

GEOMETRIC MEANING OF SASAKIAN SPACE FORMS FROM THE VIEWPOINT OF SUBMANIFOLD THEORY

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To the memory of Professor Shūkichi Tanno

Abstract

We show that M^{2n-1} is a real hypersurface all of whose geodesics orthogonal to the characteristic vector ξ are mapped to circles of the same curvature 1 in an n -dimensional nonflat complex space form $\tilde{M}_n(c)(= \mathbf{C}P^n(c)$ or $\mathbf{C}H^n(c))$ if and only if M is a Sasakian manifold with respect to the almost contact metric structure from the ambient space $\tilde{M}_n(c)$. Moreover, this Sasakian manifold M is a Sasakian space form of constant ϕ -sectional curvature $c + 1$ for each $c(\neq 0)$.

1. Introduction

We denote by $\tilde{M}_n(c)$, $n \geq 2$ a complex n -dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature $c(\neq 0)$. That is, $\tilde{M}_n(c)$ is holomorphically isometric to either an n -dimensional complex projective space $\mathbf{C}P^n(c)$ of constant holomorphic sectional curvature c or an n -dimensional complex hyperbolic space $\mathbf{C}H^n(c)$ of constant holomorphic sectional curvature c according as c is positive or negative. As is well-known, every real hypersurface M^{2n-1} of $\tilde{M}_n(c)$ admits an almost contact metric structure (ϕ, ξ, η, g) induced from this ambient space. Making use of such a structure, many geometers have studied actively real hypersurfaces in nonflat complex space forms (c.f. [9]).

On the other hand, the theory of contact geometry has been developed also by many geometers (for details, see [5]). Sasakian manifolds and Sasakian space forms are analogues to Kähler manifolds and complex space forms, respectively. J. Berndt ([4]) showed that a Sasakian space form of constant ϕ -sectional curvature $d(\neq 1)$ is realized as a Hopf hypersurface with two distinct constant principal curvatures in a nonflat complex space form (see Lemma 2). In this context we

2000 *Mathematics Subject Classification.* Primary 53C40, Secondary 53C22, 53D10.

Key words and phrases. nonflat complex space forms, real hypersurfaces, totally η -umbilic hypersurfaces, almost contact metric structure, Sasakian manifolds, Sasakian space forms, geodesics, structure torsion, Berger spheres, circles.

Received May 26, 2009; revised December 1, 2009.

study Sasakian space forms from submanifold theoretical point of view. It is well-known that a $(2n - 1)$ -dimensional Sasakian space form of constant ϕ -sectional curvature 1 is a unit sphere $S^{2n-1}(1)$. This sphere is totally umbilic in \mathbf{C}^n and is the only real hypersurface all of whose geodesics are mapped to circles of the same curvature 1 in the ambient space \mathbf{C}^n . It is hence natural to come to consider real hypersurfaces all of whose geodesics are mapped to circles of the same positive curvature in a complex n -dimensional nonflat complex space form $\tilde{M}_n(c)$. However, unfortunately there exist no such real hypersurfaces in this space $\tilde{M}_n(c)$ because this space admits no totally umbilic real hypersurfaces.

Motivated by this fact, we are interested in investigating real hypersurfaces all of whose geodesics orthogonal to the characteristic vector ξ are mapped to circles of the same positive curvature in a nonflat complex space form. The main purpose of this paper is to characterize Sasakian space forms by such a condition among real hypersurfaces in a nonflat complex space form $\tilde{M}_n(c)$ (Theorem 1 and Remark 2). This, together with results in [3], gives us many nice geometric properties of geodesics on Sasakian space forms (Proposition 2). For example, when $c > 8$, the Sasakian space form of constant ϕ -sectional curvature $c + 1$ is a Berger sphere.

In the last section, we show directly that in the case of $c > 0$ the metric of our realization of a Sasakian space form of constant ϕ -sectional curvature $c + 1$ coincides with the metric of the standard example of a Sasakian space form given in [5].

2. Preliminaries

We first review fundamental notion in contact geometry (see [5]). Let M^{2n-1} ($n \geq 2$) be a differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) . That is, this structure satisfies the following identities:

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vectors X, Y on M . We say M to be a *Sasakian manifold* if the structure tensor ϕ satisfies the differential equation

$$(2.1) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

for all tangent vectors $X, Y \in TM$, where ∇ denotes the Riemannian connection of the metric g of M . A Sasakian manifold is called a *Sasakian space form* of constant ϕ -sectional curvature c if the sectional curvature $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ satisfies $K(u, \phi u) = c$ for each unit vector u orthogonal to ξ . The following is a Sasakian analogue of Schur's Theorem.

THEOREM A. *If the ϕ -sectional curvature at each point of a Sasakian manifold M of dimension ≥ 5 does not depend on the choice of ϕ -section at that point, then it is constant on M . The curvature tensor is given by*

$$R(X, Y)Z = \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + g(Z, \phi Y)\phi X - g(Z, \phi X)\phi Y + 2g(X, \phi Y)\phi Z\},$$

where c is the constant ϕ -sectional curvature of M .

In this paper, we denote by $M^{2n-1}(c)$ a $(2n - 1)$ -dimensional Sasakian space form of constant ϕ -sectional curvature c . For the standard construction of Sasakian space forms, see pp. 114–115 in [5]. The following is the unique existence theorem of Sasakian space forms.

THEOREM B. *For any two simply connected complete Sasakian manifolds of constant ϕ -sectional curvature c , there exists an isomorphism between them which preserves their almost contact metric structures.*

We next review the fundamental theory of real hypersurfaces. Let M^{2n-1} be a real hypersurface with a unit normal local vector field \mathcal{N} of an n -dimensional Kähler manifold (\tilde{M}, g, J) with Riemannian metric g and Kähler structure J . The Riemannian connections $\tilde{\nabla}$ of \tilde{M} and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$(2.2) \quad \tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N},$$

$$(2.3) \quad \tilde{\nabla}_X \mathcal{N} = -AX$$

for arbitrary vector fields X and Y on M , where g is the Riemannian metric of M induced from the ambient space \tilde{M} and A is the shape operator of M in \tilde{M} . An eigenvector X of the shape operator A is called a *principal curvature vector* of M in \tilde{M} and an eigenvalue λ of A is called a *principal curvature* of M in \tilde{M} . We denote by V_λ the eigenspace associated to the principal curvature λ , namely we set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$.

It is well-known that M has an almost contact metric structure induced from the Kähler structure of the ambient space \tilde{M} . That is, we have a quartet (ϕ, ξ, η, g) defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

It follows from (2.2), (2.3) and $\tilde{\nabla}J = 0$ that

$$(2.4) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

$$(2.5) \quad \nabla_X \xi = \phi AX.$$

We here clarify the meaning of the condition that a real hypersurface M is a Sasakian manifold with respect to the almost contact metric structure of the

ambient Kähler manifold \tilde{M} . On an orientable connected real hypersurface M in a Kähler manifold \tilde{M} , we have an almost contact metric structure (ϕ, ξ, η, g) associated with a unit normal vector \mathcal{N} of M in \tilde{M} . Clearly the quartet $(\phi, -\xi, -\eta, g)$ is also an almost contact metric structure on M which is associated with a unit normal $-\mathcal{N}$. We call a real hypersurface M *Sasakian* if M satisfies either (2.1) or

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$$

for all vectors $X, Y \in TM$.

In the following, we consider real hypersurfaces in an n -dimensional nonflat complex space form $\tilde{M}_n(c) (= \mathbf{C}P^n(c)$ or $\mathbf{C}H^n(c)$). Denoting the curvature tensor of M by R , we have the equation of Gauss given by

$$(2.6) \quad g(R(X, Y)Z, W) \\ = (c/4)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(\phi Y, Z)g(\phi X, W) \\ - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\} \\ + g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W).$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M . For a Hopf hypersurface M^{2n-1} ($n \geq 2$) in a nonflat complex space form $\tilde{M}_n(c)$, the principal curvature δ corresponding to the characteristic vector field ξ is locally constant on M . Furthermore, every tube of sufficiently small constant radius around each Kähler submanifold of a nonflat complex space form $\tilde{M}_n(c)$ is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurfaces is natural in the theory of real hypersurfaces in a nonflat complex space form (see [9]).

In $\mathbf{C}P^n(c)$ ($n \geq 2$), a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

- (A₁) A geodesic sphere of radius r , where $0 < r < \pi/\sqrt{c}$;
- (A₂) A tube of radius r around totally geodesic $\mathbf{C}P^\ell(c)$ ($1 \leq \ell \leq n-2$), where $0 < r < \pi/\sqrt{c}$;
- (B) A tube of radius r around complex hyperquadric $\mathbf{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$;
- (C) A tube of radius r around $\mathbf{C}P^1(c) \times \mathbf{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and $n(\geq 5)$ is odd;
- (D) A tube of radius r around complex Grassmann $\mathbf{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;
- (E) A tube of radius r around Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.

These real hypersurfaces are said to be of types (A₁), (A₂), (B), (C), (D) and (E). Summing up real hypersurfaces of types (A₁) and (A₂), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are 2, 3, 3, 5, 5, 5, respectively. The principal curvatures of these real hypersurfaces in $\mathbf{C}P^n(c)$ are given as follows (see [9]):

	(A ₁)	(A ₂)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)$
λ_2	—	$-\frac{\sqrt{c}}{2} \tan\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r\right)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2} \tan\left(\frac{\sqrt{c}}{2}r\right)$
δ	$\sqrt{c} \cot(\sqrt{c}r)$	$\sqrt{c} \cot(\sqrt{c}r)$	$\sqrt{c} \cot(\sqrt{c}r)$	$\sqrt{c} \cot(\sqrt{c}r)$

One should notice that in $CP^n(c)$ a tube of radius r ($0 < r < \pi/\sqrt{c}$) around totally geodesic $CP^\ell(c)$ ($0 \leq \ell \leq n-1$) is congruent to a tube of radius $((\pi/\sqrt{c}) - r)$ around totally geodesic $CP^{n-\ell-1}(c)$.

In $CH^n(c)$ ($n \geq 2$), a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (cf. [9]):

- (A₀) A horosphere in $CH^n(c)$;
- (A_{1,0}) A geodesic sphere of radius r ($0 < r < \infty$);
- (A_{1,1}) A tube of radius r around totally geodesic $CH^{n-1}(c)$, where $0 < r < \infty$;
- (A₂) A tube of radius r around totally geodesic $CH^\ell(c)$ ($1 \leq \ell \leq n-2$), where $0 < r < \infty$;
- (B) A tube of radius r around totally real totally geodesic $RH^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A₀), (A₁), (A₁), (A₂) and (B). Here, type (A₁) means either type (A_{1,0}) or type (A_{1,1}). Summing up real hypersurfaces of types (A₀), (A₁) and (A₂), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}) \log_e(2 + \sqrt{3})$ has two distinct constant principal curvatures. Except this real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}) \log_e(2 + \sqrt{3})$, the numbers of distinct principal curvatures of these real hypersurfaces are 2, 2, 2, 3, 3, respectively. The principal curvatures of these real hypersurfaces in $CH^n(c)$ are given as follows (cf. [9]):

	(A ₀)	(A _{1,0})	(A _{1,1})	(A ₂)	(B)
λ_1	$\frac{\sqrt{ c }}{2}$	$\frac{\sqrt{ c }}{2} \coth\left(\frac{\sqrt{ c }}{2}r\right)$	$\frac{\sqrt{ c }}{2} \tanh\left(\frac{\sqrt{ c }}{2}r\right)$	$\frac{\sqrt{ c }}{2} \coth\left(\frac{\sqrt{ c }}{2}r\right)$	$\frac{\sqrt{ c }}{2} \coth\left(\frac{\sqrt{ c }}{2}r\right)$
λ_2	—	—	—	$\frac{\sqrt{ c }}{2} \tanh\left(\frac{\sqrt{ c }}{2}r\right)$	$\frac{\sqrt{ c }}{2} \tanh\left(\frac{\sqrt{ c }}{2}r\right)$
δ	$\sqrt{ c }$	$\sqrt{ c } \coth(\sqrt{ c }r)$	$\sqrt{ c } \coth(\sqrt{ c }r)$	$\sqrt{ c } \coth(\sqrt{ c }r)$	$\sqrt{ c } \tanh(\sqrt{ c }r)$

A real hypersurface M of a nonflat complex space form $\tilde{M}_n(c)$, $n \geq 2$ is called *totally η -umbilic* if its shape operator A is of the form $A = \alpha I + \beta \eta \otimes \xi$ for some smooth functions α and β on M . This definition is equivalent to saying that $Au = \alpha u$ for each vector u on M which is orthogonal to the characteristic vector ξ of M , where α is a smooth function on M . It is known that every totally η -umbilic hypersurface is a member of Hopf hypersurfaces with constant principal curvatures. The following classification theorem of totally η -umbilic hypersurfaces M shows that these two functions α and β are automatically constant on M (see [9]):

THEOREM C. *Let M^{2n-1} , $n \geq 2$ be a totally η -umbilic hypersurface of a nonflat complex space form $\tilde{M}_n(c)$ with shape operator $A = \alpha I + \beta \eta \otimes \xi$. Then M is locally congruent to one of the following:*

- (P) *A geodesic sphere of radius r ($0 < r < \pi/\sqrt{c}$) in $\mathbf{CP}^n(c)$, where $\alpha = (\sqrt{c}/2) \cot(\sqrt{cr}/2)$ and $\beta = -(\sqrt{c}/2) \tan(\sqrt{cr}/2)$;*
- (H_i) *A horosphere in $\mathbf{CH}^n(c)$, where $\alpha = \beta = \sqrt{|c|}/2$;*
- (H_{ii}) *A geodesic sphere of radius r ($0 < r < \infty$) in $\mathbf{CH}^n(c)$, where $\alpha = (\sqrt{|c|}/2) \coth(\sqrt{|c|r}/2)$ and $\beta = (\sqrt{|c|}/2) \tanh(\sqrt{|c|r}/2)$;*
- (H_{iii}) *A tube of radius r ($0 < r < \infty$) around totally geodesic complex hyperplane $\mathbf{CH}^{n-1}(c)$ in $\mathbf{CH}^n(c)$, where $\alpha = (\sqrt{|c|}/2) \tanh(\sqrt{|c|r}/2)$ and $\beta = (\sqrt{|c|}/2) \coth(\sqrt{|c|r}/2)$.*

Every totally η -umbilic hypersurface has two distinct constant principal curvatures α and $\alpha + \beta (= \delta)$. For the later use we prepare the following lemma (see [9]).

LEMMA 1. *For a real hypersurface M in a nonflat complex space form $\tilde{M}_n(c)$ ($n \geq 2$), the following conditions are mutually equivalent.*

- (1) *M is of type (A).*
- (2) *$\phi A = A\phi$.*
- (3) *$g((\nabla_X A)Y, Z) = (c/4)(-\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y))$ for arbitrary vectors X, Y and Z on M .*

In this paper, real hypersurfaces of types (A), (B), (C), (D) and (E) in $\tilde{M}_n(c)$ are said to be *standard real hypersurfaces*. It is well-known that every standard real hypersurface M is a homogeneous real hypersurface of $\tilde{M}_n(c)$, namely M is an orbit of some subgroup of the full isometry group $I(\tilde{M}_n(c))$ of $\tilde{M}_n(c)$ (see [9]).

At the end of this section we review the definition of circles in Riemannian geometry. A real smooth curve $\gamma = \gamma(s)$ parameterized by its arclength s in a Riemannian manifold M with Riemannian connection ∇ is called a *circle of curvature k* if it satisfies the ordinary differential equations $\nabla_{\dot{\gamma}}\dot{\gamma} = kY_s$, $\nabla_{\dot{\gamma}}Y_s = -k\dot{\gamma}$ with a field Y_s of unit vectors along γ . Here $k(\geq 0)$ is constant and Y_s is called the unit principal normal vector of γ . A circle of null curvature is nothing but a geodesic. The definition of a circle is equivalent to saying that it is a curve

$\gamma = \gamma(s)$ on M with Riemannian metric g satisfying the ordinary differential equation

$$(2.7) \quad \nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}\dot{\gamma}) + g(\nabla_{\dot{\gamma}}\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma})\dot{\gamma} = 0.$$

3. Geodesics on Sasakian hypersurface in $\tilde{M}_n(c)$

We shall start by classifying all Sasakian real hypersurfaces in a nonflat complex space form $\tilde{M}_n(c)$. The following lemma is essentially due to Berndt [4]. We here give its complete proof for the sake of readers' convenience.

LEMMA 2. *Let M^{2n-1} ($n \geq 2$) be a connected Sasakian real hypersurface of a nonflat complex space form $\tilde{M}_n(c)$. Then M is locally congruent to one of the following homogeneous real hypersurfaces of the ambient space $\tilde{M}_n(c)$:*

- i) *A geodesic sphere $G(r)$ of radius r with $\tan(\sqrt{c}r/2) = \sqrt{c}/2$ ($0 < r < \pi/\sqrt{c}$) in $\mathbf{C}P^n(c)$;*
- ii) *A horosphere in $\mathbf{C}H^n(-4)$;*
- iii) *A geodesic sphere $G(r)$ of radius r with $\tanh(\sqrt{|c|r}/2) = \sqrt{|c|}/2$ ($0 < r < \infty$) in $\mathbf{C}H^n(c)$ ($-4 < c < 0$);*
- iv) *A tube of radius r around totally geodesic $\mathbf{C}H^{n-1}(c)$ with $\tanh(\sqrt{|c|r}/2) = 2/\sqrt{|c|}$ ($0 < r < \infty$) in $\mathbf{C}H^n(c)$ ($c < -4$).*

In these cases, M has constant ϕ -sectional curvature $c + 1$.

Proof. Suppose that our real hypersurface M is a Sasakian manifold. Then it follows from (2.1) and (2.4) that

$$(3.1) \quad g(X, Y)\xi - \eta(Y)X = \eta(Y)AX - g(AX, Y)\xi$$

for all vectors $X, Y \in TM$. Setting $X = Y = \xi$ in Equation (3.1), we see that ξ is principal. Hence we can choose a principal curvature vector u orthogonal to ξ . Then, setting $Y = \xi$ in Equation (3.1), we find that $Au = -u$, so that the tangent bundle TM of M is decomposed as $TM = \{\xi\}_{\mathbf{R}} \oplus V_{-1}$, where $V_{-1} = \{X \in TM \mid AX = -X\}$. This, together with Theorem C, shows that our real hypersurface M is a totally η -umbilic hypersurface with coefficients $\alpha = -1$ and $\beta = c/4$ in $\tilde{M}_n(c)$. Here, we change the unit normal vector \mathcal{N} into $-\mathcal{N}$ for each hypersurface in Theorem C. Then we know that M is locally congruent to one of i), ii), iii) and iv) in Lemma 2. Next, for each unit vector u perpendicular to ξ , we compute the ϕ -sectional curvature $K(u, \phi u)$ of our real hypersurface M . It follows from Equation (2.6) and the equality $A = -I + (c/4)\eta \otimes \xi$ that $K(u, \phi u) = c + 1$.

Conversely, we suppose that our real hypersurface M is locally congruent to one of i), ii), iii) and iv) in Lemma 2. We then see that the shape operator A of our real hypersurface M is of the form $A = -I + (c/4)\eta \otimes \xi$ by changing \mathcal{N} into $-\mathcal{N}$ for each hypersurface in Theorem C. This, combined with (2.4), yields

(2.1), so that M is a Sasakian manifold. Thus we can obtain the conclusion of our Theorem. \square

Remark 1. (1) Each real hypersurface i), ii), iii) and iv) in Lemma 2 is complete and simply connected. In particular, when $c > 0$ or $-4 < c < 0$, it is compact.

(2) The proof of Lemma 2 shows that M is a Sasakian manifold if and only if M is totally η -umbilic in $\tilde{M}_n(c)$ with shape operator $A = -I + (c/4)\eta \otimes \xi$.

The following theorem is closely related to Lemma 2 in the case of $c > 0$ (for details, see [6]).

THEOREM D. *Let M be a real hypersurface in $\mathbf{CP}^n(c)$, $n \geq 3$ on which the ϕ -sectional curvature H is constant. Then M is locally congruent to one of the following:*

- (1) *A geodesic sphere $G(r)$ ($0 < r < \pi/\sqrt{c}$) with $H = c + (c/4) \cot^2(\sqrt{c}r/2) > c$;*
- (2) *A ruled real hypersurface with $H = c$;*
- (3) *A real hypersurface on which there is an integrable distribution of codimension two such that its each leaf lies on some totally geodesic $\mathbf{CP}^{n-1}(c)$ as a ruled real hypersurface with $H = c$.*

Note that a real hypersurface M in Theorem D is Sasakian if and only if M is of Case i) in Lemma 2.

We are now in a position to prove the following which gives a geometric meaning of all Sasakian real hypersurfaces M in a nonflat complex space form $\tilde{M}_n(c)$ in terms of the extrinsic shape of some geodesics on M .

THEOREM 1. *Let M be a connected real hypersurface in a nonflat complex space form $\tilde{M}_n(c)$. Then the following three conditions are mutually equivalent:*

- (1) *M is a Sasakian manifold;*
- (2) *M is a Sasakian space form of constant ϕ -sectional curvature $c + 1$;*
- (3) *There exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ orthogonal to ξ at each point p of M satisfying the following two conditions:*
 - i) *All geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ ($1 \leq i \leq 2n - 2$) are mapped to circles of the same curvature 1 in $\tilde{M}_n(c)$;*
 - ii) *All geodesics $\gamma_{ij} = \gamma_{ij}(s)$ on M with $\gamma_{ij}(0) = p$ and $\dot{\gamma}_{ij}(0) = (v_i + v_j)/\sqrt{2}$ ($1 \leq i < j \leq 2n - 2$) are mapped to circles of the same curvature 1 in $\tilde{M}_n(c)$.*

Proof. By virtue of the discussion in the proof of Lemma 2, we only need to verify that Conditions (1) and (3) are equivalent.

Suppose Condition (1). The shape operator A of M is expressed as $A = -I + (c/4)\eta \otimes \xi$. We take a unit vector v orthogonal to ξ at an arbitrary fixed

point p of M . It satisfies $Av = -v$. Let $\gamma = \gamma(s)$ be the geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ on M . We then have

$$\begin{aligned} \nabla_{\dot{\gamma}}g(\dot{\gamma}, \xi) &= g(\dot{\gamma}, \nabla_{\dot{\gamma}}\xi) = g(\dot{\gamma}, \phi A\dot{\gamma}) \quad (\text{from (2.5)}) \\ &= g(\dot{\gamma}, A\phi\dot{\gamma}) \quad (\text{from Lemma 1}) \\ &= -g(\phi A\dot{\gamma}, \dot{\gamma}) = 0. \end{aligned}$$

This, together with $g(\dot{\gamma}(0), \xi) = g(v, \xi) = 0$, shows that the tangent vector $\dot{\gamma}(s)$ is perpendicular to $\xi_{\gamma(s)}$ for each s . Hence, again by using Lemma 1 we see that

$$\nabla_{\dot{\gamma}}\|A\dot{\gamma} + \dot{\gamma}\|^2 = 2g((\nabla_{\dot{\gamma}}A)\dot{\gamma}, A\dot{\gamma} + \dot{\gamma}) = 2g((\nabla_{\dot{\gamma}}A)\dot{\gamma}, A\dot{\gamma}) + 2g((\nabla_{\dot{\gamma}}A)\dot{\gamma}, \dot{\gamma}) = 0.$$

This, combined with $A\dot{\gamma}(0) + \dot{\gamma}(0) = Av + v = 0$, yields that $A\dot{\gamma}(s) + \dot{\gamma}(s) = 0$ for each s . Therefore, from (2.2) and (2.3) we have

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = g(A\dot{\gamma}, \dot{\gamma})\mathcal{N} = -\mathcal{N}$$

and

$$\tilde{\nabla}_{\dot{\gamma}}(-\mathcal{N}) = -\dot{\gamma}.$$

Thus we can see that the geodesic γ on a Sasakian real hypersurface M is a mapped to a circle of the same curvature 1 in the ambient space $\tilde{M}_n(c)$, and get Condition (3).

Next, we take orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ at a point p of a real hypersurface M satisfying Condition (3). Then, from (2.7) they satisfy

$$(3.2) \quad \tilde{\nabla}_{\dot{\gamma}_i}\tilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i = -\dot{\gamma}_i.$$

On the other hand, from (2.2) and (2.3) we have

$$(3.3) \quad \tilde{\nabla}_{\dot{\gamma}_i}\tilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i = g((\nabla_{\dot{\gamma}_i}A)\dot{\gamma}_i, \dot{\gamma}_i)\mathcal{N} - g(A\dot{\gamma}_i, \dot{\gamma}_i)A\dot{\gamma}_i.$$

Comparing the tangential components of (3.2) and (3.3), we see that

$$g(A\dot{\gamma}_i, \dot{\gamma}_i)A\dot{\gamma}_i = \dot{\gamma}_i,$$

so that at $s = 0$ we get

$$g(Av_i, v_i)Av_i = v_i \quad \text{for } 1 \leq i \leq 2n - 2,$$

which yields that

$$(3.4) \quad Av_i = v_i \quad \text{or} \quad Av_i = -v_i \quad \text{for } 1 \leq i \leq 2n - 2.$$

This implies that ξ is a principal curvature vector, because $\langle A\xi, v_i \rangle = \langle \xi, Av_i \rangle = 0$ for $1 \leq i \leq 2n - 2$. Therefore M is a Hopf hypersurface with at most three distinct principal curvatures 1, -1 and $\delta = g(A\xi, \xi)$ at its each point. On the other hand, applying the same discussion as above to Condition ii) of (3) in Theorem 1, we get the following corresponding to Equation (3.4):

$$(3.5) \quad A((v_i + v_j)/\sqrt{2}) = (v_i + v_j)/\sqrt{2} \quad \text{or} \quad A((v_i + v_j)/\sqrt{2}) = -(v_i + v_j)/\sqrt{2}$$

for $1 \leq i < j \leq 2n - 2$. Thus, from (3.4) and (3.5) we can see that either $Av_i = v_i$ ($1 \leq i \leq 2n - 2$) or $Av_i = -v_i$ ($1 \leq i \leq 2n - 2$) holds. This implies that our real hypersurface M is totally η -umbilic with coefficient $\alpha = \pm 1$ in the ambient space $\tilde{M}_n(c)$. We hence get Condition (1). \square

Remark 2. As an immediate consequence of the proof of Theorem 1, we know that on each Sasakian real hypersurface M^{2n-1} ($n \geq 2$) every geodesic $\gamma = \gamma(s)$ whose initial vector $\dot{\gamma}(0)$ is orthogonal to $\xi_{\gamma(0)}$ is mapped to a circle of the same curvature 1 in a nonflat complex space form $\tilde{M}_n(c)$.

In consideration of Lemma 1 and the proof of Theorem 1 we find the following:

PROPOSITION 1. *Let M be a connected real hypersurface of a nonflat complex space form $\tilde{M}_n(c)$. Then M is of type either (A_0) or type (A_1) if and only if there exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ orthogonal to ξ at each point p of M satisfying the following two conditions:*

- i') *All geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ ($1 \leq i \leq 2n - 2$) are mapped to circles of positive curvature in $\tilde{M}_n(c)$;*
- ii') *All geodesics $\gamma_{ij} = \gamma_{ij}(s)$ on M with $\gamma_{ij}(0) = p$ and $\dot{\gamma}_{ij}(0) = (v_i + v_j)/\sqrt{2}$ ($1 \leq i < j \leq 2n - 2$) are mapped to circles of positive curvature in $\tilde{M}_n(c)$.*

The following theorem tells us that Theorem 1 is no longer true if we remove Condition ii) of (3) in the assumption of Theorem 1.

THEOREM 2. *For a connected real hypersurface M^{2n-1} ($n \geq 2$) in a nonflat complex space form $\tilde{M}_n(c)$, the following two conditions are mutually equivalent:*

- (1) *M^{2n-1} is a Sasakian space form of constant ϕ -sectional curvature $c + 1$ or it is locally congruent to a tube of radius $\pi/4$ around totally geodesic $CP^\ell(4)$ ($1 \leq \ell \leq n - 2$) in $CP^n(4)$;*
- (2) *There exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ orthogonal to ξ at each point p of M satisfying that all geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ ($1 \leq i \leq 2n - 2$) are mapped to circles of the same curvature 1 in $\tilde{M}_n(c)$.*

Proof. We suppose Condition (2). The discussion in the proof of Theorem 1 shows that M has constant ϕ -sectional curvature $c + 1$ or is a Hopf hypersurface with at most three distinct (constant) principal curvatures 1, -1 and $\delta = g(A\xi, \xi)$ at its each point. In the following, we shall consider the latter case. The classification theorems of Hopf hypersurfaces with constant principal curvatures imply that this non-Sasakian Hopf hypersurface M is one of the hypersurfaces of types (A_0) , (A_1) , (A_2) and (B) . This, together with the tables of principal curvatures in Section 2, implies that the real hypersurface M is locally congruent to a tube of radius $\pi/4$ around totally geodesic $CP^\ell(4)$ ($1 \leq \ell \leq n - 2$) in $CP^n(4)$. Hence we obtain Condition (1).

Condition (1) follows from Condition (2) by using Lemma 2 and the proof of Theorem 1. \square

- Remark 3.* (1) Theorems 1 and 2 are local statements. If we add the condition that M is complete and simply connected to the assumptions, then these theorems are global results.
- (2) Every circle of curvature 1 in Theorems 1 and 2 lies on a totally real totally geodesic surface $\mathbf{RP}^2(c/4)$ (resp. $\mathbf{RH}^2(c/4)$) of constant sectional curvature $c/4$ in $\mathbf{CP}^n(c)$ (resp. $\mathbf{CH}^n(c)$). It is a simple curve, but is not necessarily closed. A circle of curvature 1 in Theorems 1 and 2 is closed if and only if $c > -4$. In particular, when $c = -4$, every circle of curvature 1 in Theorems 1 and 2 is a horocycle on $\mathbf{CH}^n(-4)$ (for details, see [1, 2]).

4. The length spectrum of a Sasakian space form

In this section, let M denote an arbitrary totally η -umbilic hypersurface of a nonflat complex space form $\tilde{M}_n(c)$ with $n \geq 2$. In order to investigate geometric properties of geodesics on a complete and simply connected Sasakian space form $M^{2n-1}(c+1)$ of constant ϕ -sectional curvature $c+1$, we study those of geodesics on this hypersurface M . The following discussion is indebted to [3].

Our hypersurface M is a *nice* Riemannian homogeneous manifold. Such a manifold M is known as an example of a naturally reductive Riemannian homogeneous manifold (see [7, 8]). This fact implies that each geodesic γ on the hypersurface M is a homogeneous curve, namely the curve γ is an orbit of some one-parameter subgroup of the isometry group $I(M)$ of M , so that it is a simple curve.

We recall the congruence theorem for geodesics on the hypersurface M . To do this, for a geodesic γ on M , we define its *structure torsion* ρ_γ by $\rho_\gamma = g(\dot{\gamma}, \xi_\gamma)$. Clearly, it satisfies $-1 \leq \rho_\gamma \leq 1$. Moreover, the computation in the proof of Theorem 1 shows that ρ_γ is constant along γ .

For geodesics on the hypersurface M , we can classify them by means of their structure torsions as follows. We say that two smooth curves γ_1, γ_2 on M are congruent to each other if there exists an isometry φ of M satisfying $\gamma_2(s) = (\varphi \circ \gamma_1)(s)$ for all s . When these curves γ_1, γ_2 are geodesics on the hypersurface M , it is known that they are congruent to each other with respect to the isometry group $I(M)$ of M if and only if their structure torsions ρ_{γ_1} and ρ_{γ_2} satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.

As a matter of course, for each $c(< 0)$ every horosphere of $\mathbf{CH}^n(c)$ has no closed geodesics, because each geodesic on every horosphere is a horocycle in the ambient space $\mathbf{CH}^n(c)$. But for other totally η -umbilic hypersurfaces M , using the above congruence theorem, we can investigate the number of congruence classes of closed geodesics on M . We emphasize that each totally η -umbilic hypersurface M , which is not congruent to a horosphere of $\mathbf{CH}^n(c)$, has countably infinite congruence classes of closed geodesics.

We here pay particular attention to totally η -umbilic hypersurfaces of $\mathbf{CP}^n(c)$. Every geodesic sphere $G(r)$ ($0 < r < \pi/\sqrt{c}$) of $\mathbf{CP}^n(c)$ is a Riemannian

homogeneous space, which is not isometric but diffeomorphic to a standard sphere, with maximal sectional curvature $K = c + (c/4) \cot^2(\sqrt{cr}/2)$ and minimal sectional curvature $(c/4) \cot^2(\sqrt{cr}/2)$. Moreover, when the radius r satisfies $\tan^2(\sqrt{cr}/2) > 2$, this geodesic sphere $G(r)$ in $\mathbf{CP}^n(c)$ is so-called a Berger sphere. That is, the sectional curvatures of $G(r)$ with $\tan^2(\sqrt{cr}/2) > 2$ lie in the interval $[\delta K, K]$ with some $\delta \in (0, 1/9)$ but it has closed geodesics of length shorter than $2\pi/\sqrt{K}$ (see [10]). Indeed, if we take an arbitrary integral curve of the characteristic vector field ξ on $G(r)$, then it is a closed geodesic whose length is $2\pi \sin(\sqrt{cr})/\sqrt{c}$. This length is shorter than $2\pi/\sqrt{K}$ if $\tan^2(\sqrt{cr}/2) > 2$. Furthermore, this closed geodesic is the only closed geodesic on $G(r)$ with respect to the isometry group $\mathbf{I}(G(r))$ of $G(r)$ of length less than $2\pi/\sqrt{K}$.

We here clarify some fundamental properties of geodesics on a complete and simply connected Sasakian space form $M^{2n-1}(c+1)$ of constant ϕ -sectional curvature $c+1$ for each $c(\neq 0)$. By virtue of the above discussion and Lemma 2 we have the following:

- PROPOSITION 2.** (1) *Every geodesic γ on $M^{2n-1}(c+1)$ is a homogeneous curve, that is the curve γ is an orbit of some one-parameter subgroup of the isometry group $\mathbf{I}(M^{2n-1}(c+1))$;*
 (2) *Two geodesics γ_1, γ_2 on $M^{2n-1}(c+1)$ are congruent to each other with respect to the isometry group $\mathbf{I}(M^{2n-1}(c+1))$ of $M^{2n-1}(c+1)$ if and only if their structure torsions ρ_{γ_1} and ρ_{γ_2} satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$;*
 (3) *$M^{2n-1}(-3)$ has no closed geodesics and every geodesic on this manifold can be considered as a horocycle in $\mathbf{CH}^n(-4)$;*
 (4) *$M^{2n-1}(c+1)$, $c \neq -4$ has countably infinite congruence classes of closed geodesics;*
 (5) *When $c > 8$, $M^{2n-1}(c+1)$ is a Berger sphere.*

For other detailed information on the length spectrum, see [3].

5. The metric of a certain Sasakian space form $M^{2n-1}(c+1)$

In this section, we pay particular attention to the metric of a complete and simply connected Sasakian space form with constant ϕ -sectional curvature $c+1$ for each $c > 0$.

We investigate the metric g of i) in Lemma 2. To do this, we must clarify the relation between the metrics of $\mathbf{CP}^n(c)$ and $S^{2n+1}(c/4)$ of constant sectional curvature $c/4$ through the Hopf fibration $\pi : S^{2n+1}(c/4) \rightarrow \mathbf{CP}^n(c)$. We denote by $S^m[R]$ an m -dimensional standard sphere of radius R . So we can set

$$S^{2n+1}[R] = \{z = (z_0, \dots, z_n) \in \mathbf{C}^{n+1} \mid |z_0|^2 + \dots + |z_n|^2 = R^2\} = S^{2n+1}\left(\frac{1}{R^2}\right).$$

As is well-known, the standard inner product \langle , \rangle of \mathbf{C}^{n+1} is given by $\langle X, Y \rangle = \Re(\sum_{i=0}^n X^i \bar{Y}^i)$. Note that the horizontal part $\mathcal{H}_{\hat{z}}$ of $\pi : S^{2n+1}[R] \rightarrow \mathbf{C}P^n(4/R^2)$ at $\hat{z} \in S^{2n+1}[R]$ is expressed as

$$\mathcal{H}_{\hat{z}} = \{(\hat{z}, \hat{X}) \in \{\hat{z}\} \times \mathbf{C}^{n+1} \mid \langle \hat{z}, \hat{X} \rangle = \langle i\hat{z}, \hat{X} \rangle = 0\}.$$

Then the meric g defines the metric \hat{g} of $S^{2n+1}[R]$ which degenerates along the vertical vector $i\hat{z}$ as follows. For any $(\hat{z}, \hat{X}), (\hat{z}, \hat{Y}) \in T_{\hat{z}}S^{2n+1}[R] = \{\hat{X} \in \mathbf{C}^{n+1} \mid \langle \hat{z}, \hat{X} \rangle = 0\}$, we take two horizontal vectors

$$\hat{X} - \left\langle \frac{i\hat{z}}{R}, \hat{X} \right\rangle \frac{i\hat{z}}{R} = \hat{X} - \frac{1}{R^2} \langle i\hat{z}, \hat{X} \rangle i\hat{z}, \quad \hat{Y} - \left\langle \frac{i\hat{z}}{R}, \hat{Y} \right\rangle \frac{i\hat{z}}{R} = \hat{Y} - \frac{1}{R^2} \langle i\hat{z}, \hat{Y} \rangle i\hat{z} \in \mathcal{H}_{\hat{z}}.$$

By direct computation we have

$$\begin{aligned} \hat{g}((\hat{z}, \hat{X}), (\hat{z}, \hat{Y})) &= \left\langle \hat{X} - \frac{1}{R^2} \langle i\hat{z}, \hat{X} \rangle i\hat{z}, \hat{Y} - \frac{1}{R^2} \langle i\hat{z}, \hat{Y} \rangle i\hat{z} \right\rangle \\ &= \langle \hat{X}, \hat{Y} \rangle - \frac{2}{R^2} \langle i\hat{z}, \hat{X} \rangle \langle i\hat{z}, \hat{Y} \rangle + \frac{1}{R^4} \langle i\hat{z}, \hat{X} \rangle \langle i\hat{z}, \hat{Y} \rangle \|i\hat{z}\|^2 \\ &= \langle \hat{X}, \hat{Y} \rangle - \frac{1}{R^2} \langle i\hat{z}, \hat{X} \rangle \langle i\hat{z}, \hat{Y} \rangle, \end{aligned}$$

so that

$$(5.1) \quad \hat{g}((\hat{z}, \hat{X}), (\hat{z}, \hat{Y})) = \langle \hat{X}, \hat{Y} \rangle - \frac{1}{R^2} \langle i\hat{z}, \hat{X} \rangle \langle i\hat{z}, \hat{Y} \rangle$$

for each $(\hat{z}, \hat{X}), (\hat{z}, \hat{Y}) \in T_{\hat{z}}S^{2n+1}[R]$.

On the other hand, we denote by $G(r) = G(r; 4/R^2)$ a geodesic sphere of radius r ($0 < r < R\pi/2$) in $\mathbf{C}P^n(4/R^2)$. The inverse image $\pi^{-1}(G(r))$ is expressed as

$$\pi^{-1}(G(r)) = S^1[R_1] \times S^{2n-1}[R_2], \quad R_1^2 + R_2^2 = R^2.$$

We here remark that $R_1 = R \cos(r/R)$ and $R_2 = R \sin(r/R)$. Indeed, we can set $R_1 = R \cos \theta$, $R_2 = R \sin \theta$. It is known that the hypersurface $\pi^{-1}(G(r))$ of $S^{2n+1}[R]$ has two constant principal curvatures $-R_2/(RR_1)$, $R_1/(RR_2)$. This shows that the principal curvatures λ_1, δ of the geodesic sphere $G(r)$ are expressed as $\lambda_1 = R_1/(RR_2)$, $\delta = (R_1/(RR_2)) - (R_2/(RR_1))$. On the other hand, λ_1 can be expressed as $\lambda_1 = (\sqrt{c}/2) \cot(\sqrt{cr}/2) = (1/R) \cot(r/R)$. We hence find that $\theta = r/R$.

Next, in order to define a diffeomorphism $\varphi : S^{2n-1}(1) (= S^{2n-1}[1]) \rightarrow G(r)$, we consider a diffeomorphism $\tilde{\varphi} : S^{2n-1}(1) \rightarrow \tilde{\varphi}(S^{2n-1}(1))$ ($\subset S^1[R_1] \times S^{2n-1}[R_2]$) given by $\tilde{\varphi}(z) = (R_1, R_2 z)$. Then we get a desirable mapping $\varphi : S^{2n-1}(1) \rightarrow G(r)$ as $\varphi = \pi \circ \tilde{\varphi}$. We here note that

$$(5.2) \quad (d\tilde{\varphi})_z(z, X) = (0, R_2 X).$$

We shall compute the metric $g_{\#}$ on $S^{2n-1}(1)$ defined by $g_{\#} = \varphi^*g$, which is the pullback of g by φ . It follows from (5.1) and (5.2) that

$$\begin{aligned} g_{\#}((z, X), (z, Y)) &= \hat{g}(((R_1, R_2z), (0, R_2X)), ((R_1, R_2z), (0, R_2Y))) \\ &= \langle (0, R_2X), (0, R_2Y) \rangle \\ &\quad - \frac{1}{R^2} \langle i(R_1, R_2z), (0, R_2X) \rangle \langle i(R_1, R_2z), (0, R_2Y) \rangle \\ &= R_2^2 \langle X, Y \rangle - \frac{R_2^4}{R^2} \langle iz, X \rangle \langle iz, Y \rangle \\ &= R_2^2 (\langle X, Y \rangle - \langle iz, X \rangle \langle iz, Y \rangle) + \frac{R_1^2 R_2^2}{R^2} \langle iz, X \rangle \langle iz, Y \rangle. \end{aligned}$$

On the other hand, Remark 1(2) gives us equations $R_1 = RR_2$, $R_2/(RR_1) = c/4$ and $R_1^2 + R_2^2 = R^2$. Thus we see $R = 2/\sqrt{c}$, $R_1 = 4/\sqrt{c(c+4)}$, $R_2 = 2/\sqrt{c+4}$. We hence know that the metric g of the geodesic sphere $G(r)$ is realized as the following metric $g_{\#}$ which is nothing but the deformation of the standard metric g_0 of $S^{2n-1}(1)$:

$$g_{\#} = \frac{4}{c+4} (g_0 - \eta \otimes \eta) + \left(\frac{4}{c+4} \right)^2 \eta \otimes \eta.$$

Therefore we conclude that the metric g of i) in Lemma 2 coincides with the well-known metric on a standard example of a complete and simply connected Sasakian space form with constant ϕ -sectional curvature $c+1$ for each $c > 0$ (see [5]).

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