$, $\psi_2: CP^2\to S^7(1)$, $\psi_3: QP^2\to S^{13}(1)$ and $\psi_4: Cay\,P^2\to S^{25}(1)$ the corresponding isometric embeddings, and by $\tau_{n,m}: S^n(1)\to S^{n+m}(1)$ the inclusion. Let $\psi_1': S^2(\sqrt{3})\to S^4(1)$ be the isometric immersion defined by $\psi_1'=\psi_1\circ\pi$, where $\pi: S^2(\sqrt{3})\to RP^2$ is the canonical projection. We set

$$\phi_{1,p} = \tau_{4,p-1} \circ \psi_1 : RP^2 \to S^{2+p}, \quad p \geq 2,$$
$$\phi_{2,p} = \tau_{7,p-3} \circ \psi_2 : CP^2 \to S^{4+p}, \quad p \geq 3,$$
$$\phi_{3,p} = \tau_{13,p-5} \circ \psi_3 : QP^2 \to S^{8+p}, \quad p \geq 5,$$
$$\phi_{4,p} = \tau_{25,p-9} \circ \psi_4 : Cay\ P^2 \to S^{16+p}, \quad p \geq 9,$$
$$\phi'_{1,p} = \tau_{4,p-2} \circ \psi'_1 : S^2(\sqrt{3}) \to S^{2+p}, \quad p \geq 2.$$

Then $\phi_{i,p}$ is an isometric minimal embedding and $\phi'_{1,p}$ is an isometric minimal immersion.

In this paper, we prove a new extrinsic rigidity theorem for submanifold with parallel mean curvature in a sphere. More precisely, we obtain the following

Theorem 1.4 (Main Theorem). *Let M^n be an n-dimensional compact submanifold in a unit sphere $S^{n+p}(1)$ with parallel mean curvature vector field of norm H. If*

$$\tilde{\sigma}(u) \leq C(n,p,H), \quad \textit{for all } u \in UM,$$

then M is one of the following:
(1) *the totally umbilical sphere* $S^n_H = S^n\left(\dfrac{1}{\sqrt{1+H^2}}\right)$;
(2) *Clifford isoparametric hypersurface*

$$S^{n/2}\left(\frac{1}{\sqrt{2(1+H^2+H\sqrt{1+H^2})}}\right) \times S^{n/2}\left(\frac{1}{\sqrt{2(1+H^2-H\sqrt{1+H^2})}}\right);$$

(3) *one of the embeddings $\phi_{i,p}$, $i = 1,2,3,4$ or the immersion $\phi'_{1,p}$.*
Here

(1.1)
$$C(n,p,H) = \begin{cases} 1+H^2, & p=1,\ n \textit{ is even},\\ \dfrac{n}{n-1} + \dfrac{2n^2-2n+1}{2(n-1)^2}H^2 \\ \qquad - \dfrac{H}{2(n-1)^2}\sqrt{(2n-1)^2H^2+4n(n-1)}, & p=1,\ n \textit{ is odd},\\ 1+\dfrac{3n+1}{4n}H^2, & p=2 \textit{ and } H \neq 0,\\ \dfrac{1}{3}+\dfrac{7n+1}{24n}H^2, & p \geq 3 \textit{ or } p=2 \textit{ and } H=0. \end{cases}$$

Noting that

$$(1.2)\qquad C_1(n,p) = \inf_{H\geq 0} C(n,p,H) = \begin{cases} 1, & p=1 \text{ and } n \text{ is even, or } p=2 \text{ and } H\neq 0,\\ \dfrac{4n^2}{(2n-1)^2}, & p=1 \text{ and } n \text{ is odd},\\ \dfrac{1}{3}, & p\geq 3, \text{ or } p=2 \text{ and } H=0, \end{cases}$$

we have the following corollary.

Corollary 1.5. *Let M^n be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. If*

$$\tilde{\sigma}(u) \leq C_1(n,p), \quad \text{for all } u \in UM,$$

then M is either a totally umbilical sphere, a Clifford isoparametric hypersurface, or one of the embeddings $\phi_{i,p}, i = 1,2,3,4$, or the immersion $\phi'_{1,p}$.

When $H = 0$, our main theorem reduces to Gauchman's rigidity theorem for minimal submanifolds [4].

2. Preliminaries

We shall make the following conventions on the range of indices:

$$1 \leq A,B,C,\ldots \leq n+p; \quad 1\leq i,j,k,\ldots \leq n; \quad n+1 \leq \alpha,\beta,\gamma,\ldots \leq n+p.$$

Choose a local orthonormal frame field $\{e_A\}$ on S^{n+p} such that e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame field of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of S^{n+p}. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \quad h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha, \quad \xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii}e_\alpha,$$

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}),$$

$$R_{\alpha\beta kl} = \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}).$$

where h, ξ, R_{ijkl}, $R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the curvature tensor and the normal curvature tensor of M respectively. We set

$$S = \|h\|^2, \quad H = \|\xi\|, \quad H_\alpha = (h^\alpha_{ij})_{n\times n}, \quad c_\alpha = \frac{1}{n}\operatorname{tr} H_\alpha.$$

Denoting the first and second covariant derivatives of h^α_{ij} by h^α_{ijk} and h^α_{ijkl} respectively, we have

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ik} + \sum_k h^\alpha_{ik}\omega_{jk} + \sum_\beta h^\beta_{ij}\omega_{\alpha\beta}, \tag{2.1}$$

$$\sum_k h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\omega_{il} + \sum_l h^\alpha_{ilk}\omega_{jl} + \sum_l h^\alpha_{ijl}\omega_{kl} + \sum_\beta h^\beta_{ijk}\omega_{\alpha\beta}. \tag{2.2}$$

The Laplacian of h is defined by $\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. Let M be an n-dimensional submanifold with parallel mean curvature in S^{n+p}. Following [15, 16], we have

$$\sum_i h^\alpha_{iik} = 0, \quad \sum_i h^\alpha_{iikl} = 0, \quad \text{for all } k, l, \alpha, \tag{2.3}$$

$$\Delta h^\alpha_{ij} = \sum_{k,m} h^\alpha_{km}R_{mijk} + \sum_{k,m} h^\alpha_{mi}R_{mkjk} + \sum_{k,\beta} h^\beta_{ki}R_{\beta\alpha jk}, \tag{2.4}$$

$$\sum_\alpha R_{\alpha\beta kl}(\mathrm{tr}\, H_\alpha) = 0. \tag{2.5}$$

PROPOSITION 2.1. *For any real numbers s, t, x, y, we have*

$$sx^2 + txy \le \frac{1}{2}(s + \sqrt{s^2+t^2})(x^2+y^2), \tag{2.6}$$

and the equality holds if and only if

$$\begin{cases} (\sqrt{s^2+t^2} - s)^{1/2}x = (\sqrt{s^2+t^2}+s)^{1/2}y, & \text{for } t \ge 0, \\ (\sqrt{s^2+t^2} - s)^{1/2}x = -(\sqrt{s^2+t^2}+s)^{1/2}y, & \text{for } t < 0. \end{cases} \tag{2.7}$$

Proof. Notice that

$$\begin{aligned}(s + \sqrt{s^2+t^2})(x^2+y^2) - 2(sx^2+txy) \\ = (\sqrt{s^2+t^2} - s)x^2 - 2txy + (s + \sqrt{s^2+t^2})y^2.\end{aligned} \tag{2.8}$$

When $t \ge 0$, the RHS of (2.8) can be written as

$$((\sqrt{s^2+t^2} - s)^{1/2}x - (\sqrt{s^2+t^2}+s)^{1/2}y)^2 \ge 0. \tag{2.9}$$

When $t < 0$, the RHS of (2.8) can be written as

$$((\sqrt{s^2+t^2} - s)^{1/2}x + (\sqrt{s^2+t^2}+s)^{1/2}y)^2 \ge 0. \tag{2.10}$$

Thus we get the desired inequality, and the equality in (2.6) holds if and only if (2.9) or (2.10) becomes equality. □

Proposition 2.2. *Let $f(x)$ be a function defined by*

$$f(x) = x - \sqrt{2cx + c^2 + H^2}, \quad x \in [-H, H].$$

Then we have

$$f(x) \geq \begin{cases} -\frac{H^2}{2c} - c, & \text{for } c \geq \frac{H}{2}, \\ c - 2H, & \text{for } c < \frac{H}{2}. \end{cases} \tag{2.11}$$

Proof. Suppose that x_0 is a minimum point of f. Then $\left.\frac{\partial f}{\partial x}\right|_{x_0} = 0$. This implies $x_0 = -\frac{H^2}{2c}$. Noting that $x \in [-H, H]$, we get the desired inequality. □

3. Maximal directions

Let $x \in M$. A vector $u \in UM_x$ is called a maximal direction of $\tilde{\sigma}$ at x if $\tilde{\sigma}(u) = \max_{v \in UM_x} \tilde{\sigma}(v)$.

Choose an adapted orthonormal frame $\{e_1, \dots, e_{n+p}\}$ at x such that when restricted to M, the vectors $e_1, \dots, e_n$ are tangent to M. Assume that e_1 is a maximal direction at x, $\tilde{\sigma}(e_1) \neq 0$, and $e_{n+1} = \frac{\tilde{h}(e_1, e_1)}{\|\tilde{h}(e_1, e_1)\|}$, $e_{n+2} = \frac{\xi - \langle \xi, e_{n+1} \rangle e_{n+1}}{\|\xi - \langle \xi, e_{n+1} \rangle e_{n+1}\|}$ (if ξ is not parallel to e_{n+1}). By our choices of e_{n+1} and e_{n+2}, we have

$$\tilde{h}_{11}^{\alpha} = 0, \quad \textit{if } \alpha \neq n+1; \qquad \textit{and} \qquad c_{\alpha} = 0 \quad \textit{if } \alpha \neq n+1, n+2. \tag{3.1}$$

Since e_1 is a maximal direction, at the point x for any $t \in \mathbf{R}$ we have

$$\|\tilde{h}(e_1 + te_i, e_1 + te_i)\|^2 \leq (1 + t^2)^2 (\tilde{h}_{11}^{n+1})^2. \tag{3.2}$$

Expanding in terms of t, we obtain

$$4t\tilde{h}_{11}^{n+1}\tilde{h}_{1i}^{n+1} + O(t^2) \leq 0.$$

It follows that

$$\tilde{h}_{1i}^{n+1} = 0, \quad i = 2, \dots, n. \tag{3.3}$$

We now choose an adapted frame at $x \in M$ such that in addition to (3.1) and (3.3),

$$\tilde{h}_{ij}^{n+1} = h_{ij}^{n+1} = 0, \quad i \neq j. \tag{3.4}$$

Once more expanding (3.2) in terms of t, we can get

$$-2t^2 \left[\tilde{h}_{11}^{n+1} (\tilde{h}_{11}^{n+1} - \tilde{h}_{ii}^{n+1}) - 2 \sum_{\alpha \neq n+1} (\tilde{h}_{1i}^{\alpha})^2 \right] + O(t^3) \leq 0.$$

It follows that

$$2\sum_{\alpha\neq n+1}(\tilde{h}^{\alpha}_{1i})^2 \le \tilde{h}^{n+1}_{11}(\tilde{h}^{n+1}_{11}-\tilde{h}^{n+1}_{ii}),\quad i=2,\dots,n. \tag{3.5}$$

Define a tensor field $\tilde{T}=(\tilde{T}_{ijkl})$ on M by

$$\tilde{T}_{ijkl}=\sum_{\alpha}\tilde{h}^{\alpha}_{ij}\tilde{h}^{\alpha}_{kl}.$$

It is obvious that $\tilde{\sigma}(u)=\tilde{T}(u,u,u,u)$.

LEMMA 3.1. *Let u be a maximal direction at $x\in M$. Assume that $\tilde{\sigma}(u)\neq 0$. Let $e_1,\dots,e_{n+p}$ be an adapted frame at x such that*

$$e_1=u,\quad e_{n+1}=\frac{\tilde{h}(e_1,e_1)}{\|\tilde{h}(e_1,e_1)\|},\quad e_{n+2}=\frac{\xi-\langle\xi,e_{n+1}\rangle e_{n+1}}{\|\xi-\langle\xi,e_{n+1}\rangle e_{n+1}\|},$$

$h^{n+1}_{ij}=0$ for $i\neq j$, and ξ is not parallel to e_{n+1}. At the point x,

(1) *If $p=1$, then*

$$\frac{1}{2}(\triangle\tilde{T})_{1111}\ge \tilde{h}^{n+1}_{11}\left[n\tilde{h}^{n+1}_{11}(1+H^2+c_{n+1}\tilde{h}^{n+1}_{11})-(\tilde{h}^{n+1}_{11}+c_{n+1})\sum_k(\tilde{h}^{n+1}_{kk})^2\right]; \tag{3.6}$$

(2) *If $p=2$ and $H\neq 0$, then*

$$\frac{1}{2}(\triangle\tilde{T})_{1111}\ge \tilde{h}^{n+1}_{11}\Bigg[n\tilde{h}^{n+1}_{11}(1+H^2+c_{n+1}\tilde{h}^{n+1}_{11}) -(\tilde{h}^{n+1}_{11}+c_{n+1})\sum_k(\tilde{h}^{n+1}_{kk})^2-c_{n+2}\sum_k\tilde{h}^{n+1}_{kk}\tilde{h}^{n+2}_{kk}\Bigg]; \tag{3.7}$$

(3) *If $p\ge 3$, or $p=2$ and $H=0$, then*

$$\frac{1}{2}(\triangle\tilde{T})_{1111}\ge \tilde{h}^{n+1}_{11}\Bigg[n\tilde{h}^{n+1}_{11}(1+H^2+c_{n+1}\tilde{h}^{n+1}_{11}-(\tilde{h}^{n+1}_{11})^2) -(2\tilde{h}^{n+1}_{11}+c_{n+1})\sum_k(\tilde{h}^{n+1}_{kk})^2-c_{n+2}\sum_k\tilde{h}^{n+1}_{kk}\tilde{h}^{n+2}_{kk}\Bigg], \tag{3.8}$$

and equality holds if and only if

$$(\tilde{h}^{n+1}_{11}-\tilde{h}^{n+1}_{kk})\left[\tilde{h}^{n+1}_{11}(\tilde{h}^{n+1}_{11}-\tilde{h}^{n+1}_{kk})-2\sum_{\alpha}(\tilde{h}^{\alpha}_{1k})^2\right]=0, \tag{3.9}$$

and $\tilde{h}^{\alpha}_{11k}=0$, for any k, α.

Proof. We have

$$\frac{1}{2}(\triangle\tilde{T})_{1111} = \tilde{h}_{11}^{n+1}\triangle\tilde{h}_{11}^{n+1} + \sum_{i,\alpha}(\tilde{h}_{11i}^{\alpha})^2. \tag{3.10}$$

From (2.3), it is easy to see that $\triangle c_{n+1} = \frac{1}{n}\sum_{k,i} h_{kkii}^{n+1} = 0$. From (2.4), (3.1) and (3.4), noting that $\sum_k \tilde{h}_{kk}^{\alpha} = 0$ for any α, we have

$$\begin{aligned}
\triangle\tilde{h}_{11}^{n+1} &= \triangle h_{11}^{n+1} \\
&= \sum_{k,m} h_{km}^{n+1} R_{m11k} + \sum_{k,m} h_{m1}^{n+1} R_{mk1k} + \sum_{k,\alpha} h_{1k}^{\alpha} R_{\alpha n+1,1k} \\
&= \sum_k (h_{11}^{n+1} - h_{kk}^{n+1}) R_{1k1k} + \sum_{k,\alpha} h_{1k}^{\alpha} \sum_l (h_{l1}^{\alpha} h_{lk}^{n+1} - h_{lk}^{\alpha} h_{l1}^{n+1}) \\
&= \sum_k (h_{11}^{n+1} - h_{kk}^{n+1}) \left[1 - (\delta_{1k})^2 + \sum_{\alpha} h_{11}^{\alpha} h_{kk}^{\alpha} - 2\sum_{\alpha}(h_{1k}^{\alpha})^2\right] \\
&= \sum_k (\tilde{h}_{11}^{n+1} - \tilde{h}_{kk}^{n+1}) \left[1 + h_{11}^{n+1} h_{kk}^{n+1} + c_{n+2} h_{kk}^{n+2} - 2\sum_{\alpha\neq n+1}(h_{1k}^{\alpha})^2\right] \\
&= n\tilde{h}_{11}^{n+1}(1 + H^2 + c_{n+1}\tilde{h}_{11}^{n+1}) - (\tilde{h}_{11}^{n+1} + c_{n+1})\sum_k (\tilde{h}_{kk}^{n+1})^2 \\
&\quad - c_{n+2}\sum_k \tilde{h}_{kk}^{n+1}\tilde{h}_{kk}^{n+2} - 2\sum_k (\tilde{h}_{11}^{n+1} - \tilde{h}_{kk}^{n+1}) \sum_{\alpha\neq n+1} (\tilde{h}_{1k}^{\alpha})^2.
\end{aligned}$$

(1) If $p = 1$, then $c_{n+2} = 0$, and the last two terms above vanish.

(2) If $p = 2$ and $H \neq 0$, it follows from (2.5) and (3.1) that $R_{(n+1)(n+2)kl} = 0$. Hence the last term above vanishes again.

(3) If $p \geq 3$, or $p = 2$ and $H = 0$, by (3.5), we get

$$\begin{aligned}
\triangle\tilde{h}_{11}^{n+1} \geq\ & n\tilde{h}_{11}^{n+1}(1 + H^2 + c_{n+1}\tilde{h}_{11}^{n+1} - (\tilde{h}_{11}^{n+1})^2) \\
& - (2\tilde{h}_{11}^{n+1} + c_{n+1})\sum_k (\tilde{h}_{kk}^{n+1})^2 - c_{n+2}\sum_k \tilde{h}_{kk}^{n+1}\tilde{h}_{kk}^{n+2}.
\end{aligned}$$

Substituting this into (3.10), we get the desired results. □

Lemma 3.2. *Let* $\{e_1, \ldots, e_{n+p}\}$ *be an adapted frame at* $x \in M$ *as in Lemma* 3.1. *Assume* $\tilde{\sigma}(u) \neq 0$.

(1) *If* $p = 1$, $n(= 2m)$ *is even,* $\tilde{\sigma}(u) \leq 1 + H^2$ *for any* $u \in UM_x$, *then* $(\triangle\tilde{T})_{1111} \geq 0$. *If equality* $(\triangle\tilde{T})_{1111} = 0$ *holds, after suitable rearrangement of* $e_2, \ldots, e_{2m}$ *we have*

$$\tilde{h}_{11}^{n+1} = \cdots = \tilde{h}_{mm}^{n+1} = -\tilde{h}_{m+1,m+1}^{n+1} = \cdots = -\tilde{h}_{2m,2m}^{n+1} = \sqrt{1 + H^2}. \tag{3.11}$$

(2) *If* $p = 1$, $n(= 2m+1)$ *is odd and*

$$\tilde{\sigma}(u) \le \frac{n}{n-1} + \frac{2n^2 - 2n + 1}{2(n-1)} H^2 - \frac{H}{2(n-1)^2}\sqrt{(2n-1)^2 H^2 + 4n(n-1)},$$

for any $u \in UM_x$, *then* $(\triangle\tilde{T})_{1111} \ge 0$. *If equality* $(\triangle\tilde{T})_{1111} = 0$ *holds, after suitable rearrangement of* $e_2, \dots, e_{2m}, e_{2m+1}$ *we have*

$$\begin{aligned} \tilde{h}_{11}^{n+1} &= \cdots = \tilde{h}_{mm}^{n+1} = -\tilde{h}_{m+1,m+1}^{n+1} = \cdots = -\tilde{h}_{2m,2m}^{n+1} \\ &= \frac{-H + \sqrt{(2n-1)^2 H^2 + 4n(n-1)}}{2(n-1)}, \\ \tilde{h}_{nn}^{n+1} &= 0, \quad c_{n+1} = -H. \end{aligned} \tag{3.12}$$

(3) *If* $p = 2$ *and* $H \ne 0$, $\tilde{\sigma}(u) \le 1 + \frac{3n+1}{4n} H^2$ *for any* $u \in UM_x$, *then* $(\triangle\tilde{T})_{1111} > 0$.

(4) *If* $p \ge 3$, *or* $p = 2$ *and* $H = 0$, *and if* $\tilde{\sigma}(u) \le \frac{1}{3} + \frac{7n+1}{24n} H^2$ *for any* $u \in UM_x$, *then* $(\triangle\tilde{T})_{1111} \ge 0$. *If equality* $(\triangle\tilde{T})_{1111} = 0$ *holds, then* $n(= 2m)$ *must be even and* $H = 0$, *after suitable rearrangement of* $e_2, \dots, e_{2m}$, *we have*

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1,m+1}^{n+1} = \cdots = -h_{2m,2m}^{n+1} = \frac{\sqrt{3}}{3}. \tag{3.13}$$

Proof. First we prove assertions (1) and (2). Since e_1 is a maximal direction of $\tilde{\sigma}$, we have

$$-\tilde{h}_{11}^{n+1} \le \tilde{h}_{ii}^{n+1} \le \tilde{h}_{11}^{n+1}, \quad i = 2, \dots, n. \tag{3.14}$$

By Definition 1.1 we have

$$\sum_{i=2}^{n} \tilde{h}_{ii}^{n+1} = -\tilde{h}_{11}^{n+1}. \tag{3.15}$$

It is easy to see that the convex function $f(\tilde{h}_{22}^{n+1}, \dots, \tilde{h}_{nn}^{n+1}) = \sum_{i=2}^{n} (\tilde{h}_{ii}^{n+1})^2$ subject to the linear constraints (3.14) and (3.15) attains its maximal value $f_{max} = (n-1)(\tilde{h}_{11}^{n+1})^2$ when (after suitable rearrangement of $e_1, \dots, e_{2m}$)

$$\tilde{h}_{11}^{n+1} = \cdots = \tilde{h}_{mm}^{n+1} = -\tilde{h}_{m+1,m+1}^{n+1} = \cdots = -\tilde{h}_{2m,2m}^{n+1}, \quad \text{if } n = 2m; \tag{3.16}$$

and $f_{max} = (n-2)(\tilde{h}_{11}^{n+1})^2$ when

$$\begin{aligned} \tilde{h}_{11}^{n+1} &= \cdots = \tilde{h}_{mm}^{n+1} = -\tilde{h}_{m+1,m+1}^{n+1} \\ &= \cdots = -\tilde{h}_{2m,2m}^{n+1}, \tilde{h}_{nn}^{n+1} = 0, \quad \text{if } n = 2m+1. \end{aligned} \tag{3.17}$$

Thus, by Lemma 3.1(1) we get

$$\frac{1}{2}(\triangle\tilde{T})_{1111} \ge \begin{cases} n(\tilde{h}_{11}^{n+1})^2[1+H^2-\tilde{\sigma}(e_1)], & \text{if } p=1,\ n=2m, \quad (3.18)\\ (\tilde{h}_{11}^{n+1})^2[n(1+H^2)+c_{n+1}\tilde{h}_{11}^{n+1} \\ \qquad -(n-1)(\tilde{h}_{11}^{n+1})^2], & \text{if } p=1,\ n=2m+1. \quad (3.19)\end{cases}$$

Assertion (1) follows from (3.16) and (3.18) immediately.

If $p=1$, $n=2m+1$, we have $c_{n+1}=\pm H$, and if

$$\tilde{h}_{11}^{n+1} \le \frac{-H+\sqrt{(2n-1)^2H^2+4n(n-1)}}{2(n-1)},$$

then

$$n(1+H^2)+c_{n+1}\tilde{h}_{11}^{n+1}-(n-1)(\tilde{h}_{11}^{n+1})^2 \ge 0.$$

So assertion (2) follows from (3.17) and (3.19).

Secondly we prove assertion (3). We take the following steps: first prove $(\triangle\tilde{T})_{1111}\ge 0$, then show that the equality can not attain. Since e_1 is a maximal direction of $\tilde{\sigma}$, we have

$$(\tilde{h}_{ii}^{n+1})^2+(\tilde{h}_{ii}^{n+2})^2 \le (\tilde{h}_{11}^{n+1})^2, \quad i=2,\dots,n. \tag{3.20}$$

Therefore, by Proposition 2.1, we have

$$\begin{aligned}(\tilde{h}_{11}^{n+1}+c_{n+1})(\tilde{h}_{kk}^{n+1})^2+c_{n+2}\tilde{h}_{kk}^{n+1}\tilde{h}_{kk}^{n+2} \\ \le \frac{1}{2}(\tilde{h}_{11}^{n+1})^2[\tilde{h}_{11}^{n+1}+c_{n+1}+\sqrt{(\tilde{h}_{11}^{n+1})^2+2c_{n+1}\tilde{h}_{11}^{n+1}+H^2}],\ k=2,\dots,n.\end{aligned} \tag{3.21}$$

Substituting this into (3.7), we obtain

$$\begin{aligned}\frac{1}{2}(\triangle\tilde{T})_{1111} \ge (\tilde{h}_{11}^{n+1})^2\Big[n(1+H^2)-\frac{n+1}{2}(\tilde{h}_{11}^{n+1})^2 \\ +\frac{n-1}{2}\tilde{h}_{11}^{n+1}(c_{n+1}-\sqrt{(\tilde{h}_{11}^{n+1})^2+2c_{n+1}\tilde{h}_{11}^{n+1}+H^2})\Big].\end{aligned} \tag{3.22}$$

By Proposition 2.2, the function

$$f(c_{n+1})=c_{n+1}-\sqrt{(\tilde{h}_{11}^{n+1})^2+2c_{n+1}\tilde{h}_{11}^{n+1}+H^2}, \quad c_{n+1}\in[-H,H],$$

satisfies

$$f(c_{n+1}) \ge \begin{cases} -\dfrac{H^2}{2\tilde{h}_{11}^{n+1}}-\tilde{h}_{11}^{n+1}, & \text{if } \tilde{h}_{11}^{n+1}\ge \dfrac{H}{2}, \\ \tilde{h}_{11}^{n+1}-2H, & \text{if } \tilde{h}_{11}^{n+1}<\dfrac{H}{2}.\end{cases}$$

Substituting the above into (3.22), we have

$$\frac{1}{2}(\triangle\tilde{T})_{1111} \geq \begin{cases} (\tilde{h}_{11}^{n+1})^2\left[n + \dfrac{3n+1}{4n}H^2 - n\tilde{\sigma}(e_1)\right], & \tilde{h}_{11}^{n+1} \geq \dfrac{H}{2}, \quad (3.23) \\ (\tilde{h}_{11}^{n+1})^2[n(1+H^2) - (n-1)H\tilde{h}_{11}^{n+1} - (\tilde{h}_{11}^{n+1})^2], & \tilde{h}_{11}^{n+1} < \dfrac{H}{2}. \quad (3.24) \end{cases}$$

Assume $\tilde{h}_{11}^{n+1} \leq \sqrt{1 + \frac{3n+1}{4n}H^2}$ as in assertion (3). If $\frac{H}{2} \leq \tilde{h}_{11}^{n+1} \leq \sqrt{1 + \frac{3n+1}{4n}H^2}$, from (3.23) we obtain $(\triangle\tilde{T})_{1111} \geq 0$. If $\tilde{h}_{11}^{n+1} < \frac{H}{2}$, then

$$\begin{aligned} n(1+H^2) - (n-1)H\tilde{h}_{11}^{n+1} - (\tilde{h}_{11}^{n+1})^2 &> n(1+H^2) - \frac{n-1}{2}H^2 - \frac{H^2}{4} \\ &= n + \frac{2n+1}{4}H^2 > 0. \end{aligned}$$

Thus from (3.24), $(\triangle\tilde{T})_{1111} > 0$. In summary, it follows that $(\triangle\tilde{T})_{1111} \geq 0$.

If $(\triangle\tilde{T})_{1111} = 0$, then (3.20)–(3.24) all become equalities. By Proposition 2.1, (3.21) becomes equality if and only if (after suitable rearrangement of e_i)

$$\begin{cases} \tilde{h}_{22}^{n+1} = \cdots = \tilde{h}_{ss}^{n+1} = -\tilde{h}_{s+1,s+1}^{n+1} = \cdots = -\tilde{h}_{nn}^{n+1} = C_1(\tilde{h}_{11}^{n+1}, c_{n+1}, H), \\ \tilde{h}_{22}^{n+2} = \cdots = \tilde{h}_{ss}^{n+2} = -\tilde{h}_{s+1,s+1}^{n+2} = \cdots = -\tilde{h}_{nn}^{n+2} = C_2(\tilde{h}_{11}^{n+1}, c_{n+1}, H), \end{cases} \tag{3.25}$$

where

$$\begin{cases} C_1(\tilde{h}_{11}^{n+1}, c_{n+1}, H) = \tilde{h}_{11}^{n+1}\sqrt{\frac{1}{2}[1 + (\tilde{h}_{11}^{n+1} + c_{n+1})/\sqrt{(\tilde{h}_{11}^{n+1})^2 + 2c_{n+1}\tilde{h}_{11}^{n+1} + H^2}]}, \\ C_2(\tilde{h}_{11}^{n+1}, c_{n+1}, H) \\ \qquad = sgn(c_{n+2}) \cdot \tilde{h}_{11}^{n+1}\sqrt{\frac{1}{2}[1 - (\tilde{h}_{11}^{n+1} + c_{n+1})/\sqrt{(\tilde{h}_{11}^{n+1})^2 + 2c_{n+1}\tilde{h}_{11}^{n+1} + H^2}]}, \end{cases}$$

and $sgn(c_{n+2})$ is the sign function of c_{n+2}. On the other hand, by Definition 1.1 and (3.1) we have

$$\sum_{k=2}^{n} \tilde{h}_{kk}^{n+1} = -\tilde{h}_{11}^{n+1}, \quad \sum_{k=2}^{n} \tilde{h}_{kk}^{n+2} = 0. \tag{3.26}$$

Hence, by substituting (3.25) into (3.26), we have

$$\begin{cases} \tilde{h}_{11}^{n+1} + (2s-n-1)C_1(\tilde{h}_{11}^{n+1}, c_{n+1}, H) = 0, \\ (2s-n-1)C_2(\tilde{h}_{11}^{n+1}, c_{n+1}, H) = 0. \end{cases} \tag{3.27}$$

Noting $\tilde{h}_{11}^{n+1} \neq 0$, we have

$$C_2(\tilde{h}_{11}^{n+1}, c_{n+1}, H) = 0,$$

which is equivalent to

$$c_{n+1} = \pm H, \quad c_{n+2} = 0. \tag{3.28}$$

Since (3.23) becomes equality, we have

$$\tilde{h}_{11}^{n+1} = \sqrt{1 + \frac{3n+1}{4n}H^2}\left(> \frac{H}{2}\right). \tag{3.29}$$

Together with Proposition 2.2, we get

$$c_{n+1} = -\frac{H^2}{2\tilde{h}_{11}^{n+1}}(\leq 0). \tag{3.30}$$

From (3.28)–(3.30), we obtain $H = 0$, which is a contradiction with the assumption $H \neq 0$. So $(\triangle\tilde{T})_{1111} \neq 0$. This proves assertion (3).

Finally we prove assertion (4). From Proposition 2.1 and (3.20) we have

$$\begin{aligned}(2\tilde{h}_{11}^{n+1} + c_{n+1})(\tilde{h}_{kk}^{n+1})^2 + c_{n+2}\tilde{h}_{kk}^{n+1}\tilde{h}_{kk}^{n+2} \\ \leq \frac{1}{2}(\tilde{h}_{11}^{n+1})^2(2\tilde{h}_{11}^{n+1} + c_{n+1} + \sqrt{4(\tilde{h}_{11}^{n+1})^2 + 4c_{n+1}\tilde{h}_{11}^{n+1} + H^2}), \\ k = 2, \ldots, n.\end{aligned} \tag{3.31}$$

Substituting this into (3.8), we obtain

$$\begin{aligned}\frac{1}{2}(\triangle\tilde{T})_{1111} \geq (\tilde{h}_{11}^{n+1})^2[n(1+H^2) - (2n+1)(\tilde{h}_{11}^{n+1})^2 \\ + \frac{n-1}{2}\tilde{h}_{11}^{n+1}(c_{n+1} - \sqrt{4(\tilde{h}_{11}^{n+1})^2 + 4c_{n+1}\tilde{h}_{11}^{n+1} + H^2})].\end{aligned} \tag{3.32}$$

By Proposition 2.2, the function

$$g(c_{n+1}) = c_{n+1} - \sqrt{4(\tilde{h}_{11}^{n+1})^2 + 4c_{n+1}\tilde{h}_{11}^{n+1} + H^2}, \quad c_{n+1} \in [-H, H],$$

satisfies

$$g(c_{n+1}) \geq \begin{cases} -\dfrac{H^2}{4\tilde{h}_{11}^{n+1}} - 2\tilde{h}_{11}^{n+1}, & \tilde{h}_{11}^{n+1} \geq \dfrac{H}{4}, \\ 2\tilde{h}_{11}^{n+1} - 2H, & \tilde{h}_{11}^{n+1} < \dfrac{H}{4}. \end{cases}$$

Substituting this into (3.32), we get

$$\frac{1}{2}(\triangle\tilde{T})_{1111} \geq \begin{cases} (\tilde{h}_{11}^{n+1})^2\left[n + \dfrac{7n+1}{8}H^2 - 3n(\tilde{h}_{11}^{n+1})^2\right], & \tilde{h}_{11}^{n+1} \geq \dfrac{H}{4}, \\ (\tilde{h}_{11}^{n+1})^2[n(1+H^2) - (n-1)H\tilde{h}_{11}^{n+1} - (n+2)(\tilde{h}_{11}^{n+1})^2], & \tilde{h}_{11}^{n+1} < \dfrac{H}{4}. \end{cases} \tag{3.33}$$

Assume $\tilde{h}_{11}^{n+1} \le \sqrt{\frac{1}{3} + \frac{7n+1}{24n} H^2}$ as in assertion (4). If $\frac{H}{4} \le \tilde{h}_{11}^{n+1} \le \sqrt{\frac{1}{3} + \frac{7n+1}{24n} H^2}$, then from (3.33) we obtain $(\triangle \tilde{T})_{1111} \ge 0$. If $\tilde{h}_{11}^{n+1} < \frac{H}{4}$, we have

$$n(1+H^2) - (n-1)H\tilde{h}_{11}^{n+1} - (n+2)(\tilde{h}_{11}^{n+1})^2 > n + \frac{11n+2}{16}H^2 > 0.$$

It's seen from (3.33) that $(\triangle \tilde{T})_{1111} > 0$. In summary, we have $(\triangle \tilde{T})_{1111} \ge 0$.

If $(\triangle \tilde{T})_{1111} = 0$, then (3.31)–(3.33) all become equalities. By Proposition 2.1, (3.31) becomes equality if and only if (after suitable rearrangement of e_i)

$$\begin{cases} \tilde{h}_{22}^{n+1} = \cdots = \tilde{h}_{tt}^{n+1} = -\tilde{h}_{t+1,t+1}^{n+1} = \cdots = -\tilde{h}_{nn}^{n+1} = C_1'(\tilde{h}_{11}^{n+1}, c_{n+1}, H), \\ \tilde{h}_{22}^{n+2} = \cdots = \tilde{h}_{tt}^{n+2} = -\tilde{h}_{t+1,t+1}^{n+2} = \cdots = -\tilde{h}_{nn}^{n+2} = C_2'(\tilde{h}_{11}^{n+1}, c_{n+1}, H), \end{cases} \tag{3.34}$$

where

$$\begin{cases} C_1'(\tilde{h}_{11}^{n+1}, c_{n+1}, H) = \tilde{h}_{11}^{n+1}\sqrt{\frac{1}{2}[1 + (2\tilde{h}_{11}^{n+1} + c_{n+1})/\sqrt{4(\tilde{h}_{11}^{n+1})^2 + 4c_{n+1}\tilde{h}_{11}^{n+1} + H^2}]}, \\ C_2'(\tilde{h}_{11}^{n+1}, c_{n+1}, H) \\ \qquad = sgn(c_{n+2}) \cdot \tilde{h}_{11}^{n+1}\sqrt{\frac{1}{2}[1 - (2\tilde{h}_{11}^{n+1} + c_{n+1})/\sqrt{4(\tilde{h}_{11}^{n+1})^2 + 4c_{n+1}\tilde{h}_{11}^{n+1} + H^2}]}. \end{cases}$$

Substituting (3.34) into (3.26) we obtain

$$\begin{cases} \tilde{h}_{11}^{n+1} + (2t-n-1)C_1'(\tilde{h}_{11}^{n+1}, c_{n+1}, H) = 0, \\ (2t-n-1)C_2'(\tilde{h}_{11}^{n+1}, c_{n+1}, H) = 0. \end{cases} \tag{3.35}$$

Noting $\tilde{h}_{11}^{n+1} \ne 0$, we have

$$C_2'(\tilde{h}_{11}^{n+1}, c_{n+1}, H) = 0,$$

which is equivalent to

$$c_{n+1} = \pm H, \quad c_{n+2} = 0. \tag{3.36}$$

Substituting (3.36) into the first identity of (3.35), we obtain

$$n = 2t. \tag{3.37}$$

Combining (3.34), (3.36), (3.37), we get that $n(=2m)$ is even and

$$\begin{cases} \tilde{h}_{11}^{n+1} = \cdots = \tilde{h}_{mm}^{n+1} = -\tilde{h}_{m+1,m+1}^{n+1} = \cdots = -\tilde{h}_{2m,2m}^{n+1}, \\ \tilde{h}_{11}^{n+2} = \cdots = \tilde{h}_{mm}^{n+2} = -\tilde{h}_{m+1,m+1}^{n+2} = \cdots = -\tilde{h}_{2m,2m}^{n+2} = 0. \end{cases} \tag{3.38}$$

Since (3.33) becomes equality, we get

$$\tilde{h}_{11}^{n+1} = \sqrt{\frac{1}{3} + \frac{7n+1}{24n} H^2} \left(> \frac{H}{4} \right). \tag{3.39}$$

Together with Proposition 2.1, we have

$$c_{n+1} = -\frac{H^2}{4\tilde{h}_{11}^{n+1}} (\leq 0). \tag{3.40}$$

From (3.36), (3.39) and (3.40), we obtain $H = 0$, and $\tilde{h}_{11}^{n+1} = h_{11}^{n+1} = \sqrt{\frac{1}{3}}$. Assertion (4) is proved. □

Let $\tilde{L}(x)$ be a function defined by $\tilde{L}(x) = \max_{u \in UM_x} \tilde{\sigma}(u)$ on M. We have

LEMMA 3.3. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. Assume that one of the following conditions is satisfied.*

(1) *If $p = 1$, $n(=2m)$ is even, $\tilde{\sigma}(u) \leq 1 + H^2$ for all $u \in UM$,*

(2) *If $p = 1$, $n(=2m+1)$ is odd,*

$$\tilde{\sigma}(u) \leq \frac{n}{n-1} + \frac{2n^2 - 2n + 1}{2(n-1)} H^2 - \frac{H}{2(n-1)^2} \sqrt{(2n-1)^2 H^2 + 4n(n-1)}, \quad \text{for all } u \in UM.$$

(3) *If $p = 2$, $H \neq 0$, $\tilde{\sigma}(u) \leq 1 + \frac{3n+1}{4n} H^2$ for all $u \in UM$,*

(4) *If $p \geq 3$, or $p = 2$ and $H = 0$, and if $\tilde{\sigma}(u) \leq \frac{1}{3} + \frac{7n+1}{24n} H^2$ for all $u \in UM$.*

Then $\tilde{L}(x)$ is a constant function on M.

Proof. $\tilde{L}(x)$ is obviously a continuous function. Using the maximum principle, it suffices to show that $\tilde{L}(x)$ is a subharmonic function in the generalized sense. Fix $x \in M$ and let e_1 be a maximal direction of $\tilde{\sigma}(u)$ at x. In an open neighborhood U_x of x within the cut-locus of x, we denote by u_y the tangent vector to M obtained by parallel transport of $e_1 = u(x)$ along the unique geodesic connecting x and y. Define $\tilde{g}_x(y) = \tilde{\sigma}(u(y))$ on U_x, then

$$\begin{aligned} \triangle \tilde{g}_x(x) &= \triangle[\tilde{T}(u(y), u(y), u(y), u(y))]_{y=x} \\ &= \sum_i (\nabla_i^2 \tilde{T})(e_1, e_1, e_1, e_1) \\ &= (\triangle \tilde{T})_{1111}(x). \end{aligned} \tag{3.41}$$

If $\|\tilde{h}(e_1, e_1)\| \neq 0$, by Lemma 3.2, we have $(\triangle \tilde{T})_{1111}(x) \geq 0$. If $\|\tilde{h}(e_1, e_1)\| = 0$, then $\tilde{h} \equiv 0$ at x, i.e.,

$$\begin{cases} h_{ii}^{\alpha} = c_{\alpha}, & \text{for any } \alpha, i, \\ h_{ij}^{\alpha} = 0, & \text{for any } \alpha, i \neq j. \end{cases}$$

Substituting quantities above into (2.4), it is easy to see that $\triangle h_{ij}^{\alpha} = 0$ for any α, i, j. Therefore,

$$(\triangle \tilde{T})_{1111}(x) = 2\sum_{i,\alpha}(\tilde{h}_{11i}^{\alpha})^2 \geq 0.$$

Thus, we have in any case $\triangle \tilde{g}_x(x) = (\triangle \tilde{T})_{1111}(x) \geq 0$.

For the Laplacian of continuous functions, we have the generalized definition

$$\triangle \tilde{L} = C \lim_{r \to 0} \frac{1}{r^2}\left(\int_{B(x,r)} \tilde{L} \Big/ \int_{B(x,r)} 1 - \tilde{L}(x)\right),$$

where C is a positive constant and $B(x,r)$ denotes the geodesic ball of radius r centered at x. With this definition $\tilde{L}$ is subharmonic on M if and only if $\triangle \tilde{L}(x) \geq 0$ at each point $x \in M$. Since $\tilde{g}_x(x) = \tilde{L}(x)$ and $\tilde{g}_x \leq \tilde{L}$ on U_x, we have $\triangle \tilde{L}(x) \geq \triangle \tilde{g}_x(x) \geq 0$. Thus, $\tilde{L}(x)$ is subharmonic and hence constant on M.

4. Proof of Main Theorem

To prove Theorem 1.4 (Main Theorem), we need to prove the following results.

LEMMA 4.1. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. Assume that one of the following is satisfied*:

(1) *If* $p = 1$, $n(= 2m)$ *is even*, $\tilde{\sigma}(u) < 1 + H^2$ *for all* $u \in UM$,

(2) *If* $p = 1$, $n(= 2m + 1)$ *is odd*,

$$\tilde{\sigma}(u) < \frac{n}{n-1} + \frac{2n^2 - 2n + 1}{2(n-1)} H^2 - \frac{H}{2(n-1)^2}\sqrt{(2n-1)^2 H^2 + 4n(n-1)}, \quad \text{for all } u \in UM,$$

(3) *If* $p = 2$, $H \neq 0$, $\tilde{\sigma}(u) \leq 1 + \frac{3n+1}{4n} H^2$ *for all* $u \in UM$,

(4) *If* $p \geq 3$, *or* $p = 2$ *and* $H = 0$, *and if* $\tilde{\sigma}(u) < \frac{1}{3} + \frac{7n+1}{24n} H^2$ *for all* $u \in UM$,

then M is the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{1 + H^2}}\right)$.

Proof. Let e_1 be a maximal direction of $\tilde{\sigma}$ at $x \in M$. Assume that M is not an umbilical submanifold, i.e., $\tilde{\sigma}(e_1) \neq 0$. Define $\tilde{g}_x(y) = \tilde{\sigma}(u(y))$ as in the proof of Lemma 3.3. By Lemma 3.3, $\tilde{g}_x(x)$ is a maximum of $\tilde{g}_x$, thus $(\triangle \tilde{T})_{1111} = \triangle \tilde{g}_x(x) \leq 0$.

On the other hand, by Lemma 3.2, we have $(\triangle \tilde{T})_{1111} > 0$, for $p = 2$ and $H \neq 0$; $(\triangle \tilde{T})_{1111} \geq 0$, for $p \neq 2$, or $p = 2$ and $H = 0$. In case (3), we already have the contradiction. In the other cases, we obtain $(\triangle\tilde{T})_{1111} = 0$ on M. By Lemma 3.2, we have

$$(4.1)\quad \tilde{\sigma}(e_1) = \begin{cases} 1 + H^2, & \text{if } p = 1,\ n \text{ is even}, \\ \dfrac{n}{n-1} + \dfrac{2n^2 - 2n + 1}{2(n-1)} H^2 \\ \quad - \dfrac{H}{2(n-1)^2}\sqrt{(2n-1)^2 H^2 + 4n(n-1)}, & \text{if } p = 1,\ n \text{ is odd}, \\ \dfrac{1}{3}, & \text{if } p \geq 3, \text{ or } p = 2 \text{ and } H = 0. \end{cases}$$

This contradicts to assumptions in (1) and (2) respectively. Moreover, if $p \geq 3$ and $(\triangle\tilde{T})_{1111}(x) = 0$ on M, from Lemma 3.2 we have $H = 0$. Hence (4.1) contradicts to the assumption in (4). Therefore, $\tilde{\sigma}(u) = 0$ for any $u \in UM$, and M is totally umbilical in S^{n+p}. □

THEOREM 4.2. *Let M be a compact hypersurface with constant mean curvature in S^{n+1}. Assume that $n(= 2m)$ is even.*

(1) *If $\tilde{\sigma}(u) < 1 + H^2$ for any $u \in UM$, then M is the totally umbilic sphere* $S^n\left(\dfrac{1}{\sqrt{1+H^2}}\right)$;

(2) *If $\max_{u \in UM} \tilde{\sigma}(u) = 1 + H^2$, then M is the Clifford isoparametric hypersurface*

$$S^m\left(\frac{1}{\sqrt{2(1 + H^2 + H\sqrt{1+H^2})}}\right) \times S^m\left(\frac{1}{\sqrt{2(1 + H^2 - H\sqrt{1+H^2})}}\right).$$

Proof. The assertion of (1) follows from Lemma 4.1. We prove (2). As in Lemma 4.1, we get $(\triangle\tilde{T})_{1111} = 0$.

From (3.11) we obtain

$$(4.2)\qquad \tilde{h}^{n+1}_{aa} = -\tilde{h}^{n+1}_{rr} = \sqrt{1+H^2}, \quad \textit{for } a = 1, \ldots, m,\ r = m+1, \ldots, n.$$

Correspondingly,

$$(4.3)\qquad h^{n+1}_{aa} = c_{n+1} + \sqrt{1+H^2}, \quad h^{n+1}_{rr} = c_{n+1} - \sqrt{1+H^2},$$

where $c_{n+1} = \pm H$ is a constant.

By Lemma 3.1, $\tilde{h}^{n+1}_{11k} = 0$. In this case every e_i is a maximal direction, so we have $\tilde{h}^{n+1}_{iik} = 0$ for any i, k. From (2.3) we have $h^{n+1}_{iik} = \nabla_k(h^{n+1}_{ii}) = \nabla_k(\tilde{h}^{n+1}_{ii} + c_{n+1}) = \tilde{h}^{n+1}_{iik} = 0$. By polarization, we get $h^{n+1}_{ijk} = 0$ for any i, j, k. From (2.1) and (3.4), we obtain

$$0 = \sum_k h^{n+1}_{ijk}\omega_k = \sum_l h^{n+1}_{il}\omega_{lj} + \sum_l h^{n+1}_{lj}\omega_{li} = (h^{n+1}_{ii} - h^{n+1}_{jj})\omega_{ij}.$$

Therefore, $\omega_{ar} = 0$. Then M is locally a Riemannian product $V_1 \times V_2$ with $dim\ V_1 = dim\ V_2 = m$. The curvatures of V_1 and V_2 are

(4.4)
$$\begin{aligned} R_{abcd} &= \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} + h^{n+1}_{ac}h^{n+1}_{bd} - h^{n+1}_{ad}h^{n+1}_{bc} \\ &= (1 + (c_{n+1} + \sqrt{1+H^2})^2)(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \quad \textit{for}\ 1 \le a,b,c,d \le m, \\ R_{rstw} &= (1 + (c_{n+1} - \sqrt{1+H^2})^2)(\delta_{rt}\delta_{sw} - \delta_{rw}\delta_{st}), \quad \textit{for}\ m+1 \le r,s,t,w \le n. \end{aligned}$$

Thus V_1 and V_2 are two spaces with constant curvatures $2(1+H^2) + 2H\sqrt{1+H^2}$ and $2(1+H^2) - 2H\sqrt{1+H^2}$ respectively. The compactness of M allows us to complete the proof. □

THEOREM 4.3. *Let M be a compact hypersurface with constant mean curvature in S^{n+1}. Assume that $n(= 2m+1)$ is odd. If*

$$\tilde{\sigma}(u) \le \frac{n}{n-1} + \frac{2n^2-2n+1}{2(n-1)}H^2 - \frac{H}{2(n-1)^2}\sqrt{(2n-1)^2H^2 + 4n(n-1)},$$

then M is the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{1+H^2}}\right)$.

Proof. It follows from Lemma 3.3 that $\tilde{L}(x) = \max_{u\in UM_x}\tilde{\sigma}(u)$ is a constant function on M. Assume that

$$\tilde{L}(x) \equiv \frac{n}{n-1} + \frac{2n^2-2n+1}{2(n-1)}H^2 - \frac{H}{2(n-1)^2}\sqrt{(2n-1)^2H^2 + 4n(n-1)}.$$

As in the proof of Lemma 4.1, we have $(\triangle\tilde{T})_{1111} \equiv 0$ on M. From (3.12) we have

$$\begin{cases} h^{n+1}_{aa} = -H + [-H + \sqrt{(2n-1)^2H^2 + 4n(n-1)}]/2(n-1), \\ \qquad \textit{for}\ a = 1,\dots,m, \\ h^{n+1}_{rr} = -H - [-H + \sqrt{(2n-1)^2H^2 + 4n(n-1)}]/2(n-1), \\ \qquad \textit{for}\ r = m+1,\dots,2m, \\ h^{n+1}_{nn} = -H. \end{cases} \tag{4.5}$$

Since h^{n+1}_{nn} is a constant, from (2.1) and (3.4) we have

$$\sum_k h^{n+1}_{nnk}\omega_k = -2\sum_i h^{n+1}_{ni}\omega_{in} = 0.$$

Therefore

$$h^{n+1}_{nnk} = 0, \quad k = 1,\dots,n. \tag{4.6}$$

Moreover, as in the proof of Theorem 4.2, we have $h_{ijk}^{n+1} = 0$, for $i, j = 1, \dots, 2m$, $k = 1, \dots, n$. Hence, $h_{ijk}^{n+1} = 0$, for $i, j, k = 1, \dots, n$. Thus, it is easy to see from (2.1) that

$$\omega_{ar} = \omega_{an} = \omega_{rn} = 0. \tag{4.7}$$

Then we have

$$\begin{aligned} 0 &= d\omega_{an} \\ &= -\sum_{b=1}^{m} \omega_{ab} \wedge \omega_{bn} - \sum_{s=m+1}^{2m} \omega_{as} \wedge \omega_{sn} - \omega_{a,n+1} \wedge \omega_{n+1,n} + \omega_a \wedge \omega_n \\ &= (h_{aa}^{n+1} h_{nn}^{n+1} + 1)\omega_a \wedge \omega_n. \end{aligned}$$

Similarly we obtain $(h_{rr}^{n+1} h_{nn}^{n+1} + 1)\omega_r \wedge \omega_n = 0$. Then we have

$$\begin{cases} 1 + h_{aa}^{n+1} h_{nn}^{n+1} = 0, \\ 1 + h_{rr}^{n+1} h_{nn}^{n+1} = 0. \end{cases} \tag{4.8}$$

This implies $h_{aa}^{n+1} = h_{rr}^{n+1}$. Hence, we have

$$\begin{aligned} &-H + [-H + \sqrt{(2n-1)^2 H^2 + 4n(n-1)}]/2(n-1) \\ &\qquad = -H - [-H + \sqrt{(2n-1)^2 H^2 + 4n(n-1)}]/2(n-1). \end{aligned}$$

Thus, $(2n-1)^2 H^2 + 4n(n-1) = H^2$, i.e., $4n(n-1)(1+H^2) = 0$, which is a contradiction. Therefore,

$$\tilde{\sigma}(u) < \frac{n}{n-1} + \frac{2n^2 - 2n + 1}{2(n-1)} H^2 - \frac{H}{2(n-1)^2} \sqrt{(2n-1)^2 H^2 + 4n(n-1)}$$

for any $u \in UM$. It follows from Lemma 4.1 that M is an totally umbilical sphere. □

THEOREM 4.4. *Let M be an n-dimensional compact submanifold with parallel mean curvature in S^{n+2}. If $H \neq 0$ and*

$$\tilde{\sigma}(u) \le 1 + \frac{3n+1}{4n} H^2, \quad \textit{for any } u \in UM,$$

then M is the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{1+H^2}}\right)$.

Proof. It follows from Lemma 4.1. □

THEOREM 4.5. *Let M be an n-dimensional compact submanifold with parallel mean curvature in S^{n+p}, where $p \ge 3$, or $p = 2$ and $H = 0$. If*

$$\tilde{\sigma}(u) \le \frac{1}{3} + \frac{7n+1}{24n} H^2, \quad \textit{for any } u \in UM,$$

then M is either the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{1+H^2}}\right)$, one of the embeddings $\phi_{i,p}$, $i = 1,2,3,4$, or the minimal immersion $\phi'_{1,p}$.

Proof. In the case where $\tilde{\sigma}(u) < \frac{1}{3} + \frac{7n+1}{24n}H^2$, for any $u \in UM$, it follows from Lemma 4.1 that M is the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{1+H^2}}\right)$.

In the case $\tilde{L}(x) = \max_{u \in UM_x} \tilde{\sigma}(u) \equiv \frac{1}{3} + \frac{7n+1}{24n}H^2$, as in the proof of Lemma 4.1, we obtain $(\triangle \tilde{T})_{1111} = 0$ on M. By Lemma 3.2, we see that $n(=2m)$ is even and $H = 0$, thus $\max_{u \in UM_x} \tilde{\sigma}(u) = \max_{u \in UM_x} \sigma(u) = \frac{1}{3}$. By the same argument as in [4], one see that the isometric immersion is a $\frac{\sqrt{3}}{3}$-isotropic minimal immersion with parallel second fundamental tensor. Using Sakamoto's classification [8] of all $\frac{\sqrt{3}}{3}$-isotropic minimal immersions into a unit sphere with parallel second fundamental form, we conclude that M is one of $\phi_{i,p}$, $i = 1,2,3,4$, or $\phi'_{1,p}$. This proves Theorem 4.5. □

Proof of Main Theorem. It follows from Theorems 4.2, 4.3, 4.4 and 4.5. □

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Hong-Wei Xu
Center of Mathematical Sciences
Zhejiang University
Hangzhou 310027
P.R. China
E-mail: xuhw@cms.zju.edu.cn

Fei Huang
Department of Mathematics
Zhejiang University
Hangzhou 310027
P.R. China
E-mail: rileyhuang@163.com

Fei Xiang
Center of Mathematical Sciences
Zhejiang University
Hangzhou 310027
P.R. China
E-mail: xiangf@cms.zju.edu.cn