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INVARIANTS OF AMPLE LINE BUNDLES ON PROJECTIVE VARIETIES AND THEIR APPLICATIONS, II*†‡

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Abstract

Let X be a smooth complex projective variety of dimension n and let $L_1, \dots, L_{n-i}$ be ample line bundles on X, where i is an integer with $0 \le i \le n-1$. In the previous paper, we defined the i-th sectional geometric genus $g_i(X, L_1, \dots, L_{n-i})$ of $(X, L_1, \dots, L_{n-i})$. In this part II, we will investigate a lower bound for $g_i(X, L_1, \dots, L_{n-i})$. Moreover we will study the first sectional geometric genus of $(X, L_1, \dots, L_{n-1})$.

Introduction

This is the continuation of [13]. This paper (Part II) consists of section 3, 4, 5 and 6. Let X be a smooth complex projective variety of dimension n and let $L_1, \dots, L_{n-i}$ be ample line bundles on X, where i is an integer with $0 \le i \le n-1$. In [13], we defined the ith sectional geometric genus $g_i(X, L_1, \dots, L_{n-i})$. This invariant is thought to be a generalization of the ith sectional geometric genus $g_i(X, L)$ of polarized varieties (X, L). Furthermore in [13], we showed some fundamental properties of this invariant. In this paper and [14], we will study projective varieties more deeply by using some properties of the ith sectional geometric genus of multi-polarized varieties which have been proved in [13]. In this paper, we will mainly study a lower bound of $g_i(X, L_1, \dots, L_{n-i})$ and some properties of the case where $i = 1$. The content of this paper is the following.

In section 3 we will give some results and definitions which will be used in this paper.

In section 4, we will investigate a lower bound for the ith sectional geometric genus of multi-polarized variety $(X, L_1, \dots, L_{n-i})$. In particular, we will study a relation between $g_i(X, L_1, \dots, L_{n-i})$ and $h^i(\mathcal{O}_X)$.

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* *Key words and phrases.* Polarized varieties, ample line bundles, nef and big line bundles, sectional genus, ith sectional geometric genus, ith sectional H-arithmetic genus, ith sectional arithmetic genus, adjoint bundles.

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In section 5, we will study the nefness of $K_X + L_1 + \cdots + L_t$ for $t \geq n-2$. This investigation will make us possible to study a lower bound for $g_1(X, L_1, \dots, L_{n-1})$ (see section 6) and some properties of $g_2(X, L_1, \dots, L_{n-2})$ (see [14]).

In section 6, we mainly consider the case where $(X, L_1, \dots, L_{n-1})$ is a multi-polarized manifold of type $n-1$ by using results in section 5, and we will make a study of the following:

(1) The non-negativity of $g_1(X, L_1, \dots, L_{n-1})$.
(2) A classification of $(X, L_1, \dots, L_{n-1})$ with $g_1(X, L_1, \dots, L_{n-1}) \leq 1$.
(3) Under the assumption that $|L_j|$ is base point free for any j with $1 \leq j \leq n-1$, we will prove that $g_1(X, L_1, \dots, L_{n-1}) \geq h^1(\mathcal{O}_X)$. Moreover we will classify $(X, L_1, \dots, L_{n-1})$ with $g_1(X, L_1, \dots, L_{n-1}) = h^1(\mathcal{O}_X)$.
(4) Assume that $n = 3$, $h^0(L_1) \geq 2$ and $h^0(L_2) \geq 1$. Then we will prove $g_1(X, L_1, L_2) \geq h^1(\mathcal{O}_X)$. Furthermore we will classify multi-polarized 3-folds (X, L_1, L_2) with $g_1(X, L_1, L_2) = h^1(\mathcal{O}_X)$, $h^0(L_1) \geq 2$ and $h^0(L_2) \geq 3$.

In this paper we use the same notation as in [13].

3. Preliminaries for the second part

NOTATION 3.1. Let X be a projective variety of dimension n, let i be an integer with $0 \leq i \leq n-1$, and let $L_1, \dots, L_{n-i}$ be line bundles on X. Then $\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}})$ is a polynomial in $t_1, \dots, t_{n-i}$ of total degree at most n. So we can write $\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}})$ uniquely as follows.

$$\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}}) = \sum_{p=0}^{n} \sum_{\substack{p_1 \geq 0, \dots, p_{n-i} \geq 0 \\ p_1 + \cdots + p_{n-i} = p}} \chi_{p_1, \dots, p_{n-i}}(L_1, \dots, L_{n-i}) \binom{t_1 + p_1 - 1}{p_1} \cdots \binom{t_{n-i} + p_{n-i} - 1}{p_{n-i}}.$$

DEFINITION 3.1 ([13, Definition 2.1]). Let X be a projective variety of dimension n, let i be an integer with $0 \leq i \leq n$, and let $L_1, \dots, L_{n-i}$ be line bundles on X.

(1) The *ith sectional H-arithmetic genus* $\chi_i^H(X, L_1, \dots, L_{n-i})$ is defined by the following:

$$\chi_i^H(X, L_1, \dots, L_{n-i}) = \begin{cases} \chi_{\underbrace{1, \dots, 1}_{n-i}}(L_1, \dots, L_{n-i}) & \text{if } 0 \leq i \leq n-1, \\ \chi(\mathcal{O}_X) & \text{if } i = n. \end{cases}$$

(2) The *ith sectional geometric genus* $g_i(X, L_1, \dots, L_{n-i})$ is defined by the following:

$$g_i(X, L_1, \dots, L_{n-i}) = (-1)^i (\chi_i^H(X, L_1, \dots, L_{n-i}) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

(3) The *ith sectional arithmetic genus* $p_a^i(X, L_1, \dots, L_{n-i})$ is defined by the following:

$$p_a^i(X, L_1, \dots, L_{n-i}) = (-1)^i(\chi_i^H(X, L_1, \dots, L_{n-i}) - h^0(\mathcal{O}_X)).$$

Remark 3.1. Let X be a *smooth* projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X with $1 \le r \le n$. Then in [10, Definition 2.1], we defined the *ith c_r-sectional geometric genus $g_i(X, \mathcal{E})$ of $(X, \mathcal{E})$* for every integer i with $0 \le i \le n-r$. Let i be an integer with $0 \le i \le n-1$. Here we note that if $i = 1$, then $g_1(X, \mathcal{E})$ is the genus defined in [15, Definition 1.1], and moreover if $r = n-1$, then $g_1(X, \mathcal{E})$ is the curve genus $g(X, \mathcal{E})$ of $(X, \mathcal{E})$ which was defined in [1] and has been studied by many authors (see [22], [23] and so on). Let $L_1, \dots, L_{n-i}$ be ample line bundles on X. By setting $\mathcal{E} := L_1 \oplus \cdots \oplus L_{n-i}$, we see that $g_i(X, \mathcal{E}) = g_i(X, L_1, \dots, L_{n-i})$. In particular if $i = 1$, then $g_1(X, L_1, \dots, L_{n-1})$ is equal to the curve genus of $(X, \mathcal{E})$.

Definition 3.2. Let X and Y be smooth projective varieties with $\dim X > \dim Y \ge 1$. Then a morphism $f : X \to Y$ is called a *fiber space* if f is surjective with connected fibers. Let L be a Cartier divisor on X. Then (f, X, Y, L) is called a *polarized* (resp. *quasi-polarized*) fiber space if $f : X \to Y$ is a fiber space and L is ample (resp. nef and big).

Definition 3.3. Let $(X, L_1, \dots, L_k)$ be an n-dimensional polarized manifold of type k, where k is a positive integer. Then $(X, L_1, \dots, L_k)$ is called a *scroll* (resp. *quadric fibration*, *Del Pezzo fibration*) *over a normal variety W* if there exists a fiber space $f : X \to W$ such that $\dim W = n-k+1$ (resp. $n-k$, $n-k-1$) and $K_X + L_1 + \cdots + L_k = f^*(A)$ for an ample line bundle A on W. We say that a polarized manifold (X, L) is a *scroll* (resp. *quadric fibration*, *Del Pezzo fibration*) *over a normal variety Y with* $\dim Y = m$ if there exists a fiber space $f : X \to Y$ such that $K_X + (n-m+1)L = f^*(A)$ (resp. $K_X + (n-m)L = f^*(A)$, $K_X + (n-m-1)L = f^*(A)$) for an ample line bundle A on Y.

Theorem 3.1. *Let (X, L) be a polarized manifold with $n = \dim X \ge 3$. Then (X, L) is one of the following types*:

(1) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$.
(2) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1))$.
(3) *A scroll over a smooth curve.*
(4) $K_X \sim -(n-1)L$, *that is, (X, L) is a Del Pezzo manifold.*
(5) *A quadric fibration over a smooth curve.*
(6) *A scroll over a smooth surface.*
(7) *Let (X', L') be a reduction of (X, L).*
  (7-1) $n = 4$, $(X', L') = (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2))$.
  (7-2) $n = 3$, $(X', L') = (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$.
  (7-3) $n = 3$, $(X', L') = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$.

(7-4) $n = 3$, *X' is a $\mathbf{P}^2$-bundle over a smooth curve and* $(F', L'|_{F'})$ *is isomorphic to* $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ *for any fiber F' of it.*
(7-5) $K_{X'} + (n-2)L'$ *is nef.*

Proof. See [2, Proposition 7.2.2, Theorem 7.2.4, Theorem 7.3.2 and Theorem 7.3.4]. □

Notation 3.2. Let X be a projective manifold of dimension n.

- $\equiv$ denotes the numerical equivalence.
- $Z_{n-1}(X)$: the group of Weil divisors.
- $N_1(X) := (\{1\text{-cycles}\}/\equiv) \otimes \mathbf{R}$.
- $NE(X)$: the convex cone in $N_1(X)$ generated by the effective 1-cycles.
- $\overline{NE}(X)$: the closure of $NE(X)$ in $N_1(X)$ with respect to the Euclidean topology.
- $\rho(X) := \dim_{\mathbf{R}} N_1(X)$.
- If C is a 1-dimensional cycle in X, then we denote $[C]$ its class in $N_1(X)$.
- Let D be an effective divisor on X and $D = \sum_i a_i D_i$ its prime decomposition, where $a_i \geq 1$ for any i. Then we write $D_{\text{red}} = \sum_i D_i$.
- $\mathfrak{S}_l$ denotes the symmetric group of order l.

Definition 3.4 ([27, (1.9)]). Let X be a projective manifold of dimension n and let R be an extremal ray. Then the *length* $l(R)$ is defined by the following:

$$l(R) = \min\{-K_X C \mid C \text{ is a rational curve with } [C] \in R\}.$$

Remark 3.2. By the cone theorem (see [24, Theorem (1.4)], [18] and [20]), $l(R) \leq n+1$ holds.

Proposition 3.1. *Let X be a projective manifold of dimension n.*
(1) *If there exists an extremal ray R with $l(R) = n+1$, then* Pic $X \cong \mathbf{Z}$ *and $-K_X$ is ample.*
(2) *If there exists an extremal ray R with $l(R) = n$, then* Pic $X \cong \mathbf{Z}$ *and $-K_X$ is ample, or $\rho(X) = 2$ and there exists a morphism $cont_R : X \to B$ onto a smooth curve B whose general fiber is a smooth $(n-1)$-manifold that satisfies conditions of* (1).

Proof. See [27, Proposition 2.4]. □

Lemma 3.1. *Let (f, X, Y, L) be a quasi-polarized fiber space, where X is a normal projective variety with only $\mathbf{Q}$-factorial canonical singularities and Y is a smooth projective variety with* $\dim X = n > \dim Y \geq 1$. *Assume that $K_{X/Y} + tL$ is f-nef, where t is a positive integer. Then $(K_{X/Y} + tL)L^{n-1} \geq 0$. Moreover if* $\dim Y = 1$, *then $K_{X/Y} + tL$ is nef.*

Proof. For any ample Cartier divisor A on X and any natural number p, $K_{X/Y} + tL + (1/p)A$ is f-nef by assumption. Let m be a natural number such that $m(K_{X/Y}+tL+(1/p)A)$ is a Cartier divisor. Since $m(K_{X/Y}+tL+(1/p)A)-K_X$ is f-ample, by the base point free theorem ([19, Theorem 3-1-1]),

$$f^*f_*\mathcal{O}_X\left(lm\left(K_{X/Y}+tL+\frac{1}{p}A\right)\right)\to\mathcal{O}_X\left(lm\left(K_{X/Y}+tL+\frac{1}{p}A\right)\right)$$

is surjective for any $l\gg 0$.

Let $\mu: X_1\to X$ be a resolution of X. We put $h=f\circ\mu$. Since

$$\mu^*f^*f_*\mathcal{O}_X\left(lm\left(K_{X/Y}+tL+\frac{1}{p}A\right)\right)=h^*h_*\mathcal{O}_{X_1}\left(lm\left(K_{X_1/Y}+\mu^*\left(tL+\frac{1}{p}A\right)\right)\right),$$

we have

$$(1)\quad h^*h_*\mathcal{O}_{X_1}\left(lm\left(K_{X_1/Y}+\mu^*\left(tL+\frac{1}{p}A\right)\right)\right)\to\mu^*\mathcal{O}_X\left(lm\left(K_{X/Y}+tL+\frac{1}{p}A\right)\right)$$

is surjective. We note that $h_*\mathcal{O}_{X_1}(lm(K_{X_1/Y}+\mu^*(tL+(1/p)A)))$ is weakly positive by [8, Theorem A' in Page 358] because $\mu^*\mathcal{O}_X(lm(tL+(1/p)A))$ is semiample. (For the definition of weak positivity, see [26].) Hence by [8, Remark 1.3.2 (1)] and (1) above $\mu^*\mathcal{O}_X(lm(K_{X/Y}+tL+(1/p)A))$ is pseudo-effective. Since p is any natural number, we get $(K_{X/Y}+tL)L^{n-1}=\mu^*(K_{X/Y}+tL)(\mu^*L)^{n-1}\geq 0$.

If $\dim Y=1$, then we see that $h_*\mathcal{O}_{X_1}(lm(K_{X_1/Y}+\mu^*(tL+(1/p)A)))$ is semipositive by [8, Theorem A' in page 358] since semi-positivity and weak positivity are equivalent for torsion free sheaves on nonsingular curves. Hence by (1) above $K_{X/Y}+tL+(1/p)A$ is nef for any natural number p. Since p is any natural number, $K_{X/Y}+tL$ is nef. □

LEMMA 3.2. *Let X and Y be smooth projective varieties with* $\dim X>\dim Y\geq 1$ *and let $f: X\to Y$ be a surjective morphism with connected fibers. Then $q(X)\leq q(F)+q(Y)$, where F is a general fiber of f.*

Proof. See [8, Theorem B in Appendix] or [3, Theorem 1.6]. □

LEMMA 3.3. *Let X be a smooth projective variety, and let D_1 and D_2 be effective divisors on X. Then $h^0(D_1+D_2)\geq h^0(D_1)+h^0(D_2)-1$.*

Proof. See [11, Lemma 1.12] or [21, 15.6.2 Lemma]. □

NOTATION 3.3. Let X be a smooth projective variety of dimension n and let i be an integer with $1\leq i\leq n-1$. Let $L_1,\dots,L_{n-i}$ be nef and big line bundles on X. Assume that $\mathrm{Bs}|L_j|=\emptyset$ for every integer j with $1\leq j\leq n-i$. Then by Bertini's theorem, for every integer j with $1\leq j\leq n-i$, there exists a general member $X_j\in|L_j|_{X_{j-1}}|$ such that X_j is a smooth projective variety of dimension $n-j$. (Here we set $X_0:=X$.) Namely there exists an $(n-i)$-ladder $X\supset X_1\supset\cdots\supset X_{n-i}$ such that a projective variety X_j is smooth with $\dim X_j=n-j$.

4. Properties of the sectional geometric genus

In this section we study the relationship between $g_i(X, L_1, \dots, L_{n-i})$ and $h^i(\mathcal{O}_X)$.

Lemma 4.1. *Let X be a projective variety of dimension n, and let s be an integer with $0 \le s \le n-1$. Let $L_1, \dots, L_s$ be Cartier divisors on X. Assume the following conditions*:

(a) *There exists an irreducible and reduced divisor $X_{k+1} \in |L_{k+1}|_{X_k}|$ for any integer k with $0 \le k \le s-2$. (Here we put $X_0 := X$.)*

(b) *$h^j(-\sum_{m=1}^{s} t_m L_m) = 0$ for any integer j and t_m with $0 \le j \le n-1$, $t_m \ge 0$ for any m, and $\sum_{m=1}^{s} t_m > 0$.*

(c) *$h^0(L_s|_{X_{s-1}}) > 0$ and there exists a member $X_s \in |L_s|_{X_{s-1}}|$.*

Then

(1) *$h^j(-\sum_{m=k+1}^{s} u_m L_m|_{X_k}) = 0$ for any integer k, j and u_m with $1 \le k \le s-1$, $0 \le j \le n-k-1$, $u_m \ge 0$ for any m, and $\sum_{m=k+1}^{s} u_m > 0$.*

(2) *$h^j(\mathcal{O}_X) = h^j(\mathcal{O}_{X_1}) = \cdots = h^j(\mathcal{O}_{X_{s-1}})$ for any integer j with $0 \le j \le n-s$.*

(3) *$h^{n-s}(\mathcal{O}_{X_{s-1}}) \le h^{n-s}(\mathcal{O}_{X_s})$.*

Proof. (1) First we study the case where $k = 1$. By the above (b) and the exact sequence

$$0 \to \mathcal{O}_X\left(-L_1 - \sum_{m=2}^{s} u_m L_m\right) \to \mathcal{O}_X\left(-\sum_{m=2}^{s} u_m L_m\right) \to \mathcal{O}_{X_1}\left(-\sum_{m=2}^{s} u_m L_m|_{X_1}\right) \to 0,$$

we have $h^j(-\sum_{m=2}^{s} u_m L_m|_{X_1}) = 0$ for any integer j and u_m with $0 \le j \le n-2$, $u_m \ge 0$ for any m, and $\sum_{m=2}^{s} u_m > 0$.

Assume that (1) is true for any integer k with $k \le l-1$, where l is an integer with $2 \le l \le s-1$. We consider the case where $k = l$. By the exact sequence

$$0 \to \mathcal{O}_{X_{l-1}}\left(-L_l|_{X_{l-1}} - \sum_{m=l+1}^{s} u_m L_m|_{X_{l-1}}\right) \to \mathcal{O}_{X_{l-1}}\left(-\sum_{m=l+1}^{s} u_m L_m|_{X_{l-1}}\right)$$
$$\to \mathcal{O}_{X_l}\left(-\sum_{m=l+1}^{s} u_m L_m|_{X_l}\right) \to 0,$$

we have $h^j(-\sum_{m=l+1}^{s} u_m L_m|_{X_l}) = 0$ for any integer j and u_m with $0 \le j \le n-l-1$, $u_m \ge 0$ for any m, and $\sum_{m=l+1}^{s} u_m > 0$. Hence we get the assertion.

Next we prove (2) and (3). By (1) above, we obtain $h^j(-L_{k+1}|_{X_k}) = 0$ for any integer j and k with $0 \le k \le s-1$ and $0 \le j \le n-k-1$. Hence by the exact sequence

$$0 \to \mathcal{O}(-L_{k+1}|_{X_k}) \to \mathcal{O}_{X_k} \to \mathcal{O}_{X_{k+1}} \to 0,$$

we get the assertion. $\square$

LEMMA 4.2. *Let X be a projective variety of dimension n, and let L be a Cartier divisor on X. Assume that $h^0(L) > 0$ and $h^{n-1}(-L) = 0$. Then $g_{n-1}(X, L) = h^{n-1}(\mathcal{O}_{X_1})$, where $X_1 \in |L|$.*

Proof. We consider the exact sequence

$$0 \to -L \to \mathcal{O}_X \to \mathcal{O}_{X_1} \to 0.$$

Then

$$\begin{aligned} H^{n-1}(-L) \to H^{n-1}(\mathcal{O}_X) \to H^{n-1}(\mathcal{O}_{X_1}) \\ \to H^n(-L) \to H^n(\mathcal{O}_X) \to 0 \end{aligned}$$

is exact. Since $h^{n-1}(-L) = 0$, we see that $h^n(-L) - h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_X) = h^{n-1}(\mathcal{O}_{X_1})$. By [11, Definition 2.1 and Theorem 2.2] or [13, Corollary 2.2], we get

$$\begin{aligned} g_{n-1}(X, L) &= h^n(-L) - h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_X) \\ &= h^{n-1}(\mathcal{O}_{X_1}). \end{aligned}$$

Hence we get the assertion. □

THEOREM 4.1. *Let X be a projective variety of dimension n, and let i be an integer with $0 \le i \le n-1$. Let $L_1, \dots, L_{n-i}$ be Cartier divisors on X. Assume the following conditions*:

(a) *There exists an irreducible and reduced divisor $X_{k+1} \in |L_{k+1}|_{X_k}|$ for any integer k with $0 \le k \le n-i-2$. (Here we put $X_0 := X$.)*
(b) *$h^j(-\sum_{m=1}^{n-i} t_m L_m) = 0$ for any integer j and t_m with $0 \le j \le n-1$, $t_m \ge 0$ for any m, and $\sum_{m=1}^{n-i} t_m > 0$.*
(c) *$h^0(L_{n-i}|_{X_{n-i-1}}) > 0$ and there exists a member $X_{n-i} \in |L_{n-i}|_{X_{n-i-1}}|$.*

Then

$$g_i(X, L_1, \dots, L_{n-i}) \ge h^i(\mathcal{O}_X).$$

Proof. By Lemma 4.1 (2), we have $h^j(\mathcal{O}_X) = h^j(\mathcal{O}_{X_{n-i-1}})$ for every j with $0 \le j \le i$. Therefore

$$\begin{aligned} &(-1)^i \chi(\mathcal{O}_X) - \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X) \\ &= (-1)^i \chi(\mathcal{O}_{X_{n-i-1}}) - \sum_{j=0}^{1} (-1)^{1-j} h^{i+1-j}(\mathcal{O}_{X_{n-i-1}}). \end{aligned}$$

By [13, Lemma 2.4] we also get

$$\begin{aligned} \chi_{1,\dots,1}(L_1, \dots, L_{n-i}) &= \chi_{1,\dots,1}(L_2|_{X_1}, \dots, L_{n-i}|_{X_1}) \\ &= \cdots \\ &= \chi_1(L_{n-i}|_{X_{n-i-1}}). \end{aligned}$$

Hence by [11, Definition 2.1] and Definition 3.1 (2) we have

$$g_i(X, L_1, \ldots, L_{n-i}) = g_i(X_{n-i-1}, L_{n-i}|_{X_{n-i-1}}).$$

Here we note that by Lemma 4.1 (1) we have $h^j(-L_{n-i}|_{X_{n-i-1}}) = 0$ for any integer j with $0 \le j \le i$. By Lemma 4.2 we see that $g_i(X_{n-i-1}, L_{n-i}|_{X_{n-i-1}}) = h^i(\mathcal{O}_{X_{n-i}})$. Hence by Lemma 4.1 (2) and (3) we get

$$\begin{aligned} g_i(X, L_1, \ldots, L_{n-i}) &= g_i(X_{n-i-1}, L_{n-i}|_{X_{n-i-1}}) \\ &= h^i(\mathcal{O}_{X_{n-i}}) \\ &\ge h^i(\mathcal{O}_X). \end{aligned}$$

Hence we obtain the assertion. □

LEMMA 4.3. *Let X be a projective variety of dimension n, and let i be an integer with $0 \le i \le n-1$. Let $L_1, \ldots, L_{n-i}$ be Cartier divisors on X. Then the following are equivalent*: (*Here $\chi^i(\mathcal{O}_X) := \sum_{j=0}^{i}(-1)^j h^j(\mathcal{O}_X)$.*)

(a) $g_i(X, L_1, \ldots, L_{n-i}) \ge h^i(\mathcal{O}_X)$.
(b) $(-1)^i \chi_i^H(X, L_1, \ldots, L_{n-i}) \ge (-1)^i \chi^i(\mathcal{O}_X)$.
(c) $p_a^i(X, L_1, \ldots, L_{n-i}) \ge (-1)^i(\chi^i(\mathcal{O}_X) - 1)$.

Proof. By definition, we get

$$\begin{aligned} g_i(X, L_1, \ldots, L_{n-i}) - h^i(\mathcal{O}_X) &= (-1)^i(\chi_{1,\ldots,1}(L_1, \ldots, L_{n-i}) - \chi(\mathcal{O}_X)) \\ &\quad + \sum_{j=0}^{n-i-1} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X) \\ &= (-1)^i(\chi_i^H(X, L_1, \ldots, L_{n-i}) - \chi(\mathcal{O}_X)) \\ &\quad + \sum_{j=0}^{n-i-1} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X) \\ &= (-1)^i \chi_i^H(X, L_1, \ldots, L_{n-i}) - (-1)^i \chi^i(\mathcal{O}_X), \end{aligned}$$

and

$$\begin{aligned} &p_a^i(X, L_1, \ldots, L_{n-i}) - (-1)^i(\chi^i(\mathcal{O}_X) - 1) \\ &\quad = (-1)^i(\chi_i^H(X, L_1, \ldots, L_{n-i}) - 1) - (-1)^i(\chi^i(\mathcal{O}_X) - 1) \\ &\quad = (-1)^i \chi_i^H(X, L_1, \ldots, L_{n-i}) - (-1)^i \chi^i(\mathcal{O}_X). \end{aligned}$$

Hence we get the assertion. □

COROLLARY 4.1. *Let X be a projective variety of dimension n, and let i be an integer with $0 \le i \le n-1$. Let $L_1, \ldots, L_{n-i}$ be Cartier divisors on X. Assume the following conditions*:

(a) *There exists an irreducible and reduced divisor* $X_{k+1} \in |L_{k+1}|_{X_k}|$ *for any integer* k *with* $0 \leq k \leq n-i-1$. (*Here we put* $X_0 := X$.)

(b) $h^j(-\sum_{m=1}^{n-i} t_m L_m) = 0$ *for any integer* j *and* t_m *with* $0 \leq j \leq n-1$, $t_m \geq 0$ *for any* m, *and* $\sum_{m=1}^{n-i} t_m > 0$.

(c) $h^0(L_{n-i}|_{X_{n-i-1}}) > 0$ *and there exists a member* $X_{n-i} \in |L_{n-i}|_{X_{n-i-1}}|$.

Then we get the following: (*Here* $\chi^i(\mathcal{O}_X) := \sum_{j=0}^{i}(-1)^j h^j(\mathcal{O}_X)$.)

(1) $(-1)^i \chi_i^H(X, L_1, \ldots, L_{n-i}) \geq (-1)^i \chi^i(\mathcal{O}_X)$.

(2) $p_a^i(X, L_1, \ldots, L_{n-i}) \geq (-1)^i(\chi^i(\mathcal{O}_X) - 1)$.

Proof. By Lemma 4.3 and Theorem 4.1, we get the assertion. □

If X is normal, then we get the following.

Corollary 4.2. *Let* X *be a normal projective variety of dimension* $n \geq 3$. *Let* i *be an integer with* $0 \leq i \leq n-1$. *Let* $L_1, L_2, \ldots, L_{n-i}$ *be ample line bundles on* X *such that* $\mathrm{Bs}|L_j| = \emptyset$ *for every integer* j *with* $1 \leq j \leq n-i$. *Assume that* $h^j(-\sum_{k=1}^{n-i} t_k L_k) = 0$ *for any integer* j *and* t_k *with* $0 \leq j \leq n-1$, $t_k \geq 0$ *for any* k, *and* $\sum_{k=1}^{n-i} t_k > 0$. *Then*

$$g_i(X, L_1, \ldots, L_{n-i}) \geq h^i(\mathcal{O}_X).$$

Proof. If $i = n-1$, then by [12, Corollary 2.9] we get $g_{n-1}(X, L_1) \geq h^{n-1}(\mathcal{O}_X)$.

If $i = 0$, then $g_0(X, L_1, \ldots, L_n) = L_1 \cdots L_n \geq 1 = h^0(\mathcal{O}_X)$.

So we may assume that $1 \leq i \leq n-2$. For every integer k with $1 \leq k \leq n-i-1$, let $X_k \in |L_k|_{X_{k-1}}|$ be a general member. Then since $\mathrm{Bs}|L_k|_{X_{k-1}}| = \emptyset$, we see that X_k is a normal projective variety (for example, see [6, (0.2.9) Fact and (4.3) Theorem] or [2, Theorem 1.7.1]). Since L_{n-i} is ample with $\mathrm{Bs}|L_{n-i}| = \emptyset$, we have $h^0(L_{n-i}|_{X_{n-i-1}}) > 0$ and $|L_{n-i}|_{X_{n-i-1}}| \neq \emptyset$. Hence by Theorem 4.1, we get the assertion. □

Here we propose the following conjecture, which is a multi-polarized version on [11, Conjecture 4.1].

Conjecture 4.1. *Let* n *and* i *be integers with* $n \geq 2$ *and* $0 \leq i \leq n-1$. *Let* $(X, L_1, \ldots, L_{n-i})$ *be an* n*-dimensional multi-polarized manifold of type* $(n-i)$. *Then* $g_i(X, L_1, \ldots, L_{n-i}) \geq h^i(\mathcal{O}_X)$ *holds.*

Proposition 4.1. *Let* X *be a normal projective variety of dimension* $n \geq 2$. *Let* i *be an integer with* $0 \leq i \leq n-1$. *Let* $L_1, \ldots, L_{n-i-1}, A, B$ *be ample Cartier divisors on* X. *Assume that* $h^j(-(\sum_{p=1}^{n-i-1} t_p L_p) - aA - bB) = 0$ *for any integers* j, a, b *and* t_p *with* $0 \leq j \leq n-1$, $a \geq 0$, $b \geq 0$, $t_p \geq 0$, *and* $a + b + \sum t_p > 0$, *and that* $\mathrm{Bs}|L_j| = \emptyset$ *for* $1 \leq j \leq n-i-1$, $\mathrm{Bs}|A| = \emptyset$, *and* $\mathrm{Bs}|B| = \emptyset$. *Then*

$$g_i(X, A+B, L_1, \ldots, L_{n-i-1}) \geq g_i(X, A, L_1, \ldots, L_{n-i-1}) + g_i(X, B, L_1, \ldots, L_{n-i-1}).$$

Proof. We note that by [13, Corollary 2.4]

$$g_i(X, A+B, L_1, \ldots, L_{n-i-1}) = g_i(X, A, L_1, \ldots, L_{n-i-1}) + g_i(X, B, L_1, \ldots, L_{n-i-1}) + g_{i-1}(X, A, B, L_1, \ldots, L_{n-i-1}) - h^{i-1}(\mathcal{O}_X).$$

By assumption and Corollary 4.2 we have

$$g_{i-1}(X, A, B, L_1, \ldots, L_{n-i-1}) \geq h^{i-1}(\mathcal{O}_X).$$

Hence we get the assertion. □

Remark 4.1. If $i = 1$, then by [13, Corollary 2.4] for any ample Cartier divisors $A, B, L_1, \ldots, L_{n-2}$ we have

$$g_1(X, A+B, L_1, \ldots, L_{n-2}) \geq g_1(X, A, L_1, \ldots, L_{n-2}) + g_1(X, B, L_1, \ldots, L_{n-2})$$

because $ABL_1 \cdots L_{n-2} \geq 1 = h^0(\mathcal{O}_X)$.

5. Adjunction theory of multi-polarized manifolds

In this section, we are going to investigate the nefness of $K_X + L_1 + \cdots + L_k$. Results in this section will be used when we study the ith sectional geometric genus of multi-polarized manifolds in this paper and the Part III [14].

5.1. The nefness of $K_X + L_1 + \cdots + L_t$ for $t \geq n-1$

By putting $\mathcal{E} := L_1 \oplus \cdots \oplus L_l$ for $l = n+1, n, n-1$, we can get the following theorem by using a result of Ye and Zhang [28, Theorems 1, 2 and 3]. Here $\mathfrak{S}_l$ denotes the symmetric group of order l (see Notation 3.2).

Theorem 5.1.1. (1) *Let* $(X, L_1, \ldots, L_{n+1})$ *be an n-dimensional multi-polarized manifold of type* $n+1$ *with* $n \geq 3$. *Then* $K_X + L_1 + \cdots + L_{n+1}$ *is nef.*

(2) *Let* $(X, L_1, \ldots, L_n)$ *be an n-dimensional multi-polarized manifold of type n with* $n \geq 3$. *Then* $K_X + L_1 + \cdots + L_n$ *is nef unless*

$$(X, L_1, \ldots, L_n) \cong (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \ldots, \mathcal{O}_{\mathbf{P}^n}(1)).$$

(3) *Let X be a smooth projective variety of dimension* $n \geq 3$. *Let* $L_1, L_2, \ldots, L_{n-1}$ *be ample line bundles on X. If* $K_X + L_1 + L_2 + \cdots + L_{n-1}$ *is not nef, then there exists* $\sigma \in \mathfrak{S}_{n-1}$ *such that* $(X, L_{\sigma(1)}, L_{\sigma(2)}, \ldots, L_{\sigma(n-1)})$ *is one of the following*:
(A) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n}(1), \ldots, \mathcal{O}_{\mathbf{P}^n}(1))$.
(B) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(1), \ldots, \mathcal{O}_{\mathbf{P}^n}(1))$.
(C) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1), \mathcal{O}_{\mathbf{Q}^n}(1), \ldots, \mathcal{O}_{\mathbf{Q}^n}(1))$.
(D) *X is a* $\mathbf{P}^{n-1}$*-bundle over a smooth projective curve B and* $L_j|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ *for any fiber F and every integer j with* $1 \leq j \leq n-1$.

5.2. The nefness of $K_X + L_1 + \cdots + L_{n-2}$

THEOREM 5.2.1. *Let X be a smooth projective variety of dimension $n \geq 4$ and let $L_1, \ldots, L_{n-2}$ be ample line bundles on X. Assume the following*:

(a) $K_X + L_1 + \cdots + L_{n-2}$ *is not nef.*

(b) $K_X + (n-1)L_j$ *is nef for every integer j with $1 \leq j \leq n-2$.*

Then $(X, L_1, \ldots, L_{n-2})$ is one of the following.

(1) *There exists a multi-polarized manifold $(Y, A_1, \ldots, A_{n-2})$ of type $(n-2)$ such that $(Y, A_1, \ldots, A_{n-2})$ is a reduction of $(X, L_1, \ldots, L_{n-2})$ (see* [13, Definition 1.5]) *and $K_Y + (n-1)A_j$ is ample for every integer j.*

(2) $K_X + (n-1)L_j = \mathcal{O}_X$ *for every j with $1 \leq j \leq n-2$. Moreover $L_j = L_k$ for every pair (j,k) with $j \neq k$.*

(3) $n = 4$ *and* $(X, L_1, L_2) \cong (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2), \mathcal{O}_{\mathbf{P}^4}(2))$.

(4) *There exist a smooth projective curve W and a surjective morphism $f : X \to W$ with connected fibers such that (X, L_i) is a quadric fibration over W with respect to f for every integer i with $1 \leq i \leq n-2$.*

(5) *There exist a smooth projective surface S and a surjective morphism $f : X \to S$ with connected fibers such that f is a $\mathbf{P}^{n-2}$-bundle over S and (X, L_j) is a scroll over S with respect to f for every integer j with $1 \leq j \leq n-2$, where F is its fiber.*

Proof. By assumption, there exists an extremal ray R such that $(K_X + L_1 + \cdots + L_{n-2})R < 0$. Here we may assume that $L_1R \leq L_2R \leq \cdots \leq L_{n-2}R$. Then $(K_X + (n-2)L_1)R \leq (K_X + L_1 + \cdots + L_{n-2})R < 0$ and $K_X + (n-2)L_1$ is not nef. There exists a rational curve C with $[C] \in R$ such that $0 < -K_XC \leq n+1$, and

$$(5.2.1.a) \qquad 0 > (K_X + L_1 + \cdots + L_{n-2})C \geq (K_XC) + (n-2).$$

So we get $-K_XC \geq n-1$.

(A) The case where there exists an extremal rational curve C such that $K_XC = -n-1$.

In this case $0 > (K_X + (n-2)L_1)C = -n-1 + (n-2)L_1C$.

(A.1) Assume that $L_1C \geq 2$. Then $-n-1+2n-4 \leq -n-1+(n-2)L_1C < 0$. In particular $n = 4$ by assumption.

By Proposition 3.1 (1), we get $\operatorname{Pic}(X) \cong \mathbf{Z}$ in this case. Since $K_X + (n-1)L_1 = K_X + 3L_1$ is nef by assumption and $K_X + (n-2)L_1 = K_X + 2L_1$ is not nef, we get $L_1C = 2$ and $L_1 = \mathcal{O}(1)$ or $\mathcal{O}(2)$, where $\mathcal{O}(1)$ is the ample generator of $\operatorname{Pic}(X)$.

If $L_1 = \mathcal{O}(1)$, then $\mathcal{O}(1)C = 2$ and K_XC is even because $\operatorname{Pic}(X) \cong \mathbf{Z}$ and $\mathcal{O}(1)$ is the ample generator of $\operatorname{Pic}(X)$. But then $K_XC = -n-1 = -5$ and this is impossible. Hence $L_1 = \mathcal{O}(2)$ and $\mathcal{O}(1)C = 1$. Therefore $K_X = \mathcal{O}(-n-1) = \mathcal{O}(-5)$. We set $L_2 := \mathcal{O}(a_2)$. Since $K_X + L_1 + L_2 = \mathcal{O}(a_2 - 3)$ is not nef, we obtain $a_2 \leq 2$. By assumption, $K_X + 3L_2 = \mathcal{O}(3a_2 - 5)$ is nef. Hence $a_2 \geq 2$. Therefore $a_2 = 2$. Since $-(K_X + 4\mathcal{O}(1))$ is ample, by Kobayashi-Ochiai's theorem (see [6, (1.3) Corollary]), we have $X \cong \mathbf{P}^4$. Therefore we get the type (3).

(A.2) Assume that $L_1 C = 1$. Then $(K_X + (n-1)L_1)C = -2 < 0$. But this contradicts the assumption.

(B) The case where there exists an extremal rational curve C such that $K_X C = -n$.

In this case, $0 > (K_X + (n-2)L_1)C = -n + (n-2)L_1 C$.

(B.1) If $L_1 C \geq 2$, then $0 > (K_X + (n-2)L_1)C \geq -n + (n-2)2 = n-4 \geq 0$ and this is impossible.

(B.2) If $L_1 C = 1$, then $(K_X + (n-1)L_1)C = -n + (n-1) = -1 < 0$ and this is a contradiction.

(C) The case where $(X, L_1, \dots, L_{n-2})$ satisfies neither the case (A) nor the case (B) above.

We set $H := L_1 + \cdots + L_{n-2}$. In this case by (5.2.1.a) for every extremal rational curve B, $K_X B = -n+1$ and $L_i B = 1$ for every integer i with $1 \leq i \leq n-2$. In particular $HB = (n-2)L_i B$ for every i. Let τ_H (resp. τ_i) be the nef value of (X, H) (resp. (X, L_i)).

Claim 5.2.1. *$\tau_H = (n-1)/(n-2)$ and $\tau_i = n-1$ for every integer i with $1 \leq i \leq n-2$.*

Proof. Assume that there exists $C \in \overline{\mathrm{NE}}(X)$ such that

$$\left(K_X + \frac{n-1}{n-2}H\right)C < 0.$$

Then by the cone theorem (see also [2, Remark 4.2.6]) C can be written as $\sum_j \lambda_j C_j + \gamma$, where C_j is an extremal rational curve and γ is a 1-cycle such that the following holds:

$$\left(K_X + \frac{n-1}{n-2}H\right)\gamma = 0.$$

Hence

$$\left(K_X + \frac{n-1}{n-2}H\right)C_j < 0$$

for some j. But this is impossible because $K_X B = -n+1$ and $L_i B = 1$ for any extremal rational curve B. Therefore $\tau_H \leq (n-1)/(n-2)$. Furthermore $(K_X + aH)B < 0$ for every rational number $a < (n-1)/(n-2)$ and every extremal rational curve B. Therefore $\tau_H = (n-1)/(n-2)$. By the same arguement as above, we see that $\tau_i = n-1$. □

Let ϕ_H and ϕ_i be the nef value morphism of (X, H) and (X, L_i) respectively. Let F_H and F_i be the corresponding extremal face.

Claim 5.2.2. *$\phi_H = \phi_i$ for every integer i with $1 \leq i \leq n-2$.*

Proof. Let $C \subset X$ be an irreducible curve with $[C] \in F_H$. Then

$$\left(K_X + \frac{n-1}{n-2}H\right)C = 0.$$

Then by the cone theorem there exist extremal rational curves C_j such that $C = \sum_j \lambda_j C_j$ (see [2, Lemma 4.2.14]). Hence

$$\begin{aligned} 0 &= \left(K_X + \frac{n-1}{n-2}H\right)C \\ &= \sum_j \lambda_j \left(K_X + \frac{n-1}{n-2}H\right)C_j \\ &= \sum_j \lambda_j (K_X + (n-1)L_i)C_j \\ &= (K_X + (n-1)L_i)C. \end{aligned}$$

Therefore $[C] \in F_{L_i}$. By the same argument as above, $[C] \in F_H$ if C is a curve in X with $[C] \in F_{L_i}$. Hence $\phi_H = \phi_i$ because ϕ_H (resp. ϕ_i) is the contraction morphism of F_H (resp. F_i). □

In particular $\phi_i = \phi_j$. By Claim 5.2.1 $\tau_i = n-1$ for every integer i with $1 \leq i \leq n-2$. Hence by [2, Theorem 7.3.2], $(X, L_1, \dots, L_{n-2})$ is either of the type (1), (2), (4), or (5) in the statement of Theorem 5.2.1. Here we note that in the type (1) $K_Y + (n-1)A_j$ is ample for every j. Next we consider the type (2). Then $K_X + (n-1)L_j = \mathcal{O}_X$ for any j. Hence $(n-1)L_j = (n-1)L_k$ for $j \neq k$. Therefore $L_j \equiv L_k$. But since $h^1(\mathcal{O}_X) = 0$ and $H^2(X, \mathbf{Z})$ is torsion free in this case, we see that $L_j = L_k$.

This completes the proof of Theorem 5.2.1. □

Remark 5.2.1. In (1) of Theorem 5.2.1, we see that

$$K_Y + \frac{n-1}{n-2}(A_1 + \cdots + A_{n-2})$$

is ample. Therefore by Theorem 5.2.1 we get the following:

Let X be a smooth projective variety of dimension $n \geq 4$ and let $L_1, \dots, L_{n-2}$ be ample line bundles on X. Assume that $K_X + L_1 + \cdots + L_{n-2}$ is not nef and $K_X + (n-1)L_j$ is nef for any j. Then $(X, L_1, \dots, L_{n-2})$ is one of the following:

(I) $K_X + (n-1)L_j = \mathcal{O}_X$ for any j. Moreover $L_j = L_k$ for any (j,k) with $j \neq k$.

(II) There exist a smooth projective curve W and a surjective morphism $f : X \to W$ with connected fibers such that (X, L_i) is a quadric fibration over W with respect to f for every integer i with $1 \leq i \leq n-2$.

(III) There exist a smooth projective surface S and a surjective morphism $f : X \to S$ with connected fibers such that f is a $\mathbf{P}^{n-2}$-bundle over S

and (X, L_j) is a scroll over S with respect to f for every integer j with $1 \le j \le n-2$.

(IV) There exists a reduction $(Y, A_1, \dots, A_{n-2})$ of $(X, L_1, \dots, L_{n-2})$ such that $(Y, A_1, \dots, A_{n-2})$ satisfies one of the following.

(IV.1) $n = 4$ and $(Y, A_1, A_2) \cong (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2), \mathcal{O}_{\mathbf{P}^4}(2))$.

(IV.2) $K_Y + A_1 + \cdots + A_{n-2}$ is nef.

Remark 5.2.2. Let $(Y, A_1, \dots, A_{n-2})$ be a reduction of $(X, L_1, \dots, L_{n-2})$. If Y is not isomorphic to X, then $K_Y + A_1 + \cdots + A_{n-2} + A_j$ is ample for every integer j with $1 \le j \le n-2$.

THEOREM 5.2.2. *Let X be a smooth projective variety of dimension $n \ge 4$ and let $L_1, \dots, L_{n-2}$ be ample line bundles on X. Assume the following*:

(a) *$K_X + L_1 + \cdots + L_{n-2}$ is not nef.*

(b) *$K_X + (n-1)L_j$ is not nef for some j.*

Then there exists $\sigma \in \mathfrak{S}_{n-2}$ such that $(X, L_{\sigma(1)}, \dots, L_{\sigma(n-2)})$ is one of the following:

(1) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n}(3))$.

(2) *$n \ge 5$ and* $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(2))$.

(3) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n}(2))$.

(4) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1))$.

(5) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1), \dots, \mathcal{O}_{\mathbf{Q}^n}(1), \mathcal{O}_{\mathbf{Q}^n}(2))$.

(6) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1), \dots, \mathcal{O}_{\mathbf{Q}^n}(1))$.

(7) *X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve C and one of the following holds.* (*Here F denotes its fiber.*)

(7.1) *$L_{\sigma(j)}|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for every integer j with $1 \le j \le n-2$.*

(7.2) *$L_{\sigma(j)}|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for every integer j with $1 \le j \le n-3$ and $L_{\sigma(n-2)}|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(2)$.*

Proof. We may assume that $j = 1$ in (b). Since $K_X + (n-1)L_1$ is not nef, by [4, Theorem 1 and Theorem 2] or [16, Theorem] we see that X is isomorphic to one of the following types:

(A) $\mathbf{P}^n$.

(B) $\mathbf{Q}^n$.

(C) A $\mathbf{P}^{n-1}$-bundle over a smooth curve C.

Next we study each case.

(A) If $X \cong \mathbf{P}^n$, then we set $L_j := \mathcal{O}_{\mathbf{P}^n}(a_j)$ for $1 \le j \le n-2$. Since $K_X + (n-1)L_1$ is not nef, we have $a_1 = 1$. Here we may assume that $a_2 \le \cdots \le a_{n-2}$. Since $K_X + L_1 + \cdots + L_{n-2}$ is not nef, we get $(a_1, \dots, a_{n-4}, a_{n-3}, a_{n-2}) = (1, \dots, 1, 1, 1)$, $(1, \dots, 1, 1, 2)$, $(1, \dots, 1, 2, 2)$ or $(1, \dots, 1, 1, 3)$. We note that if $n = 4$, then $(a_1, a_2) = (2, 2)$ cannot occur.

(B) If $X \cong \mathbf{Q}^n$ with $n \ge 4$, then $\mathrm{Pic}(X) \cong \mathbf{Z}$ and we set $L_j := \mathcal{O}_{\mathbf{Q}^n}(a_j)$ for $1 \le j \le n-2$. Since $K_X + (n-1)L_1$ is not nef, we have $a_1 = 1$. Here we may assume that $a_2 \le \cdots \le a_{n-2}$. Then we note that $K_X = \mathcal{O}_{\mathbf{Q}^n}(-n)$. Since $K_X + L_1 + \cdots + L_{n-2}$ is not nef, we get $(a_1, \dots, a_{n-3}, a_{n-2}) = (1, \dots, 1, 1)$ or $(1, \dots, 1, 2)$.

(C) The case where X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve C.

(C.1) The case where $g(C) \geq 1$.

Since $K_X + (n-1)L_1$ is not nef, there exists a vector bundle $\mathscr{E}$ on C with $\mathrm{rank}(\mathscr{E}) = n$ such that $X = \mathbf{P}_C(\mathscr{E})$ and $L_1 = H(\mathscr{E})$, where $H(\mathscr{E})$ denotes the tautological line bundle on X. Then we note that $\mathscr{E}$ is ample. Let $\pi : \mathbf{P}_C(\mathscr{E}) \to C$ be its projection. Let $L_j := a_j H(\mathscr{E}) + \pi^*(B_j)$ for every integer j with $2 \leq j \leq n-2$. Here we may assume that $a_2 \leq \cdots \leq a_{n-2}$. Since $K_X + L_1 + \cdots + L_{n-2}$ is not nef, there exists an extremal rational curve B on X such that $(K_X + L_1 + \cdots + L_{n-2})B < 0$. We note that B is contained in a fiber of π. Hence

$$0 > (K_X + L_1 + \cdots + L_{n-2})B = \mathcal{O}_{\mathbf{P}^{n-1}}\left(-n + 1 + \sum_{j=2}^{n-2} a_j\right)B.$$

Hence we obtain $(a_2, \ldots, a_{n-3}, a_{n-2}) = (1, \ldots, 1, 1)$ or $(1, \ldots, 1, 2)$.

(C.2) The case where $g(C) = 0$.

There exists a vector bundle $\mathscr{E}$ on $\mathbf{P}^1$ such that $X = \mathbf{P}_C(\mathscr{E})$ and $\mathscr{E} \cong \mathcal{O}_C \oplus \mathcal{O}_C(d_1) \oplus \cdots \oplus \mathcal{O}_C(d_{n-1})$, where d_j is a non-negative integer for $1 \leq j \leq n-1$. In this case we set $L_j := \tilde{a}_j H(\mathscr{E}) + \pi^*(\tilde{B}_j)$ for $1 \leq j \leq n-2$. By [2, Lemma 3.2.4] $\tilde{a}_j > 0$ and $\tilde{b}_j > 0$ for any integer j with $1 \leq j \leq n-2$, where $\tilde{b}_j := \deg \tilde{B}_j$. Since $K_X + (n-1)L_1$ is not nef, we have $\tilde{a}_1 = 1$. We may assume that $\tilde{a}_2 \leq \cdots \leq \tilde{a}_{n-2}$.

Since $K_X \equiv -nH(\mathscr{E}) + (c_1(\mathscr{E}) - 2)F$, we have

$$K_X + L_1 + \cdots + L_{n-2} \equiv \left(-n + \sum_{j=1}^{n-2} \tilde{a}_j\right)H(\mathscr{E}) + \left(c_1(\mathscr{E}) - 2 + \sum_{j=1}^{n-2} \tilde{b}_j\right)F.$$

We note that $c_1(\mathscr{E}) - 2 + \sum_{j=1}^{n-2} \tilde{b}_j \geq 0 - 2 + (n-2) \geq 0$. Hence $K_X + L_1 + \cdots + L_{n-2}$ is not nef if and only if $-n + \sum_{j=1}^{n-2} \tilde{a}_j < 0$ because $H(\mathscr{E})$ is nef. So we get $(\tilde{a}_1, \ldots, \tilde{a}_{n-3}, \tilde{a}_{n-2}) = (1, \ldots, 1, 1)$ or $(1, \ldots, 1, 2)$.

This completes the proof. □

Remark 5.2.3. Assume that $(X, L_1, \ldots, L_k)$ is either the type (3) (D) in Theorem 5.1.1 ($k = n-1$ in this case) or (7.1) in Theorem 5.2.2 ($k = n-2$ in this case). Let $f : X \to C$ be its projection. Then for every j with $1 \leq j \leq k$ there exists an ample line bundle $B_j \in \mathrm{Pic}(C)$ such that $K_X + nL_j = f^*(B_j)$. Hence $n(K_X + L_{b_1} + \cdots + L_{b_n}) = f^*(B_{b_1} + \cdots + B_{b_n})$ for *any* $(b_1, \ldots, b_n)$ with $\{b_1, \ldots, b_n\} \subset \{1, \ldots, k\}$. On the other hand, by assumption, there exists a line bundle $D \in \mathrm{Pic}(C)$ such that $K_X + L_{b_1} + \cdots + L_{b_n} = f^*(D)$. Hence D is ample because $\deg D = \deg(B_{b_1} + \cdots + B_{b_n})/n > 0$. Therefore we see that $(X, L_{b_1}, \ldots, L_{b_n})$ is a scroll over C.

Remark 5.2.4. By Theorem 5.2.1, Remark 5.2.1 and Theorem 5.2.2, we get the following:

Let X be a smooth projective variety of dimension $n \geq 4$ and let $L_1, \dots, L_{n-2}$ be ample line bundles on X. Let $(Y, A_1, \dots, A_{n-2})$ be a reduction of $(X, L_1, \dots, L_{n-2})$. Assume that $K_X + L_1 + \cdots + L_{n-2}$ is not nef. Then there exists $\sigma \in \mathfrak{S}_{n-2}$ such that $(X, L_{\sigma(1)}, \dots, L_{\sigma(n-2)})$ is one of the following:

(1) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1))$.
(2) $n \geq 5$ and $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(2))$.
(3) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n}(2))$.
(4) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n}(3))$.
(5) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1), \dots, \mathcal{O}_{\mathbf{Q}^n}(1))$.
(6) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1), \dots, \mathcal{O}_{\mathbf{Q}^n}(1), \mathcal{O}_{\mathbf{Q}^n}(2))$.
(7) X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve C and one of the following holds. (Here F denotes its fiber).
  (7.1) $L_{\sigma(j)}|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for every integer j with $1 \leq j \leq n-2$.
  (7.2) $L_{\sigma(j)}|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for every integer j with $1 \leq j \leq n-3$ and $L_{\sigma(n-2)}|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(2)$.
(8) $K_X + (n-1)L_j = \mathcal{O}_X$ for any j. Moreover $L_j = L_k$ for any (j,k) with $j \neq k$.
(9) There exist a smooth projective curve W and a surjective morphism $f : X \to W$ with connected fibers such that (X, L_i) is a quadric fibration over W with respect to f for every integer i with $1 \leq i \leq n-2$.
(10) There exist a smooth projective surface S and a surjective morphism $f : X \to S$ with connected fibers such that f is a $\mathbf{P}^{n-2}$-bundle over S and (X, L_j) is a scroll over S with respect to f for every integer j with $1 \leq j \leq n-2$.
(11) $n = 4$ and $(Y, A_1, A_2) \cong (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2), \mathcal{O}_{\mathbf{P}^4}(2))$.
(12) $K_Y + A_1 + \cdots + A_{n-2}$ is nef.

THEOREM 5.2.3. *Let $(X, L_1, \dots, L_{n-2})$ be an n-dimensional multi-polarized manifold with $n \geq 4$. Assume that $K_X + L_1 + \cdots + L_{n-2}$ is nef. Then one of the following holds.*

(1) $K_X + L_1 + \cdots + L_{n-2} = \mathcal{O}_X$.
(2) $(X, L_1, \dots, L_{n-2})$ *is a Del Pezzo fibration over a smooth curve.*
(3) $(X, L_1, \dots, L_{n-2})$ *is a quadric fibration over a normal surface.*
(4) $(X, L_1, \dots, L_{n-2})$ *is a scroll over a normal 3-fold.*
(5) $K_X + L_1 + \cdots + L_{n-2}$ *is nef and big.*

Proof. If $K_X + L_1 + \cdots + L_{n-2}$ is ample, then $(X, L_1, \dots, L_{n-2})$ satisfies (5). So we may assume that $K_X + L_1 + \cdots + L_{n-2}$ is not ample. Then we can take the nef value morphism $\phi : X \to Y$ of $(X, L_1 + \cdots + L_{n-2})$, where Y is a normal projective variety.

Assume that $\dim Y < \dim X$. Let F be a general fiber of ϕ. Then $K_F + L_1|_F + \cdots + L_{n-2}|_F = \mathcal{O}_F$. Hence $\dim F \geq n-3$ by Remark 3.2. Namely, $\dim Y \leq 3$. Therefore we get the type (1), (2), (3) and (4).

Assume that $\dim Y = \dim X$. Then $K_X + L_1 + \cdots + L_{n-2}$ is nef and big. Therefore we get the assertion. □

6. The first sectional geometric genus

In this section, we consider the first sectional geometric genus of multi-polarized manifolds.

6.1. Fundamental results

PROPOSITION 6.1.1. *Let X be a smooth projective variety of dimension n, and let $L_1, \dots, L_{n-1}$ be line bundles on X. Then*

$$g_1(X, L_1, \dots, L_{n-1}) = 1 + \frac{1}{2}\left(K_X + \sum_{j=1}^{n-1} L_j\right) L_1 \cdots L_{n-1}.$$

Proof. We use [13, Corollary 2.7] for $i = 1$. Here we note the following: the proof of [13, Theorem 2.4] shows that the equality in [13, Corollary 2.7] holds for any line bundles $L_1, \dots, L_{n-i}$. By [13, Corollary 2.7], there are the following terms in $g_1(X, L_1, \dots, L_{n-1})$:

$$\left(\sum_{j=1}^{n-1} L_j\right) L_1 \cdots L_{n-1}$$

and

$$L_1 \cdots L_{n-1} T_1(X).$$

Here $T_1(X)$ denotes the Todd polynomial of weight 1 of the tangent bundle $\mathscr{T}_X$ (see [13, Definition 1.7]). The coefficient of $(\sum_{j=1}^{n-1} L_j) L_1 \cdots L_{n-1}$ is $1/2$ and the coefficient of $L_1 \cdots L_{n-1} T_1(X)$ is $(-1)^1/(1! \cdots 1!) = -1$. Since $T_1(X) = (1/2)c_1(X) = -(1/2)K_X$, we obtain

$$\begin{aligned} g_1(X, L_1, \dots, L_{n-1}) &= 1 + \frac{1}{2}\left(\sum_{j=1}^{n-1} L_j\right) L_1 \cdots L_{n-1} + \frac{1}{2} K_X L_1 \cdots L_{n-1} \\ &= 1 + \frac{1}{2}\left(K_X + \sum_{j=1}^{n-1} L_j\right) L_1 \cdots L_{n-1}. \end{aligned}$$

So we get the assertion. □

By setting $\mathscr{E} := L_1 \oplus \cdots \oplus L_{n-1}$, we can obtain the following theorems by Remark 3.1 and [23, Theorems 1 and 2].

THEOREM 6.1.1. *Let X be a smooth projective variety of dimension $n \geq 3$. Let $L_1, \dots, L_{n-1}$ be ample line bundles on X. Then $g_1(X, L_1, \dots, L_{n-1}) \geq 0$.*

If $g_1(X, L_1, \dots, L_{n-1}) = 0$, then $(X, L_{\sigma(1)}, \dots, L_{\sigma(n-1)})$ is one of the following: (Here $\sigma \in \mathfrak{S}_{n-1}$.)

(A) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1))$.
(B) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1))$.
(C) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1), \dots, \mathcal{O}_{\mathbf{Q}^n}(1))$.
(D) *X is a $\mathbf{P}^{n-1}$-bundle over a projective line $\mathbf{P}^1$ and $L_j|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for any fiber F and j with $1 \leq j \leq n-1$.*

Theorem 6.1.2. *Let X be a smooth projective variety of dimension $n \geq 3$ and let $L_1, \dots, L_{n-1}$ be ample line bundles on X. Assume that $g_1(X, L_1, \dots, L_{n-1}) = 1$. Then $(X, L_1, \dots, L_{n-1})$ is one of the following*:
(1) *$(X, L_1, \dots, L_{n-1})$ satisfies $K_X + L_1 + \cdots + L_{n-1} = \mathcal{O}_X$.*
(2) *X is a $\mathbf{P}^{n-1}$-bundle over an elliptic curve C and $L_j|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for any fiber F and any integer j with $1 \leq j \leq n-1$.*

Here we note that we can characterize $(X, L_1, \dots, L_{n-1})$ in the case (1) in Theorem 6.1.2.

Theorem 6.1.3. *Let X be a smooth projective variety of dimension $n \geq 3$. Let $L_1, L_2, \dots, L_{n-1}$ be ample line bundles on X. Assume that $K_X + L_1 + \cdots + L_{n-1} = \mathcal{O}_X$. Then there exists $\sigma \in \mathfrak{S}_{n-1}$ such that $(X, L_{\sigma(1)}, L_{\sigma(2)}, \dots, L_{\sigma(n-1)})$ is one of the following*:
(A) *(X, L) is a Del Pezzo manifold for some ample line bundle L on X and $L_j = L$ for every integer j with $1 \leq j \leq n-1$.*
(B) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(3), \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1))$.
(C) *$n \geq 4$ and* $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1))$.
(D) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(2), \mathcal{O}_{\mathbf{Q}^n}(1), \dots, \mathcal{O}_{\mathbf{Q}^n}(1))$.
(E) *$X \cong \mathbf{P}^2 \times \mathbf{P}^1$, $L_1 = p_1^*(\mathcal{O}_{\mathbf{P}^2}(2)) + p_2^*(\mathcal{O}_{\mathbf{P}^1}(1))$ and $L_2 = p_1^*(\mathcal{O}_{\mathbf{P}^2}(1)) + p_2^*(\mathcal{O}_{\mathbf{P}^1}(1))$, where p_i is the ith projection.*

Proof. First we note that $h^1(\mathcal{O}_X) = 0$ by assumption.
(1) Assume that $K_X + (n-1)L_j$ is nef for any j. Then

$$\sum_{j=1}^{n-1}(K_X + (n-1)L_j) = (n-1)(K_X + L_1 + \cdots + L_{n-1}) = \mathcal{O}_X.$$

Therefore $(K_X + (n-1)L_j)L_j^{n-1} = 0$. Since $K_X + (n-1)L_j$ is nef, we have $K_X + (n-1)L_j = \mathcal{O}_X$, that is, (X, L_j) is a Del Pezzo manifold. Moreover since $(n-1)L_j = (n-1)L_k$ for any $j \neq k$, we have $L_j \equiv L_k$. But since $h^1(\mathcal{O}_X) = 0$ and $H^2(X, \mathbf{Z})$ is torsion free, we have $L_j = L_k$. So we get the type (A) above.

(2) Assume that $K_X + (n-1)L_j$ is not nef for some j. Then by the adjunction theory, we see that X is one of the following type:
(2.1) $X \cong \mathbf{P}^n$.
(2.2) $X \cong \mathbf{Q}^n$.
(2.3) X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve B.

(2.1) First we consider the case where $X \cong \mathbf{P}^n$. Then by assumption we get $(L_1, \dots, L_{n-1})$ is isomorphic to

$$(\mathcal{O}_{\mathbf{P}^n}(3), \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1)) \quad \text{or} \quad (\mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(1)).$$

Here we note that $n \geq 4$ in the latter case because $K_X + (n-1)L_j$ is not nef for some j.

(2.2) Next we consider the case where $X \cong \mathbf{Q}^n$. Then by assumption we get the type (D) above.

(2.3) Finally we consider the case where X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve B. Since $h^1(\mathcal{O}_X) = 0$, we see that $B \cong \mathbf{P}^1$. Then there exists a vector bundle $\mathcal{E}$ of rank n on X such that $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_{n-1})$ and $X \cong \mathbf{P}_{\mathbf{P}^1}(\mathcal{E})$, where $a_j \geq 0$ for every j. Then by [2, Lemma 3.2.4], $aH(\mathcal{E}) + bF$ is ample if and only if $a > 0$ and $b > 0$. Here we note that by the assumption that $\mathcal{O}_X(K_X + L_1 + \cdots + L_{n-1}) = \mathcal{O}_X$, we may assume that $L_1|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(2)$ and $L_j|_F = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for any fiber F and every integer j with $2 \leq j \leq n-1$. Hence we can write $L_1 = 2H(\mathcal{E}) + \pi^*(B_1)$ and $L_j = H(\mathcal{E}) + \pi^*(B_j)$ for every integer j with $2 \leq j \leq n-1$, where $B_j \in \mathrm{Pic}(\mathbf{P}^1)$. Set $b_j := \deg B_j$. Then $b_j \geq 1$ because L_j is ample. Since $K_X = -nH(\mathcal{E}) + \pi^*(K_{\mathbf{P}^1} + \det \mathcal{E})$, we have $K_X + L_1 + \cdots + L_{n-1} = \pi^*(K_{\mathbf{P}^1} + \det \mathcal{E} + B_1 + \cdots + B_{n-1})$. Since $\deg \mathcal{E} \geq 0$, we see that

$$\begin{aligned} \deg(K_{\mathbf{P}^1} + \det \mathcal{E} + B_1 + \cdots + B_{n-1}) &= -2 + \deg \mathcal{E} + b_1 + \cdots + b_{n-1} \\ &\geq n - 3 \geq 0. \end{aligned}$$

By the assumption that $\mathcal{O}_X(K_X + L_1 + \cdots + L_{n-1}) = \mathcal{O}_X$, we get $\deg(K_{\mathbf{P}^1} + \det \mathcal{E} + B_1 + \cdots + B_{n-1}) = 0$. Hence $n = 3$, $\deg \mathcal{E} = 0$ and $b_j = 1$ for every j. In particular $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$. Therefore we get the type (E). □

Remark 6.1.1. In general, let $\mathcal{F}$ be an ample vector bundle of rank $n-1$ on a smooth projective variety X of dimension n. Then a classification of $(X, \mathcal{F})$ with $\mathcal{O}_X(K_X + \det \mathcal{F}) = \mathcal{O}_X$ has been obtained. See [25].

By Corollary 4.2, we get the following:

THEOREM 6.1.4. *Let X be a smooth projective variety of dimension $n \geq 3$, let i be an integer with $0 \leq i \leq n-1$, and let $L_1, \dots, L_{n-i}$ be ample and spanned line bundles on X. Then $g_i(X, L_1, \dots, L_{n-i}) \geq h^i(\mathcal{O}_X)$.*

By considering this theorem, it is natural to classify $(X, L_1, \dots, L_{n-i})$ such that $\mathrm{Bs}|L_j| = \emptyset$ for any j with $1 \leq j \leq n-i$ and $g_i(X, L_1, \dots, L_{n-i}) = h^i(\mathcal{O}_X)$. Here we consider the case where $i = 1$. Set $\mathcal{E} := L_1 \oplus \cdots \oplus L_{n-1}$. Then $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on X. Since, as we said in Remark 3.1, $g_1(X, L_1, \dots, L_{n-1})$ is equal to the curve genus $g(X, \mathcal{E})$ of $\mathcal{E}$, we can get the following theorem by [22, Theorem].

Theorem 6.1.5. *Let X be a smooth projective variety of dimension $n \geq 3$, and let $L_1, \dots, L_{n-1}$ be ample and spanned line bundles on X. If $g_1(X, L_1, \dots, L_{n-1}) = h^1(\mathcal{O}_X)$, then $(X, L_1, \dots, L_{n-1})$ is one of the following*:

(1) $g_1(X, L_1, \dots, L_{n-1}) = 0$.

(2) *X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve B and $L_j = H(\mathcal{E}) + f^*(D_j)$ for any j with $1 \leq j \leq n-1$, where $\mathcal{E}$ is a vector bundle of rank n on B such that $X \cong \mathbf{P}_B(\mathcal{E})$, $H(\mathcal{E})$ is the tautological line bundle on X, $f : X \to B$ is its fibration, and $D_j \in \operatorname{Pic}(B)$ for any j.*

Moreover we can also get the following theorem by Remark 3.1 and [15, Theorems 5.2 and 5.3].

Theorem 6.1.6. *Let X be a smooth projective variety of dimension $n \geq 3$. Assume that there exists a fiber space $f : X \to C$, where C is a smooth projective curve. Let $L_1, \dots, L_{n-1}$ be ample line bundles on X. Then $g_1(X, L_1, \dots, L_{n-1}) \geq g(C)$. Moreover if $g_1(X, L_1, \dots, L_{n-1}) = g(C)$, then X is a $\mathbf{P}^{n-1}$-bundle on C via f and $L_j|_F \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for any fiber F of f and every integer j with $1 \leq j \leq n-1$.*

Next we consider Conjecture 4.1 for the case where $i = 1$ and $\kappa(X) = 0$ or 1.

Theorem 6.1.7. *Let X be a smooth projective variety of dimension $n \geq 3$. Let $L_1, \dots, L_{n-1}$ be ample line bundles on X. Assume that $L_1 \cdots L_{n-1} L_j \geq 2$ for any j with $1 \leq j \leq n-1$ and $\kappa(X) = 0$ or 1. Then $g_1(X, L_1, \dots, L_{n-1}) \geq q(X)$.*

Proof. If $\kappa(X) = 0$, then $h^1(\mathcal{O}_X) \leq n$ by the classification theory of manifolds (see [17, Corollary 2]). Hence

$$\begin{aligned} g_1(X, L_1, \dots, L_{n-1}) &= 1 + \frac{1}{2}(K_X + L_1 + \cdots + L_{n-1}) L_1 \cdots L_{n-1} \\ &\geq 1 + (n-1) = n \geq h^1(\mathcal{O}_X). \end{aligned}$$

Next we consider the case where $\kappa(X) = 1$. By taking the Iitaka fibration of X, there exists a smooth projective variety X', a smooth projective curve C', a birational morphism $\mu : X' \to X$ and a fiber space $f' : X' \to C'$ such that $\kappa(F') = 0$ for any general fiber F' of f'. In this case $h^1(\mathcal{O}_{X'}) \leq h^1(\mathcal{O}_{C'}) + h^1(\mathcal{O}_{F'}) \leq g(C') + n - 1$ by Lemma 3.2 and [17, Corollary 2]. Here we note that by the proof of [8, Theorem 1.3.3] we have $K_{X'/C'}(\mu^* L_1) \cdots (\mu^* L_{n-1}) \geq 0$. We also note that

$$\begin{aligned} &g_1(X', \mu^*(L_1), \dots, \mu^*(L_{n-1})) \\ &\quad = 1 + \frac{1}{2}(K_{X'/C'} + \mu^*(L_1) + \cdots + \mu^*(L_{n-1})) \mu^*(L_1) \cdots \mu^*(L_{n-1}) \\ &\qquad + (g(C') - 1) \mu^*(L_1) \cdots \mu^*(L_{n-1}) F'. \end{aligned}$$

If $g(C') \geq 1$, then since $\mu^*(L_1)\cdots\mu^*(L_{n-1})F' \geq 1$ we see that

$$\begin{aligned} g_1(X, L_1, \dots, L_{n-1}) &= g_1(X', \mu^*L_1, \dots, \mu^*L_{n-1}) \\ &\geq g(C') + \frac{1}{2}(\mu^*L_1 + \cdots + \mu^*L_{n-1})(\mu^*L_1)\cdots(\mu^*L_{n-1}) \\ &\geq g(C') + n - 1 \\ &\geq h^1(\mathcal{O}_{X'}) = h^1(\mathcal{O}_X). \end{aligned}$$

If $g(C') = 0$, then $h^1(\mathcal{O}_{X'}) \leq n - 1$ and by assumption here we get

$$\begin{aligned} g_1(X, L_1, \dots, L_{n-1}) &= 1 + \frac{1}{2}(K_X + L_1 + \cdots + L_{n-1})L_1\cdots L_{n-1} \\ &\geq 1 + (n-1) = n > h^1(\mathcal{O}_{X'}) = h^1(\mathcal{O}_X). \end{aligned}$$

This completes the proof of Theorem 6.1.7. □

6.2. The case of 3-folds.

Here we consider the case where X is a 3-fold. The method is similar to that of [9]. We fix the notation which will be used below.

Notation 6.2.1. Let (X, L_1) be a polarized manifold with $\dim X = 3$ and $h^0(L_1) \geq 2$. Let Λ be a linear pencil which is contained in $|L_1|$ such that $\Lambda = \Lambda_M + Z$, where Λ_M is the movable part of Λ and Z is the fixed part of $|L_1|$. We will make a fiber space by using this Λ. Let $\varphi : X \dashrightarrow \mathbf{P}^1$ be the rational map associated with Λ_M, and $\theta : X' \to X$ an elimination of indeterminacy of φ. So we obtain a surjective morphism $\varphi' : X' \to \mathbf{P}^1$. By taking the Stein factorization, if necessary, there exist a smooth projective curve C, a finite morphism $\delta : C \to \mathbf{P}^1$ and a fiber space $f' : X' \to C$ such that $\varphi' = \delta \circ f'$. Let $a_\Lambda := \deg \delta$ and F' a general fiber of f'.

Theorem 6.2.1. *Let X be a smooth projective variety of dimension* 3. *Let L_1, L_2 be ample line bundles on X. Assume that $h^0(L_1) \geq 2$ and $h^0(L_2) \geq 1$. Then $g_1(X, L_1, L_2) \geq q(X)$.*

Proof. If $K_X + L_1 + L_2$ is not nef, then by Theorem 5.1.1, Remark 5.2.3 and [13, Example 2.1 (A), (B), (E) and (H)] we get $g_1(X, L_1, L_2) \geq q(X)$.

So we may assume that $K_X + L_1 + L_2$ is nef. Here we use Notation 6.2.1.

(I) If $g(C) \geq 1$, then θ is the identity mapping. By Proposition 6.1.1, we have

$$\begin{aligned} g_1(X, L_1, L_2) &= 1 + \frac{1}{2}(K_X + L_1 + L_2)L_1L_2 \\ &= 1 + \frac{1}{2}(K_{X/C} + L_1 + L_2)L_1L_2 + (g(C) - 1)L_1L_2F'. \end{aligned}$$

Since $K_{X/C}+L_1+L_2$ is f'-nef and $\dim C=1$, we see that $K_{X/C}+L_1+L_2$ is nef by Lemma 3.1. Here we note that $a_\Lambda \geq 2$ because $g(C)\geq 1$. Since $L_1-a_\Lambda F'$ is effective, we obtain

$$\begin{aligned} g_1(X,L_1,L_2) &= 1+\frac{1}{2}(K_{X/C}+L_1+L_2)L_1L_2+(g(C)-1)L_1L_2F' \\ &\geq 1+\frac{1}{2}(K_{X/C}+L_1+L_2)(a_\Lambda F')L_2+(g(C)-1)L_1L_2F' \\ &\geq g(C)+(K_{F'}+L_1|_{F'}+L_2|_{F'})L_2|_{F'}. \end{aligned}$$

If $h^1(\mathcal{O}_{F'})=0$, then $h^1(\mathcal{O}_X)=g(C)$. Moreover since $K_{F'}+L_1|_{F'}+L_2|_{F'}$ is nef, we get $g_1(X,L_1,L_2)\geq g(C)$. Hence $g_1(X,L_1,L_2)\geq g(C)=h^1(\mathcal{O}_X)$. Hence we may assume that $h^1(\mathcal{O}_{F'})>0$.

Since $h^0(L_2|_{F'})>0$ and $\dim F'=2$, we have $g(L_2|_{F'})\geq h^1(\mathcal{O}_{F'})$ ([7, Lemma 1.2 (2)]). Therefore

$$g_1(X,L_1,L_2)\geq g(C)+2h^1(\mathcal{O}_{F'})-2+(L_1|_{F'})(L_2|_{F'}).$$

Then by Lemma 3.2

$$\begin{aligned} g_1(X,L_1,L_2) &\geq g(C)+h^1(\mathcal{O}_{F'})+h^1(\mathcal{O}_{F'})-2+(L_1|_{F'})(L_2|_{F'}) \\ &\geq g(C)+h^1(\mathcal{O}_{F'}) \\ &\geq h^1(\mathcal{O}_X). \end{aligned}$$

(II) Assume that $g(C)=0$. Let D be an irreducible and reduced divisor on X such that the strict transform of D by θ is a general fiber F'. Then L_1-D is linearly equivalent to an effective divisor. Here we note that $K_X+L_1+L_2$ is nef. So we have

$$\begin{aligned} g_1(X,L_1,L_2) &= g_1(X',\theta^*L_1,\theta^*L_2) \\ &= 1+\frac{1}{2}(K_{X'}+\theta^*L_1+\theta^*L_2)(\theta^*L_1)(\theta^*L_2) \\ &= 1+\frac{1}{2}\theta^*(K_X+L_1+L_2)(\theta^*L_1)(\theta^*L_2) \\ &\geq 1+\frac{1}{2}\theta^*(K_X+L_1+L_2)(\theta^*L_2)F' \\ &= 1+\frac{1}{2}(\theta^*(K_X+D)+\theta^*(L_1-D)+\theta^*L_2)(\theta^*L_2)F'. \end{aligned}$$

Since $\theta^*(L_1-D)(\theta^*L_2)F'\geq 0$, we have

$$g(X,L_1,L_2)\geq 1+\frac{1}{2}(\theta^*(K_X+D)+\theta^*L_2)(\theta^*L_2)F'.$$

By the same argument as in the proof of [9, Claim 2.4], we can prove

$$\theta^*(K_X + D)(\theta^* L_2)F' \geq (K_{X'} + F')(\theta^* L_2)F'.$$

Hence

$$\begin{aligned} g_1(X, L_1, L_2) &\geq 1 + \frac{1}{2}(K_{X'} + F' + \theta^* L_2)(\theta^* L_2)F' \\ &= g(\theta^* L_2|_{F'}). \end{aligned}$$

Since $h^0(\theta^*(L_2)|_{F'}) > 0$, we get $g(\theta^*(L_2)|_{F'}) \geq h^1(\mathcal{O}_{F'})$ by [7, Lemma 1.2 (2)]. Therefore by Lemma 3.2

$$g_1(X, L_1, L_2) \geq g(\theta^*(L_2)|_{F'}) \geq h^1(\mathcal{O}_{F'}) \geq h^1(\mathcal{O}_{X'}) = h^1(\mathcal{O}_X).$$

This completes the proof. □

THEOREM 6.2.2. *Let X be a smooth projective variety of dimension 3 and let L_1 and L_2 be ample line bundles on X with $h^0(L_1) \geq 2$ and $h^0(L_2) \geq 1$. Let $\Lambda \subset |L_1|$ be a linear pencil, and we use Notation 6.2.1. Assume that for some $\sigma \in \mathfrak{S}_2$ $(X, L_{\sigma(1)}, L_{\sigma(2)})$ is neither of the following*:

(A) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1), \mathcal{O}_{\mathbf{P}^3}(1))$.
(B) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2), \mathcal{O}_{\mathbf{P}^3}(1))$.
(C) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1), \mathcal{O}_{\mathbf{Q}^3}(1))$.
(D) *X is a $\mathbf{P}^2$-bundle over a smooth projective curve and $L_j|_F = \mathcal{O}_{\mathbf{P}^2}(1)$ for any fiber F and $j = 1, 2$.*

Then

(1) $g_1(X, L_1, L_2) \geq a_\Lambda q(X)$ *if* $g(C) = 0$.
(2) $g_1(X, L_1, L_2) \geq q(X) + (a_\Lambda - 1)q(F')$ *if* $g(C) \geq 1$.

Proof. If $K_X + L_1 + L_2$ is not nef, then (X, L_1, L_2) is one of the types from (A) to (D) above by Theorem 5.1.1 (3). So we may assume that $K_X + L_1 + L_2$ is nef. Let Z, θ, f' and C be as in Notation 6.2.1. Let $Z = \sum_{i=1}^m b_i Z_i$, and let Z_i' be the strict transform of Z_i by θ. Let $\theta' : X'' \to X'$ be a birational morphism such that Z_i'' is a smooth surface, where Z_i'' is the strict transform of Z_i' by θ'. We can take a general element $B \in |L_1|$ such that $B = G_1 + \cdots + G_{a_\Lambda} + Z$, where each G_i is the image of a general fiber of f' by θ. Let $h := f' \circ \theta'$ and $\pi := \theta \circ \theta'$. Then the strict transform of G_i by π is a general fiber of h. Let F_i'' be the strict transform of G_i by π. We note that Z_i'' is the strict transform of Z_i by π. Then we have

$$\begin{aligned} g_1(X, L_1, L_2) = g(X'', \pi^* L_1, \pi^* L_2) &= 1 + \frac{1}{2}(K_{X''} + \pi^* L_1 + \pi^* L_2)(\pi^* L_1)(\pi^* L_2) \\ &= 1 + \frac{1}{2}\pi^*(K_X + L_1 + L_2)(\pi^* L_2)(\pi^* B) \\ &\geq 1 + \frac{1}{2}\pi^*(K_X + L_1 + L_2)(\pi^* L_2)(\pi^*(B_{\mathrm{red}})). \end{aligned}$$

Put $B_{\mathrm{nr}} := B - B_{\mathrm{red}}$. Then by the same argument as in [9, Claim 2.9] we have $B_{\mathrm{nr}} B_{\mathrm{red}} L_2 \geq 0$. Hence

$$\begin{aligned} g_1(X, L_1, L_2) &\geq 1 + \frac{1}{2}\pi^*(K_X + L_1 + L_2)(\pi^* L_2)(\pi^*(B_{\mathrm{red}})) \\ &\geq 1 + \frac{1}{2}(\pi^*(K_X + B_{\mathrm{red}}) + \pi^* L_2)(\pi^* L_2)(\pi^*(B_{\mathrm{red}})). \end{aligned}$$

Moreover since $\pi^*(B_{\mathrm{red}}) - \sum_{i=1}^{a_\Lambda} F_i'' - \sum_{i=1}^{m} Z_i''$ is a π-exceptional effective divisor, we get

$$\begin{aligned} g_1(X, L_1, L_2) &\geq 1 + \frac{1}{2}(\pi^*(K_X + B_{\mathrm{red}}) + \pi^* L_2)(\pi^* L_2)(\pi^*(B_{\mathrm{red}})) \\ &= 1 + \frac{1}{2}\sum_{i=1}^{a_\Lambda}(\pi^*(K_X + G_i) + \pi^* L_2)(\pi^* L_2) F_i'' \\ &\quad + \frac{1}{2}\sum_{i=1}^{a_\Lambda}\pi^*(B_{\mathrm{red}} - G_i)(\pi^* L_2) F_i'' \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}(\pi^*(K_X + Z_i) + \pi^* L_2)(\pi^* L_2) Z_i'' \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\pi^*(B_{\mathrm{red}} - Z_i)(\pi^* L_2) Z_i''. \end{aligned}$$

Because L_2 is ample and B is connected, we have

$$\frac{1}{2}\left(\sum_{i=1}^{a_\Lambda}\pi^*(B_{\mathrm{red}} - G_i)(\pi^* L_2) F_i'' + \sum_{i=1}^{m}\pi^*(B_{\mathrm{red}} - Z_i)(\pi^* L_2) Z_i''\right) \geq a_\Lambda + m - 1.$$

Therefore

$$\begin{aligned} g_1(X, L_1, L_2) &\geq 1 + \frac{1}{2}\sum_{i=1}^{a_\Lambda}(\pi^*(K_X + G_i) + \pi^* L_2)(\pi^* L_2) F_i'' \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}(\pi^*(K_X + Z_i) + \pi^* L_2)(\pi^* L_2) Z_i'' + (a_\Lambda + m - 1) \\ &= \sum_{i=1}^{a_\Lambda}\left(1 + \frac{1}{2}(\pi^*(K_X + G_i) + \pi^* L_2)(\pi^* L_2) F_i''\right) \\ &\quad + \sum_{i=1}^{m}\left(1 + \frac{1}{2}(\pi^*(K_X + Z_i) + \pi^* L_2)(\pi^* L_2) Z_i''\right). \end{aligned}$$

By the same argument as in the proof of [9, Claim 2.4], we can prove that

$$(\pi^*(K_X + G_i) + \pi^* L_2)(\pi^* L_2) F_i'' \geq (K_{X''} + F_i'' + \pi^* L_2)(\pi^* L_2) F_i''$$

and

$$(\pi^*(K_X + Z_i) + \pi^* L_2)(\pi^* L_2) Z_i'' \geq (K_{X''} + Z_i'' + \pi^* L_2)(\pi^* L_2) Z_i''.$$

So we obtain

$$\begin{aligned} g_1(X, L_1, L_2) &\geq \sum_{i=1}^{a_\Lambda} \left(1 + \frac{1}{2}(K_{X''} + F_i'' + \pi^* L_2)(\pi^* L_2) F_i''\right) \\ &\quad + \sum_{i=1}^{m} \left(1 + \frac{1}{2}(K_{X''} + Z_i'' + \pi^* L_2)(\pi^* L_2) Z_i''\right) \\ &= \sum_{i=1}^{a_\Lambda} g((\pi^* L_2)|_{F_i''}) + \sum_{i=1}^{m} g((\pi^* L_2)|_{Z_i''}). \end{aligned}$$

We note that $g(\pi^* L_2|_{Z_i''}) \geq 0$ for any i since $\dim Z_i'' = 2$ (for example, see [5, (4.8) Corollary]).

(I) The case where $g(C) = 0$.

Because $h^0((\pi^* L_2)|_{F_i''}) \geq 1$ and $\dim F_i'' = 2$, we have $g((\pi^* L_2)|_{F_i''}) \geq q(F_i'')$ for every i. Since $q(F_i'') \geq q(X'') = q(X') = q(X)$ for every i by Lemma 3.2, we get $g_1(X, L_1, L_2) \geq a_\Lambda q(X)$.

(II) The case where $g(C) \geq 1$.

Then θ is the identity mapping and $Z_i = Z_i'$ for every i. Since L_2 is ample and G_i is a fiber of f', there exists a Z_i such that $f'|_{Z_i} : Z_i \to C$ is surjective. We consider the fiber space $h|_{Z_i''} : Z_i'' \to C$. By [7, Theorem 2.1 and Theorem 5.5], we have $g((\pi^* L_2)|_{Z_i''}) \geq g(C)$. On the other hand, $g((\pi^* L_2)|_{F_i''}) \geq q(F_i'')$ holds because $h^0((\pi^* L_2)|_{F_i''}) \geq 1$ and $\dim F_i'' = 2$. Therefore we get $g_1(X, L_1, L_2) \geq g(C) + a_\Lambda q(F_i'')$. Since $g(C) + q(F_i'') \geq q(X'') = q(X') = q(X)$ by Lemma 3.2 and $q(F_i'') = q(F')$ for every i, we get $g_1(X, L_1, L_2) \geq q(X) + (a_\Lambda - 1) q(F')$. (Here we note that $a_\Lambda \geq 2$ in this case.)

This completes the proof of Theorem 6.2.2. □

THEOREM 6.2.3. *Let X be a smooth projective variety of dimension 3 and let L_1 and L_2 be ample line bundles on X such that $h^0(L_1) \geq 2$ and $h^0(L_2) \geq 1$. Let $\Lambda \subset |L_1|$ be a linear pencil and we use Notation 6.2.1.*

If $a_\Lambda = 1$, then $g_1(X, L_1, L_2) \geq q(X) + \frac{1}{2} GZL_2$, where G is a general element of Λ_M and Z is the fixed part of $|L_1|$, unless $(X, L_{\sigma(1)}, L_{\sigma(2)})$ is one of the following for some $\sigma \in \mathfrak{S}_2$:

(A) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1), \mathcal{O}_{\mathbf{P}^3}(1))$.

(B) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2), \mathcal{O}_{\mathbf{P}^3}(1))$.

(C) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1), \mathcal{O}_{\mathbf{Q}^3}(1))$.

(D) *X is a $\mathbf{P}^2$-bundle over a smooth projective curve and $L_j|_F = \mathcal{O}_{\mathbf{P}^2}(1)$ for any fiber F and $j = 1, 2$.*

In particular, $g_1(X, L_1, L_2) \geq q(X) + 1$ if $Z \neq 0$.

Proof. If $K_X + L_1 + L_2$ is not nef, then (X, L_1, L_2) is one of the types from (A) to (D) above by Theorem 5.1.1 (3). So we may assume that $K_X + L_1 + L_2$ is nef. We note that the strict transform of G by θ is F'. So we have

$$\begin{aligned} g_1(X, L_1, L_2) &= 1 + \frac{1}{2}(K_{X'} + \theta^*(L_1 + L_2))(\theta^* L_1)(\theta^* L_2) \\ &= 1 + \frac{1}{2}\theta^*(K_X + L_1 + L_2)(\theta^* L_1)(\theta^* L_2) \\ &\geq 1 + \frac{1}{2}\theta^*(K_X + L_1 + L_2)(\theta^* L_2)F' \\ &= 1 + \frac{1}{2}(\theta^*(K_X + G) + \theta^*(L_1 - G) + \theta^* L_2)(\theta^* L_2)F'. \end{aligned}$$

By the same argument as in the proof of [9, Claim 2.4], we can prove

$$\theta^*(K_X + G)(\theta^* L_2)F' \geq (K_{X'} + F')(\theta^* L_2)F'.$$

On the other hand, $\theta^*(L_1 - G)(\theta^* L_2)F' = ZGL_2$. Hence

$$\begin{aligned} g_1(X, L_1, L_2) &\geq 1 + \frac{1}{2}(K_{X'} + F' + \theta^* L_2)(\theta^* L_2)F' + \frac{1}{2}ZGL_2 \\ &= g((\theta^* L_2)|_{F'}) + \frac{1}{2}ZGL_2. \end{aligned}$$

Because $h^0((\theta^* L_2)|_{F'}) \geq 1$ and $\dim F' = 2$, we obtain $g((\theta^* L_2)|_{F'}) \geq q(F')$ by [7, Lemma 1.2 (2)]. Since $g(C) = 0$ in this case, we have $q(F') \geq q(X') = q(X)$. Therefore

$$g_1(X, L_1, L_2) \geq q(F') + \frac{1}{2}ZGL_2 \geq q(X) + \frac{1}{2}ZGL_2.$$

If $Z \neq 0$, then $Z \cap G \neq \phi$ since $G + Z$ is connected. Since L_2 is ample and G is a general element of Λ_M, we have $ZGL_2 > 0$. Because $g_1(X, L_1, L_2)$ is an integer, we have $g_1(X, L_1, L_2) \geq q(X) + 1$. This completes the proof. □

Theorem 6.2.4. *Let X be a smooth projective variety with* $\dim X = 3$ *and let L_1 and L_2 be ample line bundles on X with $h^0(L_1) \geq 2$ and $h^0(L_2) \geq 3$. If $g_1(X, L_1, L_2) = q(X)$, then $(X, L_{\sigma(1)}, L_{\sigma(2)})$ is one of the following types for some $\sigma \in \Sigma_2$.*

(A) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1), \mathcal{O}_{\mathbf{P}^3}(1))$.
(B) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2), \mathcal{O}_{\mathbf{P}^3}(1))$.
(C) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1), \mathcal{O}_{\mathbf{Q}^3}(1))$.
(D) *X is a $\mathbf{P}^2$-bundle over a smooth projective curve and $L_j|_F = \mathcal{O}_{\mathbf{P}^2}(1)$ for any fiber F and $j = 1, 2$.*

Proof. We use Notation 6.2.1.

If $K_X + L_1 + L_2$ is not nef, then by Theorem 5.1.1 (3) we see that (X, L) is one of the types from (A) to (D) above. So we may assume that $K_X + L_1 + L_2$ is nef. In particular we note that $g_1(X, L_1, L_2) \geq 1$.

(1) The case in which $g(C) \geq 1$.

We note that θ is the identity mapping and $a_\Lambda \geq 2$ in this case. By Theorem 6.2.2 (2), we have $q(X) = g_1(X, L_1, L_2) \geq q(X) + (a_\Lambda - 1)q(F')$. Because $a_\Lambda \geq 2$, we obtain $q(F') = 0$. Hence $q(X) \leq g(C) + q(F') = g(C)$ by Lemma 3.2. But since $g(C) \leq q(X)$, we get $q(X) = g(C)$, and $g_1(X, L_1, L_2) = q(X) = g(C)$. Then (X, L_1, L_2) is the type (D) above by Theorem 6.1.6. This is a contradiction by assumption.

(2) The case in which $g(C) = 0$.

If $a_\Lambda \geq 2$, then $q(X) = g_1(X, L_1, L_2) \geq 2q(X)$ by Theorem 6.2.2 (1). Hence $q(X) = 0$, and $g(X, L_1, L_2) = q(X) = 0$. But this is a contradiction.

So we consider the case where $a_\Lambda = 1$. By Theorem 6.2.3, we see

$$Z = 0, \tag{6.2.4.1}$$

that is, $|L_1|$ has no fixed component. By the proof of Theorem 6.2.3, we see that $g((\theta^* L_2)|_{F'}) = q(F')$. Here we note that

$$g((\theta^* L_2)|_{F'}) - q(F') = h^0(K_{F'} + (\theta^* L_2)|_{F'}) - h^0(K_{F'})$$

by the Riemann-Roch theorem and the Kawamata-Viehweg vanishing theorem. Since $h^0((\theta^* L_2)|_{F'}) \geq 2$, we have $h^0(K_{F'}) = 0$ by Lemma 3.3. Assume that $\kappa(F') \geq 0$. Then $q(F') \leq 1$ because $\chi(\mathcal{O}_{F'}) \geq 0$. Hence $g((\theta^* L_2)|_{F'}) = q(F') \leq 1$. But since $\kappa(F') \geq 0$, we have $g((\theta^* L_2)|_{F'}) \geq 2$ and this is a contradiction. Hence we have

$$\kappa(F') = -\infty. \tag{6.2.4.2}$$

Because $g((\theta^* L_2)|_{F'}) = q(F')$, we can prove the following claim.

Claim 6.2.1. $\kappa(K_{F'} + (\theta^* L_2)|_{F'}) = -\infty$.

Proof. Assume that $\kappa(K_{F'} + (\theta^* L_2)|_{F'}) \geq 0$. Then $g((\theta^* L_2)|_{F'}) \geq 1$.

Since $0 < g((\theta^* L_2)|_{F'}) = q(F')$, a $((\theta^* L_2)|_{F'})$-minimalization of $(F', (\theta^* L_2)|_{F'})$ (see [7, Definition 1.9]) is a scroll over a smooth curve B by [7, Theorem 3.1]. Hence there is a surjective morphism $h : F' \to B$ such that a general fiber F_h of h is $\mathbf{P}^1$. Hence $(K_{F'} + (\theta^* L_2)|_{F'})F_h = -1$. But this is a contradiction because F_h is nef. This completes the proof of Claim 6.2.1. □

On the other hand,

$$\begin{aligned} K_{F'} + (\theta^* L_2)|_{F'} &= (K_{X'} + F' + \theta^* L_2)_{F'} \\ &= (\theta^*(K_X + L_2) + E_\theta + F')_{F'}, \end{aligned}$$

where E_θ is a θ-exceptional effective divisor.

Let (M, A) be a reduction of (X, L_2) and let $\pi : X \to M$ be its reduction map. Assume that $K_M + A$ is nef. Then $h^0(m(K_M + A)) > 0$ for any large $m \gg 0$ by the nonvanishing theorem. Here we note that $K_X + L_2 = \pi^*(K_M + A) + E$ for an effective π-exceptional divisor E. Hence for any large m, we have

$$h^0(m(K_X + L_2)) = h^0(m\pi^*(K_M + A) + mE) > 0.$$

Therefore $h^0(m(\theta^*(K_X + L_2))_{F'}) \geq 1$. Since F' is a general fiber of f', we have $h^0((E_\theta + F')|_{F'}) \geq 1$. Hence $h^0(m(\theta^*(K_X + L_2) + E_\theta + F')|_{F'}) \geq 1$ for any large $m \gg 0$. But this is a contradiction by Claim 6.2.1. Hence $K_M + A$ is not nef, and by Theorem 3.1 we see that (M, A) is one of the following types. (Here we note that $\dim M = 3$ in this case.)

(a) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$.
(b) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$.
(c) A scroll over a smooth curve C.
(d) $K_M \sim -2A$, that is, (M, A) is a Del Pezzo manifold.
(e) A quadric fibration over a smooth curve C.
(f) A scroll over a smooth surface S.
(g) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$.
(h) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$.
(i) M is a $\mathbf{P}^2$-bundle over a smooth curve C with $(F, A|_F) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ for any fiber F of it.

If (M, A) is either of the cases (a), (b), (d), (g), and (h), then $q(X) = 0$. Hence by assumption $g_1(X, L_1, L_2) = q(X) = 0$. But this is a contradiction.

If (M, A) is either of the cases (c), (e), and (i), then $q(X) = g(C)$. Hence by assumption $g(X, L_1, L_2) = q(X) = g(C)$. So by Theorem 6.1.6, (X, L_1, L_2) is the type (D) above. But in this case $K_X + L_1 + L_2$ is not nef and this is a contradiction.

So we consider the case in which (M, A) is the case (f). Let $\varphi : M \to S$ be its $\mathbf{P}^1$-bundle, where S is a smooth surface.

Claim 6.2.2. $\kappa(S) = -\infty$.

Proof. We note that $Z = 0$ by (6.2.4.1). We take a general element $G \in |A|$. Then G is irreducible and reduced, and the strict transform of G by θ is F'. Since A is ample, $\varphi|_G : G \to S$ is surjective. Hence we obtain $\kappa(S) = -\infty$ since $\kappa(F') = -\infty$ by (6.2.4.2). This completes the proof of this claim. □

If $q(S) = 0$, then $q(X) = q(S) = 0$. Hence by assumption $g_1(X, L_1, L_2) = q(X) = q(S) = 0$. Hence (X, L_1, L_2) is one of the types from (A) to (D) above by Theorem 6.1.1. But this is a contradiction by assumption.

If $q(S) \geq 1$, we take the Albanese map of S, $\alpha : S \to B$, where B is a smooth curve. Then by assumption $g_1(X, L_1, L_2) = q(X) = q(S) = g(B)$. Hence

(X, L_1, L_2) is the type (D) above by Theorem 6.1.6. But this is a contradiction by the same reason as above. This completes the proof of Theorem 6.2.4. □

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