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# ON THE CANONICAL BUNDLE FORMULA FOR ABELIAN FIBER SPACES IN POSITIVE CHARACTERISTIC

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## Abstract

Let $X$ be a non-singular projective $(n+1)$-fold defined over an algebraically closed field $k$ of characteristic $p \geq 0$, and $B$ be a non-singular complete curve defined over $k$. A surjective morphism $f : X \to B$ is said to be an $n$-abelian fiber space if almost all fibers are $n$-dimensional abelian varieties. We examine the canonical bundle formula for $n$-abelian fiber spaces.

## Introduction

Let $k$ be an algebraically closed field of characteristic $p \geq 0$. Let $X$ be a non-singular projective $(n+1)$-fold defined over $k$, and $B$ be a non-singular complete curve defined over $k$. A surjective morphism $f : X \to B$ is said to be an $n$-abelian fiber space if $f_*\mathcal{O}_X = \mathcal{O}_B$ and almost all fibers are $n$-dimensional abelian varieties. Let $b$ be a point of $B$. We set $F_b = f^{-1}(b)$. A fiber $F_b$ is said to be a multiple fiber of $f$ with multiplicities $m$ if $m \geq 2$ and $F_b = mP$ with $P = \sum_{i=1}^{r} n_i E_i$ such that $(n_1, \ldots, n_r) = 1$, where $E_i$'s are prime divisors on $X$. Sometimes we simply call $F_b = mP$ a multiple fiber of $f$ or a multiple fiber. In this paper, we study the structure of $n$-abelian fiber spaces in positive characteristic.

In Section 1, we mainly study the structure of 2-abelian fiber spaces under certain conditions and examine the canonical bundle formula. We can similarly treat the higher dimensional case. We define the notion of tame fibers and wild fibers in the case of $n$-abelian fiber spaces like in the case of elliptic fibrations. Let $f : X \to B$ be an $n$-abelian fiber space with $(K_X^2 \cdot H^{n-1}) = 0$, where $H$ is a hyperplane section on $X$. Since $B$ is a non-singular curve, we have $R^i f_* \mathcal{O}_X = L_i \oplus T_i$, where $L_i$ is a locally free sheaf and $T_i$ is a torsion sheaf $(i = 1, 2, \ldots, n)$. We call a multiple fiber $F_b = mP$ a wild fiber if one of the following equivalent conditions is satisfied.

1. $b \in \operatorname{Supp} T_n$,
2. $\dim H^0(mP, \omega_{mP}) \geq 2$.

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If a multiple fiber is not a wild fiber, it is called a tame fiber. The main purpose of this paper is to give a canonical bundle formula for $n$-abelian fiber spaces. The following is one of the main theorems of this paper.

THEOREM 0.1. *Let $f : X \to B$ be an $n$-abelian fiber space with $(K_X^2 \cdot H^{n-1}) = 0$, where $H$ is a hyperplane section on $X$. Let $R^i f_* \mathcal{O}_X = L_i \oplus T_i$, where $L_i$ is a locally free sheaf and $T_i$ is a torsion sheaf $(i = 1, 2, \dots, n)$. Let $l(T_i)$ be the length of $T_i$. Then we have*

$$\omega_X \cong f^*(L_n^{-1} \otimes \omega_B) \otimes \mathcal{O}_X\left(\sum_{i=1}^{r} a_i P_i\right),$$

*where*

1. $m_i P_i = F_{b_i}$ *are the multiple fibers of* $f$
2. $0 \le a_i \le m_i - 1$
3. $a_i = m_i - n_i$ *if $F_{b_i}$ is a tame fiber, where* $n_i = \min\{n \in \mathbf{Z}_{>0} \mid \dim H^0(\omega_{nP_i}) > 0\}$
4. $\chi(\mathcal{O}_X) = \sum_{i=1}^{n} (-1)^i (\deg L_i + l(T_i))$.

We call $n_i$ a jumping value of the multiple fiber $m_i P_i$. We note that the condition that $(K_X^2 \cdot H^{n-1}) = 0$, in a sense, corresponds to the minimality of elliptic fibration and this condition is equivalent to the condition that $K_X$ is $f$-nef. In Section 2, we investigate special phenomena in positive characteristic. Let $f : X \to B$ be an $n$-abelian fiber spaces. By [1], we see that $\deg f_* \omega_{X/B} \ge 0$ if $\mathrm{char}(k) = 0$. In this paper, we give an example of 2-abelian fiber space $f : X \to B$ with $\deg f_* \omega_{X/B} < 0$ in positive characteristic. Next we consider a 2-abelian fiber space $f : X \to B$ with Kodaira dimension $\kappa(X) = 1$. For such an abelian fiber space $f$, there exists a positive integer $m$ such that the multicanonical system $|mK_X|$ gives a unique structure of abelian fiber space. We consider the problem: "Find the smallest integer $M$ such that the multicanonical system $|mK_X|$ gives the structure of abelian fiber space for any 2-abelian fiber space and any integer $m \ge M$." In this paper, we give an example which shows the following theorem does not hold in positive characteristic.

THEOREM 0.2 (see [1]). *Assume* $\mathrm{char}(k) = 0$. *Let $f : X \to B$ be a 2-abelian fiber space with $\kappa(X) = 1$ such that $K_X$ is $f$-nef and the jumping values for all multiple fibers are equal to 1. Then the multicanonical system $|mK_X|$ gives the structure of abelian fiber space if $m \ge 14$. Also 14 is the best possible bound.*

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## Notation

Throughout this paper, we fix an algebraically closed field $k$ of characteristic $p \geq 0$. Let $X$ be a non-singular projective $(n+1)$-fold defined over $k$, and $B$ be a non-singular complete curve defined over $k$. A surjective morphism $f : X \to B$ is said to be an $n$-abelian fiber space if $f_* \mathcal{O}_X = \mathcal{O}_B$ and almost all fibers are $n$-dimensional abelian varieties. Let $b$ be a point of $B$. We set $F_b = f^{-1}(b)$. A fiber $F_b$ is said to be a multiple fiber of $f$ with multiplicities $m$ if $m \geq 2$ and $F_b = mP$ with $P = \sum_{i=1}^{r} n_i E_i$ such that $(n_1, \dots, n_r) = 1$, where $E_i$'s are prime divisors on $X$. Sometimes we simply call $F_b = mP$ a multiple fiber of $f$ or a multiple fiber.

For a non-singular complete algebraic variety $X$ defined over $k$, we use the following notation.

$\mathcal{O}_X$ : the structure sheaf on $X$.
$\omega_X$ : a canonical sheaf of $X$.
$H^i(X, \mathcal{F})$ : the $i$-th cohomology group of a coherent sheaf $\mathcal{F}$ on $X$.
$h^i(X, \mathcal{F})$ : the dimension of $i$-th cohomology group of a coherent $\mathcal{F}$ on $X$.
$\chi(\mathcal{F})$ : the Euler characteristic of a coherent sheaf $\mathcal{F}$ on $X$.
$K_X$ : a canonical divisor of $X$.
$\Phi_{|mK_X|}$ : the rational mapping associated with the multicanonical system $|mK_X|$.
$\kappa(X)$ : the Kodaira dimension of $X$.
$g(C)$ : the genus of a non-singular curve $C$.
$\mathrm{Pic}(X)$ : the Picard group of $X$.
$\mathrm{NS}(X)$ : the Néron-Severi group of $X$.
$[\alpha]$ : the largest integer which does not exceed a real number $\alpha$.

For Cartier divisors $D$, $D'$ on $X$, we denote by $D \sim D'$ the linear equivalence. For a group $G$ and elements $\sigma_1, \dots, \sigma_t$ of $G$, we denote by $\langle \sigma_1, \dots, \sigma_t \rangle$ the subgroup generated by $\sigma_1, \dots, \sigma_t$. Sometimes, a Cartier divisor and the associated invertible sheaf will be identified.

## 1. A canonical bundle formula for abelian fiber spaces

In this section, we mainly consider 2-abelian fiber spaces. We can easily generalize almost all results in this section to $n$-abelian fiber spaces with arbitrary $n$.

The following two theorems are well known (cf [4]).

THEOREM 1.1. *If $f : X \to Y$ is a proper morphism of locally noetherian schemes and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules on $X$ for all $p \geq 0$, the direct image sheaves $R^p f_*(\mathcal{F})$ are coherent sheaves of $\mathcal{O}_Y$-modules.*

For any morphism $f : X \to Y$ and $y \in Y$, we denote by $X_y$ the fiber of $f$ over $y$, and for $\mathscr{F}$ a quasi-coherent on $X$, we denote by $\mathscr{F}_y$ the sheaf $\mathscr{F} \otimes_{\mathcal{O}_Y} k(y)$ on $X_y$.

THEOREM 1.2. *Let $f : X \to Y$ is a proper morphism of locally noetherian schemes and $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-module on $X$, flat on $Y$. Assume $Y$ is reduced and connected. Then for all $p$ the following are equivalent*:

1. $y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ *is a constant function*,
2. $R^p f_*(\mathscr{F})$ *is a locally free sheaf on $Y$, and for all $y \in Y$, the natural map*:

$$R^p f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} k(y) \to H^p(X_y, \mathscr{F}_y).$$

*is an isomorphism.*

*If these conditions are satisfied,*

$$R^{p-1} f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} k(y) \to H^{p-1}(X_y, \mathscr{F}_y).$$

*is an isomorphism for any $y \in Y$.*

Let $f : X \to B$ be an $n$-abelian fiber space. Note that $f$ is flat because $B$ is a non-singular curve, and all the fibers of $f$ are connected by Zariski's connected theorem. By the definition of $f$, there exists a finite number of points $b_1, \dots, b_r \in B$ such that, for every point $b \in B \backslash \{b_1, \dots, b_r\}$, the fiber $F_b$ is an $n$-dimensional abelian variety, and $F_{b_i}$ is a non-multiple singular fiber or $F_{b_i} = m_i P_i$ is a multiple fiber. By above two theorems, we see that $R^n f_* \mathcal{O}_X$ is a coherent $\mathcal{O}_B$-module such that $(R^n f_* \mathcal{O}_X) \otimes k(b) \cong H^n(F_b, \mathcal{O}_{F_b})$ for all $b \in B$. Since $\dim H^n(F_b, \mathcal{O}_{F_b}) = 1$ for all $b \in B \backslash \{b_1, \dots, b_r\}$, $R^n f_* \mathcal{O}_X$ is invertible over the open set $B \backslash \{b_1, \dots, b_r\}$. On the other hand, since $B$ is a non-singular curve, we have $R^i f_* \mathcal{O}_X = L_i \oplus T_i$ for $i = 1, 2, \dots, n$, where $L_i$ is a locally free sheaf of finite rank and $T_i$ is a torsion part. Putting these observations together, we see that $L_n$ is an invertible sheaf and the support of $T_n$ is contained in the set $\{b_1, \dots, b_r\}$. Since the fibers of $f$ over $b \in B \backslash \{b_1, \dots, b_r\}$ are $n$-dimensional abelian varieties, we get that $R^i f_* \mathcal{O}_X \otimes k(b) \cong H^i(F_b, \mathcal{O}_{F_b})$ over $b \in B \backslash \{b_1, \dots, b_r\}$ for $i = 1, 2, \dots, n-1$ by Theorem 1.2. On the other hand, we have $\dim H^i(F_b, \mathcal{O}_{F_b}) = \binom{n}{i}$ for all $b \in B \backslash \{b_1, \dots, b_r\}$ and $i = 1, 2, \dots, n-1$. Therefore we see that $L_i$ is a locally free sheaf of rank $\binom{n}{i}$ for $i = 1, 2, \dots, n-1$.

Now we define the notion of tame fibers and wild fibers.

DEFINITION. Let $f : X \to B$ be an $n$-abelian fiber space and let $R^i f_* \mathcal{O}_X = L_i \oplus T_i$, where $L_i$ is a locally free sheaf and $T_i$ is a torsion part. Let $b$ be a point of $B$. The fiber $F_b$ of $f$ is said to be a *wild fiber* if $b \in \operatorname{Supp} T_n$. If a multiple fiber is not a wild fiber, it is called a *tame fiber*.

*Remark*. Let $f : X \to B$ be an $n$-abelian fiber space and let $b$ be a point of $B$. By the Serra duality and Theorem 1.2, the fiber $F_b$ is a wild fiber if and only if $\dim H^n(F_b, \mathcal{O}_{F_b}) = \dim H^0(F_b, \omega_{F_b}) \geq 2$. Aerdts showed that in characteristic 0, $R^i f_* \mathcal{O}_X$ is a locally free sheaf for any $i$. Therefore, a wild fiber appears only in positive characteristic.

To give a canonical bundle formula for 2-abelian fiber space, we need the following two lemmas (due to [2]).

Lemma 1.3. *Let $f : X \to B$ be a 2-abelian fiber space and let $H$ be a hyperplane section on $X$. Let $b$ be a point of $B$ and let $D$ be a connected component of the fiber $F_b$. Then $(D^2 \cdot H) \leq 0$. Also $(D^2 \cdot H) = 0$ if and only if there exists $a \in \mathbf{Q}$ such that $D = aF_b$.*

*Proof.* We can write $F_b = \sum_{i=1}^s n_i E_i$, where $E_i$'s are integral surfaces. We see that $(E_i \cdot F_b \cdot H) = 0$ and $(E_i \cdot E_j \cdot H) \geq 0$ if $i \neq j$. We denote by $e_i$ (resp. $h$) the class of $E_i$ (resp. $H$) in $\mathrm{NS}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $M$ be a $\mathbf{Q}$-vector subspace of $\mathrm{NS}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ which is generated by $e_i$'s. Then, using the intersection form on $\mathrm{NS}(X)$, we have a symmetric bilinear form

$$M \times M \ni (x, y) \mapsto (x \cdot y \cdot h) \in \mathbf{Q}$$

which satisfies $(e_i \cdot e_j \cdot h) \geq 0$ if $i \neq j$. Replacing each $e_i$ with $n_i e_i$, we may assume that $(e_i \cdot z \cdot h) = 0$ where $z = \sum_{i=1}^s e_i$. It suffices to prove $(x \cdot x \cdot h) \leq 0$ for $x = \sum c_i e_i$ $(c_i \in \mathbf{Q})$ and to prove $(x \cdot x \cdot h) = 0$ if and only if $c_i = c_j$ whenever $(e_i \cdot e_j \cdot h) > 0$. Then

$$\begin{aligned} (x \cdot x \cdot h) &= \sum_i c_i^2 (e_i \cdot e_j \cdot h) + 2 \sum_{i<j} c_i c_j (e_i \cdot e_j \cdot h) \\ &\leq \sum_i c_i^2 (e_i \cdot e_i \cdot h) + \sum_{i<j} (c_i^2 + c_j^2)(e_i \cdot e_j \cdot h) \\ &= \sum_{i,j} c_i^2 (e_i \cdot e_j \cdot h) = \sum_i c_i^2 (e_i \cdot z \cdot h) = 0. \end{aligned}$$

Using this inequality, we see that $(x \cdot x \cdot h) = 0$ if and only if $c_i = c_j$ whenever $(e_i \cdot e_j \cdot h) > 0$. □

Lemma 1.4. *Let $f : X \to B$ be a 2-abelian fiber space and let $H$ be a hyperplane section on $X$. Let $F_b = mP$ be a multiple fiber of $f$, where $P = \sum_{i=1}^t n_i E_i$ such that $E_i$'s are integral surfaces and $(n_1, \dots, n_t) = 1$, and let $D$ be a divisor on $P$ such that $(D \cdot E_i \cdot H) = 0$ for every $i = 1, \dots, t$. Then $H^0(P, \mathcal{O}_P(D)) \neq 0$ if and only if $\mathcal{O}_P(D) \cong \mathcal{O}_P$ and $H^0(P, \mathcal{O}_P) = k$.*

*Proof.* It suffices to show that every nonzero section $s \in H^0(P, \mathcal{O}_P(D))$ generates $\mathcal{O}_P(D)$, that is, it defines an isomorphism of $\mathcal{O}_P$ onto $\mathcal{O}_P(D)$. This would show also that $H^0(P, \mathcal{O}_P)$ is a field. Since $H^0(P, \mathcal{O}_P)/k$ is a finite extension and $k$ is algebraically closed, we see that $H^0(P, \mathcal{O}_P) \cong k$.

Let $s_i = s|_{E_i} \in H^0(E_i, \mathcal{O}_P(D) \otimes \mathcal{O}_{E_i})$. Since $(D|_{E_i} \cdot H|_{E_i}) = (D \cdot E_i \cdot H) = 0$, we have that either $s_i$ is identically zero on $E_i$, or $s_i$ does not vanish anywhere on $E_i$ (in which case $s_i$ generates $\mathcal{O}_P(D) \otimes \mathcal{O}_{E_i}$). If $s_i$ is identically zero on a component $E_i$, then $s_j$ is also identically zero for every $j$, for $P$ is connected. Therefore, if $s_i$ does not vanish anywhere on $E_i$, then $s$ does not vanish anywhere on $P$, so that $s$ generates $\mathcal{O}_P(D)$.

Now assume that $s_i$ is identically zero on $E_i$ for every $i$. We shall show that this assumption leads to a contradiction. Let $k_i$ be the order of vanishing of $s$ along $E_i$. This means that $s \in \mathrm{Ker}[H^0(P, \mathcal{O}_P(D)) \to H^0(k_iE_i, \mathcal{O}_P(D) \otimes \mathcal{O}_{k_iE_i})]$ and if $k_i < n_i$, then $s \notin \mathrm{Ker}[H^0(P, \mathcal{O}_P(D)) \to H^0((k_i+1)E_i, \mathcal{O}_P(D) \otimes \mathcal{O}_{(k_i+1)E_i})]$. We put $D_1 = \sum_{i=1}^t k_iE_i$. Restricting to $H$, we get $(D_1^2 \cdot H) = 0$ by the proof of [2, Theorem 7.8]. By Lemma 1.3, there exists $c \in \mathbf{Q}$ such that $D_1 = cP$ $(0 < c \le 1)$.

Since $(n_1, \dots, n_t) = 1$, there exist $a_1, \dots, a_t \in \mathbf{Z}$ such that $\sum_{i=1}^t a_in_i = 1$. Then

$$c = \frac{a_1k_1}{a_1n_1} = \cdots = \frac{a_tk_t}{a_tn_t} = \frac{\sum_i a_ik_i}{\sum_i a_in_i} = \sum_i a_ik_i \in \mathbf{Z}.$$

Therefore $c = 1$ and so $D_1 = P$. A contradiction. □

Definition. Let $f : X \to B$ be an $n$-abelian fiber space and let $mP$ be a multiple fiber of $f$. The positive integer $\min\{n \in \mathbf{Z}_{>0} \mid \dim H^0(\omega_{nP}) > 0\}$ is called a *jumping value of the multiple fiber $mP$*.

In the following, we give a canonical bundle formula for 2-abelian fiber space.

Theorem 1.5. *Let $f : X \to B$ be a 2-abelian fiber space with $(K_X^2 \cdot H) = 0$, where $H$ is a hyperplane section on $X$. Let $R^1f_*\mathcal{O}_X = E \oplus S$ and $R^2f_*\mathcal{O}_X = L \oplus T$, where $E$ (resp. $L$) is a locally free sheaf and $S$ (resp. $T$) is a torsion part. Let $l(S)$ (resp. $l(T)$) is the length of $S$ (resp. $T$). Then we have*

$$\omega_X \cong f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X\left(\sum_{i=1}^r a_iP_i\right),$$

*where*

1. *$m_iP_i = F_{b_i}$ are the multiple fibers of $f$*
2. *$0 \le a_i \le m_i - 1$*
3. *$a_i = m_i - n_i$ if $F_{b_i}$ is a tame fiber, where $n_i$ is a jumping value of the multiple fiber $m_iP_i$*
4. *$\chi(\mathcal{O}_X) + \deg E + l(S) = \deg L + l(T)$.*

*Proof.* If $z_1, \dots, z_s$ are $s$ general points of $B$, we have an exact sequence

$$0 \to \omega_X \to \omega_X \otimes \mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right) \to \bigoplus_{i=1}^{s}(\omega_X \otimes \mathcal{O}_X(F_{z_i}) \otimes \mathcal{O}_{F_{z_i}}) \to 0$$

Since $F_{z_1}, \dots, F_{z_s}$ are abelian surfaces, we get $\omega_X \otimes \mathcal{O}_X(F_{z_i}) \otimes \mathcal{O}_{F_{z_i}} \cong \omega_{F_{z_i}} \cong \mathcal{O}_{F_{z_i}}$ for any $i$. We get an exact sequence

$$0 \to H^0(\omega_X) \to H^0\left(\omega_X \otimes \mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right) \to \bigoplus_{i=1}^{s} H^0(\mathcal{O}_{F_{z_i}}) \to H^1(\omega_X).$$

From this exact sequence we get the inequality

$$\begin{aligned}\dim H^0\left(\omega_X \otimes \mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right) &\geq \dim H^0(\omega_X) + \sum_{i=1}^{s} \dim H^0(\mathcal{O}_{F_{z_i}}) - \dim H^1(\omega_X) \\ &= \dim H^0(\omega_X) + s - \dim H^1(\omega_X).\end{aligned}$$

Therefore the complete linear system $|K_X + \sum_{i=1}^{s} F_{z_i}|$ is nonempty if $s > \dim H^1(\omega_X) - \dim H^0(\omega_X)$. Then let $D \in |K_X + \sum_{i=1}^{s} F_{z_i}|$. Now assume that there exists an irreducible component $E \subset D$ such that $E \not\subset F_z$ for any point $z \in B$. Since $(E \cdot F_z \cdot H) > 0$, we get $(D \cdot F_z \cdot H) > 0$. Since $D \sim K_X + \sum_{i=1}^{s} F_{z_i}$, we see that

$$\begin{aligned}(D \cdot F_z \cdot H) &= \left(K_X + \sum_{i=1}^{s} F_{z_i} \cdot F_z \cdot H\right) = (K_X \cdot F_z \cdot H) \\ &= (K_X + F_z \cdot F_z \cdot H) = (K_X + F_z|_{F_z} \cdot H|_{F_z}) = (K_{F_z} \cdot H|_{F_z}) > 0.\end{aligned}$$

Now we assume that $F_z$ is an abelian surface. Then, we have $(K_{F_z} \cdot H|_{F_z}) = 0$, which is a contradiction. Therefore all the components of $D$ are contained in fibers of $f$, and $K_X$ has also the same property. Put

$$K_X \sim \sum_{j=1}^{s} c_j F_{z_j} + D \quad (c_j \in \mathbf{Z}, s \geq 0),$$

with $D \geq 0$, where $D$ is contained in a finite union of fibers of $f$, but does not contain any fiber of $f$. Let $D_1, \dots, D_t$ be the connected components of the divisor $D$ such that $\operatorname{Supp}(D_i) \cap \operatorname{Supp}(D_j) = \emptyset$ if $i \neq j$. Say $D_i$ is contained in the fiber $F_{z_i}$ for some point $z_i \in B$ for each $i$. By Lemma 1.3, we get $(D_i^2 \cdot H) \leq 0$ for each $i$. By hypothesis, $(K_X^2 \cdot H) = (D^2 \cdot H) = \sum_{i=1}^{t}(D_i^2 \cdot H) = 0$. Therefore $(D_i^2 \cdot H) = 0$ for each $i$. By Lemma 1.3, there exists $d_i \in \mathbf{Q}$ such that $D_i = d_i F_{z_i}$ for each $i$. Therefore we can write

$$\omega_X \cong f^*(M) \otimes \mathcal{O}_X\left(\sum_{i=1}^{r} a_i P_i\right) \quad (0 \leq a_i \leq m_i - 1, a_i \in \mathbf{Z}), \tag{1}$$

where $M$ is an invertible sheaf on $B$.

By the proof of [2, Theorem 7.15], we get

$$f_*\mathcal{O}_X\left(\sum_{i=1}^{r} a_iP_i\right) \cong \mathcal{O}_B \quad \text{if } 0 \le a_i \le m_i - 1. \tag{2}$$

From (1), (2) and the projection formula, we get $f_*(\omega_X) \cong M$.

Now the duality theorem for a map says that

$$M \cong f_*(\omega_X) \cong \mathcal{H}om_{\mathcal{O}_B}(R^2f_*\mathcal{O}_X, \omega_B) \cong \mathcal{H}om_{\mathcal{O}_B}(L, \omega_B) \cong L^{-1} \otimes \omega_B,$$

because $R^2f_*\mathcal{O}_X = L \oplus T$ and the dual of the torsion part is zero. Therefore formula (1) becomes

$$\omega_X \cong f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X\left(\sum_{i=1}^{r} a_iP_i\right) \quad (0 \le a_i \le m_i - 1).$$

Next we consider the value of $a_i$. For simplicity, we set $b = b_1$, $a = a_1$, $n_0 = n_1$ and $F_b = mP$. Then we can write $\omega_X \cong f^*(M) \otimes \mathcal{O}_X(aP) \otimes \mathcal{O}_X(\sum_{i=2}^{r} a_iP_i)$, where $M$ is an invertible sheaf on $B$. We set $\omega_n = \omega_{nP} \cong \omega_X \otimes \mathcal{O}_X(nP) \otimes \mathcal{O}_{nP}$. To prove that $a = m - n_0$ if a multiple fiber $mP$ is a tame fiber, we need the following lemma (cf [5]).

Lemma 1.6. *Let $P$ and $\omega_n$ be as above. Then,*
1. *If $\omega_n$ is not trivial, then* $\dim H^0(\omega_n) = \dim H^0(\omega_{n-1})$.
2. *If $\omega_n$ is trivial, then* $\dim H^0(\omega_n) = \dim H^0(\omega_{n-1}) + 1$.

*Proof.* The exact sequence

$$0 \to \omega_{n-1} \to \omega_n \to \omega_n|_P \to 0$$

induces an exact sequence

$$0 \to H^0(\omega_{n-1}) \to H^0(\omega_n) \xrightarrow{\gamma} H^0(\omega_n|_P),$$

where $\gamma$ is a restriction map.

If $\omega_n$ is trivial, $\omega_n|_P$ is trivial, we get $H^0(\omega_n|_P) \cong k$ by Lemma 1.4. Therefore $\gamma$ is surjective and $\dim H^0(\omega_n) = \dim H^0(\omega_{n-1}) + 1$. If $\omega_n$ and $\omega_n|_P$ are not trivial, we get $H^0(\omega_n|_P) = 0$ by Lemma 1.4. Therefore $\dim H^0(\omega_{n-1}) = \dim H^0(\omega_n)$. If $\omega_n$ is not trivial and $\omega_n|_P$ is trivial, we get $H^0(\omega_n|_P) \cong k$ by Lemma 1.4. Assume that there exists $\tilde{\sigma} \in H^0(\omega_n)$ such that $\gamma(\tilde{\sigma}) = \sigma$ for some nonzero section $\sigma \in H^0(\omega_n|_P)$. Since $\sigma$ doesn't vanish on $P$, $\tilde{\sigma}$ doesn't vanish on $nP$. It follows that $\omega_n$ is trivial. This is a contradiction. This implies that $\gamma$ is not surjective. Therefore $\dim H^0(\omega_n) = \dim H^0(\omega_{n-1})$. □

By Lemma 1.6, $\dim H^0(\omega_{nP})$ is a non-decreasing function of $n$. Since $\dim H^0(\omega_{mP}) = \dim H^0(\omega_{F_b}) = \dim H^2(\mathcal{O}_{F_b}) = \dim(R^2f_*\mathcal{O}_X \otimes k(b)) > 0$, we have $1 \le n_0 \le m$. If $n = m - a$, $\omega_n$ is trivial. Thus, we see $n_0 = m - a$ if $mP$ is a tame fiber.

Finally, we will show $\chi(\mathcal{O}_X) + \deg E + l(S) = \deg L + l(T)$. Let $z_1, \dots, z_s$ be general points of $B$. Consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right) \to \bigoplus_{i=1}^{s} (\mathcal{O}_X(F_{z_i}) \otimes \mathcal{O}_{F_{z_i}}) \to 0.$$

Since $F_{z_i}$'s are abelian surfaces, we get $\chi(\mathcal{O}_X(F_{z_i}) \otimes \mathcal{O}_{F_{z_i}}) = 0$ by the Riemann-Roch theorem for surface. Therefore we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(\sum_{i=1}^{s} F_{z_i}))$.

Now consider the spectral sequence

$$E_2^{pq} = H^p\left(B, R^q f_* \mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right) \Rightarrow H^{p+q}\left(X, \mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right).$$

By the projection formula, we have $R^q f_* \mathcal{O}_X(\sum_{i=1}^{s} F_{z_i}) \cong R^q f_* \mathcal{O}_X \otimes \mathcal{O}_B(\sum_{i=1}^{s} z_i)$. Because of the ampleness of $\mathcal{O}_B(\sum_{i=1}^{s} z_i)$ for any sufficient large $s$, we have that $H^p(B, R^q f_* \mathcal{O}_X \otimes \mathcal{O}_B(\sum_{i=1}^{s} z_i)) = 0$ for any $p > 0$. By using this vanishing and the degenerating spetctral sequence, we get $H^0(B, R^i f_* \mathcal{O}_X(\sum_{i=1}^{s} F_{z_i})) \cong H^i(X, \mathcal{O}_X(\sum_{i=1}^{s} F_{z_i}))$ for any $i \geq 0$ and any sufficient large $s$. By the Riemann-Roch theorem for curve and the ampleness of $\mathcal{O}_B(\sum_{i=1}^{s} z_i)$ for any sufficient large $s$, we get

$$\begin{aligned}
h^0\left(\mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right) &= h^0\left(\mathcal{O}_B\left(\sum_{i=1}^{s} z_i\right)\right) \\
&= \deg\left(\mathcal{O}_B\left(\sum_{i=1}^{s} z_i\right)\right) - g(B) + 1 \\
&= s - g(B) + 1, \\
h^1\left(\mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right) &= h^0\left(B, R^1 f_* \mathcal{O}_X \otimes \mathcal{O}_B\left(\sum_{i=1}^{s} z_i\right)\right) \\
&= h^0\left(B, E \otimes \mathcal{O}_B\left(\sum_{i=1}^{s} z_i\right)\right) + l(S) \\
&= \deg(E) + 2(s - g(B) + 1) + l(S), \\
h^2\left(\mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right) &= h^0\left(B, R^2 f_* \mathcal{O}_X \otimes \mathcal{O}_B\left(\sum_{i=1}^{s} z_i\right)\right) \\
&= \deg(L) + s - g(B) + 1 + l(T), \\
h^3\left(\mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right) &= h^0\left(R^3 f_* \mathcal{O}_X \otimes \mathcal{O}_B\left(\sum_{i=1}^{s} z_i\right)\right) = 0,
\end{aligned}$$

for any sufficient large $s$. By putting these observations together, for any sufficient large $s$, we get

$$\begin{aligned}\chi(\mathcal{O}_X) &= \chi\left(\mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right)\\ &= \sum_{j=0}^{3}(-1)^j h^j\left(\mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right)\\ &= -\deg E - l(S) + \deg L + l(T).\end{aligned}$$

Therefore we conclude $\chi(\mathcal{O}_X) + \deg E + l(S) = \deg L + l(T)$. □

THEOREM 1.7. *Let $f : X \to B$ be a 2-abelian fiber space as in Theorem 1.5. Then we have $\chi(\mathcal{O}_X) = 0$.*

*Proof.* By Theorem 1.5, there exists $m \in \mathbf{Z}_{>0}$ such that $\omega_X^{\otimes m} \cong f^*(M)$ for some invertible sheaf $M$ on $B$. Let $z_1, \dots, z_s$ be general points of $B$. Consider the exact sequence

$$0 \to \omega_X^{\otimes m} \to \omega_X^{\otimes m} \otimes \mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right) \to \bigoplus_{i=1}^{s} \omega_X^{\otimes m} \otimes \mathcal{O}_X(F_{z_i}) \otimes \mathcal{O}_{F_{z_i}} \to 0.$$

By the Riemann-Roch theorem for surface, we get $\chi(\omega_X^{\otimes m} \otimes \mathcal{O}_X(F_{z_i}) \otimes \mathcal{O}_{F_{z_i}}) = 0$. Therefore $\chi(\omega_X^{\otimes m}) = \chi(\omega_X^{\otimes m} \otimes \mathcal{O}_X(\sum_{i=1}^{s} F_{z_i})) = \chi(f^*(M \otimes \mathcal{O}_B(\sum_{i=1}^{s} z_i)))$.

Consider the spectral sequence

$$\begin{aligned}E_2^{pq} &= H^p\left(B, R^q f_* f^*\left(M \otimes \mathcal{O}_B\left(\sum_{i=1}^{s} z_i\right)\right)\right)\\ &\Rightarrow H^{p+q}\left(X, f^*\left(M \otimes \mathcal{O}_B\left(\sum_{i=1}^{s} z_i\right)\right)\right).\end{aligned}$$

By the projection formula, we get $R^q f_* f^*(M \otimes \mathcal{O}_B(\sum_{i=1}^{s} z_i)) \cong R^q f_* \mathcal{O}_X \otimes (M \otimes \mathcal{O}_B(\sum_{i=1}^{s} z_i))$. By the ampleness of $\mathcal{O}_B(\sum_{i=1}^{s} z_i)$ for any sufficient large $s$, we have $H^p(B, R^q f_* \mathcal{O}_X \otimes M \otimes \mathcal{O}_B(\sum_{i=1}^{s} z_i)) = 0$ for any $p > 0$ and any sufficient large $s$. By using this vanishing and the degenerating spectral sequence, we get $H^i(X, f^*(M \otimes \mathcal{O}_B(\sum_{i=1}^{s} z_i))) \cong H^0(B, R^i f_* \mathcal{O}_X \otimes (M \otimes \mathcal{O}_B(\sum_{i=1}^{s} z_i)))$ for any $i > 0$ and any sufficient large $s$. By computating the cohomology like in the proof of Theorem 1.5, we get

$$\chi(\omega_X^{\otimes m}) = \chi\left(\omega_X^{\otimes m} \otimes \mathcal{O}_X\left(\sum_{i=1}^{s} F_{z_i}\right)\right) = -\deg E - l(S) + \deg L + l(T) = \chi(\mathcal{O}_X).$$

By the Riemann-Roch theorem,

$$\chi(\omega_X^{\otimes m}) = \frac{1}{12}m(m-1)(2m-1)(K_X^3) + (1-2m)\chi(\mathcal{O}_X) = (1-2m)\chi(\mathcal{O}_X).$$

Combining these two equalities, we have $\chi(\mathcal{O}_X) = 0$ because $m \in \mathbf{Z}_{>0}$. $\square$

DEFINITION. Let $f : X \to B$ be an $n$-abelian fiber space and let $D$ be a divisor on $X$. We say that $D$ is *$f$-nef* if for any irreducible curve $C$ on $X$ such that $f(C)$ is a point, we have $(D \cdot C) \geq 0$.

*Remark*. By the proof of Theorem 1.5, if all $P_i$ are irreducible, we see that $(K_X^2 \cdot H) = 0$. We see that the condition that $(K_X^2 \cdot H) = 0$ is equivalent to the condition that $K_X$ is $f$-nef. In the case of $n$-abelian fiber space $f : X \to B$, we can similarly give the canonical bundle formula under the condition that $(K_X^2 \cdot H^{n-1}) = 0$, where $H$ is a hyperplane section on $X$. The condition that $(K_X^2 \cdot H^{n-1}) = 0$, in a sense, corresponds to the minimality of elliptic fibration. We also see that the conditon that $(K_X^2 \cdot H^{n-1}) = 0$ is equivalent to the condition that $K_X$ is $f$-nef. But Theorem 1.5 is not true for $n$-abelian fiber spaces. In the following, we give a canonical bundle formula for $n$-abelian fiber spaces.

THEOREM 1.8. *Let $f : X \to B$ be an $n$-abelian fiber space with $(K_X^2 \cdot H^{n-1}) = 0$, where $H$ is a hyperplane section on $X$. Let $R^i f_* \mathcal{O}_X = L_i \oplus T_i$, where $L_i$ is a locally free sheaf and $T_i$ is a torsion sheaf $(i = 1, 2, \ldots, n)$. Let $l(T_i)$ be the length of $T_i$. Then we have*

$$\omega_X \cong f^*(L_n^{-1} \otimes \omega_B) \otimes \mathcal{O}_X\left(\sum_{i=1}^{r} a_i P_i\right),$$

*where*

1. *$m_i P_i = F_{b_i}$ are the multiple fibers of $f$*
2. *$0 \leq a_i \leq m_i - 1$*
3. *$a_i = m_i - n_i$ if $F_{b_i}$ is a tame fiber, where $n_i$ is a jumping value of the multiple fiber $m_i P_i$*
4. *$\chi(\mathcal{O}_X) = \sum_{i=1}^{n} (-1)^i (\deg L_i + l(T_i))$.*

Now we give here easy examples.

*Example*. (1) Assume $\mathrm{char}(k) = p > 0$. Let $E$ be an ordinary elliptic curve, and $a \in E$ a point of order $p$. Then the group $G = \langle \sigma \rangle \cong \mathbf{Z}/p\mathbf{Z}$ acts on $E$ by

$$\sigma : E \ni x \mapsto x + a \in E.$$

The group $G$ also acts on the projective line $\mathbf{P}^1$ by

$$\sigma : t \mapsto t + 1,$$

where $t$ is a coordinate of an affine line $\mathbf{A}^1$ in $\mathbf{P}^1$. Therefore the group $G$ acts on $\mathbf{P}^1 \times E \times E$. We have a 2-abelian fiber space

$$f : X = \mathbf{P}^1 \times E \times E/\langle\sigma\rangle \to \mathbf{P}^1/\langle\sigma\rangle \cong \mathbf{P}^1$$

with a wild fiber $pF_\infty$ over the point at infinity of $\mathbf{P}^1$. The canonical bundle formula is

$$K_X \cong f^*\mathcal{O}_{\mathbf{P}^1}(-1) + (p-2)F_\infty.$$

(2) ([6, Section 16]) Assume $\mathrm{char}(k) = 0$. Let $E_\rho$ be an elliptic curve with period $(1, \rho)$, where $\rho = \exp(2\pi i/3)$. Let $g$ denote the automorphism of $E_\rho \times E_\rho \times E_\rho$ defined by $g(z_1, z_2, z_3) = (\rho z_1, \rho z_2, \rho z_3)$. Let $V$ be a non-singular model of $E_\rho \times E_\rho \times E_\rho/\langle g\rangle$, obtained by the canonical resolution of singularities. The projection $p : E_\rho \times E_\rho \times E_\rho \to E_\rho \times E_\rho \times E_\rho/\langle g\rangle$ to the first factor induces a 2-abelian fiber space $f : V \to \mathbf{P}^1$. K. Ueno showed that $K_V \cong \mathcal{O}_V$. This gives an example of 2-abelian fiber space such that the jumping value of this fibration is not equal to one (for details, see [6]).

(3) ([6, Section 16]) Assume $\mathrm{char}(k) = 0$. Let $A$ be an abelian variety of dimension three and let $\iota : A \to A$ denote the standard involution on $A$. The quotient space $W = A/\langle\iota\rangle$ has $2^6$ singular points, which corresponds to the $2^6$ fixed points of the involution $\iota$ on $A$. Let $V$ be a non-singular model of $W$, obtained by the canonical resolution of singularities. Then K. Ueno showed that $\kappa(V) = 0$. More precisely,

$$mK_V \cong \sum_{i=1}^{2^6} \left(\frac{3m}{2} - 1\right) E_i,$$

where $E_i \cong \mathbf{P}^2$ appears in the canonical resolution of singularities of $W$. He also showed that $\chi(\mathcal{O}_V) = 4$. Thus from Theorem 1.8, we conclude that $K_V$ is not $f$-nef.

(4) Assume that $0 < \mathrm{char}(k) = p \equiv 1 \pmod 6$. Let $C$ be the non-singular complete model of the curve defined by the equation

$$t^2 = x^p - x.$$

The genus of $C$ is given by $g(C) = \frac{1}{2}(p-1)$. Let $E$ and $E'$ be ordinary elliptic curves, $a \in E$ a point of order $p$ and $a' \in E'$ a point of order 6. The group $\langle\sigma\rangle \cong \mathbf{Z}/p\mathbf{Z}$ and $\langle\tau\rangle \cong \mathbf{Z}/6\mathbf{Z}$ act on $C$, $E$ and $E'$ by

$$\begin{array}{lll} \sigma : (x,t) \mapsto (x+1, t) & \tau : (x,t) \mapsto (\omega x, -\omega t) & \text{on } C \\ \sigma : z \mapsto z + a & \tau : z \mapsto z & \text{on } E \\ \sigma : z' \mapsto z' & \tau : z' \mapsto z' + a' & \text{on } E', \end{array}$$

where $\omega$ is a primitive cube root of unity. Since $C \times E \to C \times E/\langle\sigma\rangle$ is an étale morphism, we have an ellpitic fibration

$$f_0 : X_0 = C \times E/\langle\sigma\rangle \to C/\langle\sigma\rangle \cong \mathbf{P}^1$$

with a canonical bundle formula which is given by

$$K_{X_0} = (p-3)E_\infty,$$

where $pE_\infty$ is a multiple fiber over the point at infinity of $\mathbf{P}^1$. We set $X = X_0 \times E'/\langle\tau\rangle$ and $f : X \to \mathbf{P}^1/\langle\tau\rangle \cong \mathbf{P}^1$. The canonical divisor of $X$ has the form $K_X = f^*\mathcal{O}_{\mathbf{P}^1}(l) + 5F_0 + bF_\infty$ for $l \in \mathbf{Z}$ and $1 \le b \le 6p$, where $6F_0 = f^{-1}(0)$ and $6pF_\infty = f^{-1}(\infty)$. Since $X_0 \times E' \to X$ is an étale morphism, we have a canonical bundle formula

$$K_X = f^*\mathcal{O}_{\mathbf{P}^1}(-1) + 5F_0 + (2p-3)F_\infty.$$

## 2. Special phenomena in positive characteristic

In this section, we investigate special phenomena in positive characteristic.

Theorem 2.1 (Fujita). *$f : M \to C$ be a Kähler fiber space over a curve $C$. Then $f_*\omega_{M/C}$ is locally free and numerically semipositive.*

Let $f : X \to B$ be a 2-abelian fiber space as in Theorem 1.5. By Theorem 2.1, we get $\deg(L^{-1}) = \deg(R^2 f_*\mathcal{O}_X)^\vee = \deg f_*\omega_{X/B} \ge 0$ if $\mathrm{char}(k) = 0$, where $^\vee$ denotes the dual. If $\mathrm{char}(k) > 0$, we have an example such that $\deg(L^{-1}) < 0$.

*Example*. Assume that $\mathrm{char}(k) = p \ge 5$. Let $C$ be a non-singular complete model of the curve defined by the equation

$$t^2 = x^{6p} - 1.$$

Then the group $G = \langle\sigma\rangle \cong \mathbf{Z}/2\mathbf{Z}$ acts on $C$ by

$$\sigma : C \ni (x,t) \mapsto (x,-t) \in C.$$

The canonical morphism

$$\pi_0 : C \to C/\langle\sigma\rangle \cong \mathbf{P}^1$$

has degree 2. Since $(2,p) = 1$, $\pi_0$ is a finite separable morphism of curves. We see that $\pi_0$ is ramified only at 6 points. By the Riemann-Hurwitz Theorem,

$$2g(C) - 2 = 2 \cdot (2g(\mathbf{P}^1) - 2) + 6.$$

Therefore the genus of $C$ is 2. Let $E$ be an ordinary elliptic curve, and $a \in E$ a point of order 2. The group $G$ also acts on $E$ by

$$\sigma : E \ni z \mapsto z + a \in E.$$

Then we have an elliptic fibration

$$f_0 : X_0 = C \times E/G \to C/G \cong \mathbf{P}^1.$$

A canonical bundle formula is given by

$$K_{X_0} = f_0^*\mathcal{O}_{\mathbf{P}^1}(-2) + E_1 + \cdots + E_6,$$

where $2E_i$ $(i = 1, \dots, 6)$ are fibers of some points of $\mathbf{P}^1$. Let $E'$ be a supersingular elliptic curve. Since $\alpha_p = \operatorname{Spec} k[\varepsilon]/(\varepsilon^p) \subset E'$, the group $\alpha_p$ acts on $E'$. The group $\alpha_p$ also acts on $C$ and $E$ by

$$C \ni (x, t) \mapsto (x + \varepsilon, t) \in C,$$
$$E \ni z \mapsto z \in E.$$

Then we have a 2-abelian fiber space

$$f : X = X_0 \times E'/\alpha_p \to \mathbf{P}^1/\alpha_p \cong \mathbf{P}^1.$$

A canonical bundle formula is given by

$$K_X = f^*\mathcal{O}_{\mathbf{P}^1}(-3) + F_1 + \cdots + F_6 + F_\infty$$

with a wild fiber $pF_\infty$ over the point at infinity of $\mathbf{P}^1$, and $F_i$ $(i = 1, \dots, 6)$ are $E_i \times E'$. Therefore, if we write $R^2 f_*\mathcal{O}_X = L \oplus T$ as in Theorem 1.5, we get $\deg(L^{-1}) = -1 < 0$.

Let $f : X \to B$ be a 2-abelian fiber space as in Theorem 1.5. By Theorem 1.5, we have $K_X = f^*(K_B - L) + \sum_{i=1}^r a_i P_i$ $(0 \le a_i \le m_i - 1)$. For $m \in \mathbf{Z}$,

$$|mK_X| = \left| f^*\left\{ m(K_B - L) + \sum_{i=1}^r \left[\frac{ma_i}{m_i}\right] P_i \right\} \right| + \sum_{i=1}^r \left( ma_i - m_i \left[\frac{ma_i}{m_i}\right] \right) P_i,$$

where $\sum_{i=1}^r \left( ma_i - m_i \left[\frac{ma_i}{m_i}\right] \right) P_i$ is the fixed part of $|mK_X|$. We see that $\kappa(X) = \kappa(mK_X) = \kappa\left( m(K_B - L) + \sum_{i=1}^r \left[\frac{ma_i}{m_i}\right] P_i \right)$. Therefore, we have

$$\begin{cases} \kappa(X) = -\infty \Leftrightarrow 2g(B) - 2 + \deg L^{-1} + \sum_{i=1}^r \dfrac{a_i}{m_i} < 0 \\ \kappa(X) = 0 \Leftrightarrow 2g(B) - 2 + \deg L^{-1} + \sum_{i=1}^r \dfrac{a_i}{m_i} = 0 \\ \kappa(X) = 1 \Leftrightarrow 2g(B) - 2 + \deg L^{-1} + \sum_{i=1}^r \dfrac{a_i}{m_i} > 0. \end{cases}$$

Now consider a 2-abelian fiber space $f : X \to B$ with $\kappa(X) = 1$, and the rational map $\Phi_{|mK_X|} : X \to \mathbf{P}^N$ induced by the complete linear system $|mK_X|$.

Proposition 2.2 (see [1]). *Let $f : X \to B$ be a 2-abelian fiber space with $\kappa(X) = 1$ as in Theorem 1.5. Assume $\Phi_{|mK_X|} : X \to \Phi_{|mK_X|}(X) \subset \mathbf{P}^N$ is a morphism. Then $\Phi_{|mK_X|}$ gives the fibration $f : X \to B$.*

*Proof.* Suppose that we have two different fibrations $f : X \to B$ and $f' : X \to B'$. First of all we shall prove that there exists $b' \in B'$ such that $F'_{b'} = f'^{-1}(b')$ satisfies $f(F'_{b'}) = b \in B$. Suppose the contrary, i.e. for any $b' \in B'$, the fiber $F'_{b'}$ projects onto $B$ under $f$. The canonical bundle formula for $X$

with respect to $f$ shows that for a generic hyperplane section $H$ of $X$, we have $(K_X \cdot F'_{b'} \cdot H) > 0$. On the other hand, using the canonical bundle formula for $X$ we have $(K_X \cdot F'_{b'} \cdot H) = 0$ (since $K_X$ consists of fibers of $f'$). A contradiction. This immediately implies that for any point $b' \in B'$ the fiber $F'_{b'}$ projects to a point under $f$. Suppose the cotrary, i.e. let $b', c' \in B'$ be two points of $B'$ such that $f(F'_{c'}) = c \in B$ and $f$ maps $F'_{b'}$ onto $B$. Let $b \in B$ be a point different from $c$. Let $H$ denote a generic hyperplane section of $X$. Note that $F'_{c'} \subset F_c$. Then by our choices $(F_b \cdot F'_{b'} \cdot H) > 0$ and $(F_b \cdot F'_{c'} \cdot H) = 0$. However, $F'_{b'}$ and $F'_{c'}$ are algebraically equivalent, so we arrive at a contradiction. Hence the statement is proved, by using the irreducibility of the pencil. □

Let $f : X \to B$ be a 2-abelian fiber space as in Theorem 1.5. Now we consider the problem: "Find the smallest integer $M$ such that the multicanonical system $|mK_X|$ gives the structure of 2-abelian fiber space $f : X \to B$ with Kodaira dimension $\kappa(X) = 1$ and for any 2-abelian fiber space and any integer $m \geq M$." This problem is equivalent to the problem: "Find the smallest integer $M$ such that

$$M(2g(B) - 2 + \deg L^{-1}) + \sum_{i=1}^{r} \left[\frac{Ma_i}{m_i}\right] \geq 2g(B) + 1$$

under the condition

$$2g(B) - 2 + \deg L^{-1} + \sum_{i=1}^{r} \frac{a_i}{m_i} > 0."$$

Theorem 2.3 (see [1]). *Assume* char$(k) = 0$. *Let $f : X \to B$ be a 2-abelian fiber space with $\kappa(X) = 1$ such that $K_X$ is $f$-nef and the jumping values for all multiple fibers are equal to* 1. *Then the multicanonical system $|mK_X|$ gives the structure of abelian fiber space if $m \geq 14$. Moreover* 14 *is the best possible bound.*

We give here an example which shows Theorem 2.3 does not hold in positive characteristic.

*Example*. Let $f : X \to \mathbf{P}^1$ be as in Example (4) of section 1. We have the canonical bundle formura

$$K_X = f^*\mathcal{O}_{\mathbf{P}^1}(-1) + 5F_0 + (2p - 3)F_\infty,$$

where $6F_0 = f^{-1}(0)$ and $6pF_\infty = f^{-1}(\infty)$. Since $-1 + \frac{5}{6} + \frac{2p-3}{6p} = \frac{p-3}{6p} > 0$, we have $\kappa(X) = 1$. Now assume $p = 7$. Then, putting $m = 14$, we see that the value of $-m + \left[\frac{5m}{6}\right] + \left[\frac{11m}{42}\right]$ is equal to 0. Therefore, this gives an example which shows that Theorem 2.3 does not hold in characteristic $p = 7$.

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