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KODAI MATH. J.
33 (2010), 240–250

ON HOPF HYPERSURFACES IN A NON-FLAT COMPLEX SPACE FORM WITH η-RECURRENT RICCI TENSOR

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Abstract

Baikoussis, Lyu and Suh [1] showed that a Hopf hypersurface M in a non-flat complex space form $M_n(c)$ with constant mean curvature and with η-recurrent Ricci tensor is locally congruent to one of real hypersurfaces of type A and B. They also conjectured that the same result can be obtained even without the constancy assumption on the mean curvature (cf. [1, Remark 5.1.]). The purpose of this paper is to answer this question in the affirmative.

1. Introduction

Let $M_n(c)$ be an n-dimensional non-flat complex space form with constant holomorphic sectional curvature $4c$. A complete and simply connected non-flat complex space form is either a complex projective space $\mathbf{C}P^n$ or a complex hyperbolic space $\mathbf{C}H^n$, according to as $c > 0$ or $c < 0$. Let M be a real hypersurface in $M_n(c)$. Then the complex structure J of $M_n(c)$ induces an almost contact metric structure $(\phi, \xi, \eta, \langle\,,\,\rangle)$ on M. If the structure vector field ξ of M is principal then M is called a *Hopf hypersurface*. Typical examples of Hopf hypersurfaces in $M_n(c)$ are the homogeneous one with constant principal curvatures, nowadays known as real hypersurfaces of type A_1, A_2, B, C, D, E when the ambient space is $\mathbf{C}P^n$; and of type A_0, A_1, A_2, B when the ambient space is $\mathbf{C}H^n$ (cf. [2, 12]).

In the following, we denote by $\Gamma(\mathscr{V})$ the module of all differentiable sections on the vector bundle $\mathscr{V}$ over M.

It is well known that there are no real hypersurfaces M in $M_n(c)$ with parallel Ricci tensor S, i.e., $\nabla S = 0$ (cf. [6]), where ∇ denotes the Levi-Civita connection on M. Consequently, it is natural to consider a weaker form of the parallelism condition on S for real hypersurfaces in $M_n(c)$. The *holomorphic distribution D* on M is the distribution that is orthogonal to ξ, i.e.,

$$D_x = \{X \in T_xM \mid \langle X, \xi\rangle = 0\}, \quad x \in M.$$

2000 *Mathematics Subject Classification*. Primary 53C40; Secondary 53C15.

Key words and phrases. Complex space form, Hopf hypersurfaces, η-recurrent Ricci tensor.
Received January 20, 2009.

In [11], Suh weakened the parallelism condition on S to so-called η-parallelism condition i.e., the Ricci tensor S is said to be *η-parallel* if

$$\langle (\nabla_X S) Y, Z \rangle = 0$$

for any X, Y and $Z \in \Gamma(D)$; and gave a classification of Hopf hypersurfaces in $M_n(c)$ with η-parallel Ricci tensor.

The Ricci tensor S of a real hypersurface M is said to be *recurrent* if there exists a 1-form ψ on M such that

$$\nabla S = S \otimes \psi.$$

The parallelism on S may be regarded as a special case of recurrence on S. The non-existence problem of real hypersurfaces with recurrent Ricci tensor in $M_n(c)$ was initiated by Hamada [5], and it has been solved in [4] and [8].

On the other hand, Baikoussis, Lyu and Suh introduced a weaker notion of η-recurrence on S, i.e., the Ricci tensor S is said to be *η-recurrent* if there exists a 1-form ψ on M such that (cf. [1])

$$\langle (\nabla_X S) Y, Z \rangle = \psi(X) \langle SY, Z \rangle$$

for any $X, Y, Z \in \Gamma(D)$, where D is the *holomorphic distribution* on M defined as follows

$$D_x = \{X \in T_x M \mid \langle X, \xi \rangle = 0\}, \quad x \in M.$$

In [1], Baikoussis, Lyu and Suh proved the following

Theorem 1.1. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 3$, with constant mean curvature. If the Ricci tensor S is η-recurrent, then M is locally congruent to one of the following real hypersurfaces*:

(a) *For $c > 0$*:
  (A_1) *a tube over hyperplane $\mathbf{C}P^{n-1}$*;
  (A_2) *a tube over totally geodesic $\mathbf{C}P^k$, where $1 \leq k \leq n-2$*;
  (B) *a tube over complex quadric Q_{n-1}*.

(b) *For $c < 0$*:
  (A_0) *a horosphere*;
  (A_1) *a geodesic hypersphere or a tube over hyperplane $\mathbf{C}H^{n-1}$*;
  (A_2) *a tube over totally geodesic $\mathbf{C}H^k$, where $1 \leq k \leq n-2$*;
  (B) *a tube over totally real hyperbolic space $\mathbf{R}H^n$*.

They also conjectured that the same result can be obtained even without the constancy assumption on the mean curvature (cf. [1, Remark 5.1.]). The purpose of this paper is to answer this question in the affirmative, i.e., we shall slightly improve Theorem 1.1 to the following

Theorem 1.2. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 3$. If the Ricci tensor S is η-recurrent, then M is locally congruent to one of the following real hypersurfaces*:

(a) *For* $c > 0$:
(A$_1$) *a tube over hyperplane* $\mathbf{C}P^{n-1}$;
(A$_2$) *a tube over totally geodesic* $\mathbf{C}P^k$, *where* $1 \leq k \leq n-2$;
(B) *a tube over complex quadric* Q_{n-1}.
(b) *For* $c < 0$:
(A$_0$) *a horosphere*;
(A$_1$) *a geodesic hypersphere or a tube over hyperplane* $\mathbf{C}H^{n-1}$;
(A$_2$) *a tube over totally geodesic* $\mathbf{C}H^k$, *where* $1 \leq k \leq n-2$;
(B) *a tube over totally real hyperbolic space* $\mathbf{R}H^n$.

2. Preliminaries

Let M be a connected real hypersurface in $M_n(c)$, $n \geq 3$, and let N be a unit normal vector field on M. Denote by $\bar{\nabla}$ and ∇ respectively the Levi-Civita connection on $M_n(c)$ and the connection induced on M. Then the Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N$$
$$\bar{\nabla}_X N = -AX$$

for any $X, Y \in \Gamma(TM)$, where $\langle\, ,\rangle$ denotes the Riemannian metric of M induced from the Riemannian metric of $M_n(c)$. Now, we define a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = \langle \xi, X\rangle. \tag{1}$$

Then the set of tensors $(\phi, \xi, \eta, \langle\, ,\rangle)$ satisfy the following

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1 \tag{2}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi, \quad \nabla_X \xi = \phi AX. \tag{3}$$

Let R be the curvature tensor of M. Then the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned} R(X,Y)Z = c\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle \phi Y, Z\rangle \phi X - \langle \phi X, Z\rangle \phi Y \\ &- 2\langle \phi X, Y\rangle \phi Z\} + \langle AY, Z\rangle AX - \langle AX, Z\rangle AY \end{aligned}$$
$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y\rangle \xi\}.$$

It follows from the Gauss equation that the Ricci tensor S of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2 X. \tag{4}$$

where $h = \operatorname{trace} A$, called the *mean curvature* on M, and the covariant derivative of the Ricci tensor S is given by

$$\begin{aligned} (\nabla_X S)Y = &-3c\{\langle \phi AX, Y\rangle \xi + \eta(Y)\phi AX\} + (Xh)AY \\ &+ (hI - A)(\nabla_X A)Y - (\nabla_X A)AY. \end{aligned} \tag{5}$$

Further, we define the second covariant derivative $\nabla_X \nabla_Y S$ by

$$(\nabla_X \nabla_Y S)Z = \nabla_X\{(\nabla_Y S)Z\} - (\nabla_{\nabla_X Y} S)Z - (\nabla_Y S)\nabla_X Z.$$

The Ricci tensor S of M is said to be *η-parallel* if

$$\langle (\nabla_X S)Y, Z\rangle = 0$$

for any X, Y and $Z \in \Gamma(D)$.

An eigenvalue of the shape operator tensor A of M is called a *principal curvature* and a *principal curvature vector* is an eigenvector of A. A real hypersurface M in $M_n(c)$ is called a *Hopf hypersurface* if the structure tensor field ξ is principal, i.e., we have $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$. The following theorem characterized Hopf hypersurfaces M in $M_n(c)$ with η-parallel Ricci tensor.

Theorem 2.1 ([11]). *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 3$, with η-parallel Ricci tensor. Then M is locally congruent to one of real hypersurfaces of type A_1, A_2 and B (for $c > 0$); or type A_0, A_1, A_2 and B (for $c < 0$).*

3. Basic properties of Hopf hypersurfaces

In this section, we shall derive some basic properties about Hopf hypersurfaces M in $M_n(c)$. In the following, we suppose M is a connected Hopf hypersurface in $M_n(c)$. Further, we denote by $\mathrm{Spec}_A(D)$ and $\mathrm{Spec}_S(D)$ respectively the spectrum of $A|_D$ and $S|_D$. For each $\lambda \in \mathrm{Spec}_A(D)$, we denote by T_λ the subbundle of D foliated by the eigenspace of $A|_D$ corresponding to λ.

We first recall

Lemma 3.1 ([7], [10]). *Let M be a Hopf hypersurface in $M_n(c)$. Then*

1. *the principal curvature α is constant*;
2. $2(A\phi A - c\phi) = \alpha(\phi A + A\phi)$;
3. *if $Y \in T_\lambda$ and $\phi Y \in T_{\tilde{\lambda}}$, then* $2(\lambda\tilde{\lambda} - c) = \alpha(\lambda + \tilde{\lambda})$
4. $\nabla_\xi A = (\alpha/2)(\phi A - A\phi)$.

Consider a unit principal vector field $Y \in T_\lambda$, it follows from the above lemma that

$$\xi\lambda = \langle (\nabla_\xi A)Y, Y\rangle = \frac{\alpha}{2}\langle (\phi A - A\phi)Y, Y\rangle = 0.$$

This implies that $\xi h = 0$. Further, by (4) we can see that each principal curvature of M induces an eigenvalue of S, i.e., $SY = \sigma Y$, where $\sigma = (2n+1)c + h\lambda - \lambda^2$, for $\lambda \in \mathrm{Spec}_A(D)$ and $Y \in T_\lambda$; and $S\xi = v\xi$, where $v = (2n-2)c + h\alpha - \alpha^2$. Since $\xi\alpha = \xi\lambda = \xi h = 0$, we also obtain $\xi\sigma = \xi v = 0$. On the other hand, it follows from the Codazzi equation, (5) and Lemma 3.1(4) that we have

$$\nabla_\xi S = \frac{\alpha}{2}(\phi S - S\phi)$$

$$(\nabla_X S)\xi = c\{-3\phi AX - (h-\alpha)\phi X + A\phi X\} + \alpha(Xh)\xi + \frac{\alpha}{2}((h-\alpha)I - A)(\phi A - A\phi)X$$

for any $X \in \Gamma(TM)$. We summarize the above observation in the following lemma.

Lemma 3.2. *Let M be a Hopf hypersurface in $M_n(c)$. Then*
1. *The principal curvatures, eigenvalues of S and the mean curvature h are constant along the integral curves of ξ;*
2. $\nabla_\xi S = (\alpha/2)(\phi S - S\phi)$;
3. *for any $X \in \Gamma(TM)$, we have*

$$(\nabla_X S)\xi = c\{-3\phi AX - (h-\alpha)\phi X + A\phi X\} + \alpha(Xh)\xi + \frac{\alpha}{2}((h-\alpha)I - A)(\phi A - A\phi)X.$$

4. Principal curvatures of Hopf hypersurfaces with η-recurrent Ricci tensor

In this section, we shall begin the proof of Theorem 1.2, which will be completed in the next section. Our plan goes as follows: we first prove that under the assumptions of Theorem 1.2, the Ricci tensor S is η-parallel; and then by invoking Theorem 2.1, we conclude that M is either of type A (i.e., A_1, A_2 for $c > 0$ and A_0, A_1, A_2 for $c < 0$) or B.

Throughout this section, we suppose M is a connected Hopf hypersurface in $M_n(c)$, $n \geq 3$, with η-recurrent Ricci tensor.

For any $\sigma \in \mathrm{Spec}_S(D)$ with $SY = \sigma Y$, where Y is a unit vector field in $\Gamma(D)$, it follows from the η-recurrency condition that

$$X\sigma = X\langle SY, Y\rangle = \langle(\nabla_X S)Y, Y\rangle = \psi(X)\langle SY, Y\rangle = \sigma\psi(X)$$

for any $X \in \Gamma(D)$. Together with the fact that $\xi\sigma = 0$, we may define a 1-form Ψ as follows: $\Psi(\xi) = 0$ and $\Psi(X) = \psi(X)$, for any $X \in \Gamma(D)$ so that we have

$$d\sigma = \sigma\Psi \tag{6}$$

and the η-recurrent condition on S can be rewritten as

$$\langle(\nabla_Y S)Z, W\rangle = \Psi(Y)\langle SZ, W\rangle \tag{7}$$

for any $Y, Z, W \in \Gamma(D)$.

Now, from the equation (6) we obtain

$$0 = d^2\sigma = d\sigma \wedge \Psi + \sigma\, d\Psi = \sigma\, d\Psi. \tag{8}$$

Next, for $\lambda_1, \dots, \lambda_{2n-2} \in \mathrm{Spec}_A(D)$ (here, each $\lambda_j \in \mathrm{Spec}_A(D)$ not necessarily distinct), we denote by $\sigma_j \in \mathrm{Spec}_S(D)$ that correspond to λ_j, i.e.,

$$\sigma_j = (2n+1)c + h\lambda_j - \lambda_j^2 \tag{9}$$

for $1 \le j \le 2n-2$. Moreover, we put $\mathscr{G}_j = \{x \in M \mid \sigma_j(x) \neq 0\}$ and $\mathscr{G}$ the union of these open sets $\mathscr{G}_j$. In the rest of this section, unless otherwise stated, we restrict our arguments on the open set $\mathscr{G}$.

We now prove the following

LEMMA 4.1. *On the open set* $\mathscr{G}$, *we have*

$$\begin{aligned}\langle (R(X,Y)S)Z, W\rangle = \langle (\phi A + A\phi)X, Y\rangle \langle (\nabla_\xi S)Z, W\rangle \\ + \langle \phi AX, Z\rangle \langle (\nabla_Y S)\xi, W\rangle - \langle \phi AY, Z\rangle \langle (\nabla_X S)\xi, W\rangle \\ + \langle \phi AX, W\rangle \langle (\nabla_Y S)Z, \xi\rangle - \langle \phi AY, W\rangle \langle (\nabla_X S)Z, \xi\rangle\end{aligned}$$

for any $X, Y, Z, W \in \Gamma(D)$.

Proof. Note that at each point $x \in \mathscr{G}$, there is at least one $\sigma \in \mathrm{Spec}_S(D)$ such that $\sigma(x) \neq 0$. Hence, on the open set $\mathscr{G}$, it follows from (8) that $d\Psi = 0$, or equivalently, $(\nabla_X \Psi)Y = (\nabla_Y \Psi)X$, for any $X, Y \in \Gamma(TM)$.

By differentiating (7) in the direction of $X \in \Gamma(D)$, we obtain

$$\begin{aligned}\langle (\nabla_X \nabla_Y S)Z + (\nabla_{\nabla_X Y} S)Z + (\nabla_Y S)\nabla_X Z, W\rangle + \langle (\nabla_Y S)Z, \nabla_X W\rangle \\ = \{(\nabla_X \Psi)Y + \Psi(\nabla_X Y)\}\langle SZ, W\rangle + \Psi(Y)\{\langle (\nabla_X S)Z, W\rangle \\ + \langle S\nabla_X Z, W\rangle + \langle SZ, \nabla_X W\rangle\}\end{aligned} \tag{10}$$

for any $Y, Z, W \in \Gamma(D)$. On the other hand, by using (2) and (3), we have

$$\begin{aligned}\nabla_X Y &= (\nabla_X Y)^\circ + \eta(\nabla_X Y)\xi, \quad (\text{where } (\nabla_X Y)^\circ = -\phi^2 \nabla_X Y) \\ &= (\nabla_X Y)^\circ - \langle \phi AX, Y\rangle \xi,\end{aligned}$$

for any $X, Y \in \Gamma(D)$. This, together with (7), (10) and the fact that $SX \perp \xi$, for $X \perp \xi$, give

$$\begin{aligned}\langle (\nabla_X \nabla_Y S)Z, W\rangle - \langle \phi AX, Y\rangle \langle (\nabla_\xi S)Z, W\rangle - \langle \phi AX, Z\rangle \langle (\nabla_Y S)\xi, W\rangle \\ - \langle \phi AX, W\rangle \langle (\nabla_Y S)Z, \xi\rangle \\ = (\nabla_X \Psi)Y \cdot \langle SZ, W\rangle + \Psi(Y)\Psi(X)\langle SZ, W\rangle.\end{aligned}$$

By taking account of the Ricci identity, $(R(X,Y)S)Z = (\nabla_X \nabla_Y S)Z - (\nabla_Y \nabla_X S)Z$ and the above equation, we obtain the statement. □

LEMMA 4.2. *If* $\lambda \in \mathrm{Spec}_A(D)$ *and* $\lambda \neq \alpha/2$, *then*

$$(\lambda - \tilde{\lambda})(h - \lambda - \tilde{\lambda})(\alpha(\lambda + \tilde{\lambda}) + 4c) = 0$$

on $\mathscr{G}$, *where* $\tilde{\lambda} = (\alpha\lambda + 2c)/(2\lambda - \alpha)$.

Proof. Let Y be a unit vector field in $\Gamma(T_\lambda)$. Then by Lemma 3.1, $A\phi Y = \tilde{\lambda}\phi Y$. Moreover, from Lemma 3.2 we obtain

$$\langle(\nabla_\xi S)Y, \phi Y\rangle = \frac{\alpha}{2}(\lambda - \tilde{\lambda})(h - \lambda - \tilde{\lambda})$$

$$\langle(\nabla_Y S)\xi, \phi Y\rangle = (-3\lambda - (h - \alpha) + \tilde{\lambda})c + \frac{\alpha}{2}(\lambda - \tilde{\lambda})(h - \alpha - \tilde{\lambda})$$

$$\langle(\nabla_{\phi Y} S)Y, \xi\rangle = (3\tilde{\lambda} + (h - \alpha) - \lambda)c + \frac{\alpha}{2}(\lambda - \tilde{\lambda})(h - \alpha - \lambda).$$

Next, by putting $X = W = \phi Y$, $Z = Y$ in Lemma 4.1, making use of the Gauss equation and the above three equations, we have

$$\begin{aligned}
&(\lambda - \tilde{\lambda})(h - \lambda - \tilde{\lambda})(4c + \lambda\tilde{\lambda}) \\
&\quad = -\frac{\alpha}{2}(\lambda + \tilde{\lambda})(\lambda - \tilde{\lambda})(h - \lambda - \tilde{\lambda}) - \tilde{\lambda}\{(-3\lambda - (h - \alpha) + \tilde{\lambda})c \\
&\qquad + \frac{\alpha}{2}(\lambda - \tilde{\lambda})(h - \alpha - \tilde{\lambda})\} - \lambda\{(3\tilde{\lambda} + (h - \alpha) - \lambda)c + \frac{\alpha}{2}(\lambda - \tilde{\lambda})(h - \alpha - \lambda)\} \\
&\quad = -(\lambda - \tilde{\lambda})(h - \lambda - \tilde{\lambda})(\alpha(\lambda + \tilde{\lambda}) + c) + \alpha(\lambda - \tilde{\lambda})\left(c + \frac{\alpha}{2}(\lambda + \tilde{\lambda}) - \lambda\tilde{\lambda}\right).
\end{aligned}$$

By using Lemma 3.1(3), this equation reduces to

$$(\lambda - \tilde{\lambda})(h - \lambda - \tilde{\lambda})(\alpha(\lambda + \tilde{\lambda}) + 4c) = 0. \qquad \square$$

LEMMA 4.3. *If* $\alpha/2 \in \mathrm{Spec}_A(D)$ *then* $\Psi = 0$ *on* $\mathscr{G}$.

Proof. Suppose $\alpha/2 \in \mathrm{Spec}_A(D)$, then by putting $\lambda = \alpha/2$ in Lemma 3.1(3) we get $\alpha^2 = -4c$ and so $c < 0$, (without lose of generality, we assume $c = -1$), hence, we get $\alpha^2 = 4$. If $\mathrm{Spec}_A(D) = \{\alpha/2\}$, then our statement is clearly true.

Now, suppose that there exists $\lambda \in \mathrm{Spec}_A(D)$, $\lambda \neq \alpha/2$ and let $Y \in \Gamma(T_\lambda)$. It follows from Lemma 3.1 that we have

$$\left(\lambda - \frac{\alpha}{2}\right)\tilde{\lambda} = \frac{\alpha}{2}\left(\lambda - \frac{2}{\alpha}\right).$$

By making use of the fact that $\alpha/2 = 2/\alpha$, we get $\tilde{\lambda} = \alpha/2$. Furthermore, since both $\lambda - \tilde{\lambda}$ and $\alpha(\lambda + \tilde{\lambda}) + 4c$ are nonzero, from Lemma 4.2 we get $h - \lambda - \tilde{\lambda} = 0$, and hence $\lambda = h - \alpha/2$, which means that M admits at most three distinct principal curvatures, α with multiplicity 1, $\lambda_1 = \alpha/2$, with multiplicity $2n - 2 - m$ and $\lambda_2 = h - \alpha/2$ with multiplicity m. Next, observe that

$$h = \alpha + (2n - 2 - m)\lambda_1 + m\lambda_2.$$

Thus, we obtain $(1 - m)h = \alpha(n - m)$ and so by (9), we obtain $\sigma_1 = -(2n + 1) - 2(n - m)/(m - 1) - 1$, which is locally a nonzero constant on $\mathscr{G}$. Consequently, we get $\Psi = 0$ by using (6). $\square$

LEMMA 4.4. *If* $\alpha/2 \notin \mathrm{Spec}_A(D)$ *then* $\Psi = 0$ *on* $\mathscr{G}$.

Proof. We consider the open subset $\mathscr{H}_j = \{x \in \mathscr{G} \mid (\lambda_j - \tilde{\lambda}_j)(h - \lambda_j - \tilde{\lambda}_j) \neq 0\}$. Then on such open subset $\mathscr{H}_j$, we have $\alpha(\lambda_j + \tilde{\lambda}_j) + 4c = 0$, so both λ_j, $\tilde{\lambda}_j$ are locally constant and $\alpha \neq 0$. Moreover, from Lemma 3.1, λ_j and $\tilde{\lambda}_j$ can also be related by $\lambda_j \tilde{\lambda}_j + c = 0$. Now, by using (6), we get

$$d[(2n+1)c + \lambda_j h - \lambda_j^2] = [(2n+1)c + \lambda_j h - \lambda_j^2]\Psi.$$

As λ_j is a constant, we have

$$\lambda_j \, dh = [(2n+1)c + \lambda_j h - \lambda_j^2]\Psi.$$

Similarly, we also have

$$\tilde{\lambda}_j \, dh = [(2n+1)c + \tilde{\lambda}_j h - \tilde{\lambda}_j^2]\Psi.$$

These imply that $(\lambda_j - \tilde{\lambda}_j)\, dh = (\lambda_j - \tilde{\lambda}_j)(h - \lambda_j - \tilde{\lambda}_j)\Psi$. Since $\lambda_j \neq \tilde{\lambda}_j$, we obtain

$$dh = (h - \lambda_j - \tilde{\lambda}_j)\Psi.$$

On the other hand, taking account of $\lambda_j \tilde{\lambda}_j = -c$, we have

$$(\lambda_j + \tilde{\lambda}_j)\, dh = 4nc\Psi + (\lambda_j + \tilde{\lambda}_j)(h - \lambda_j - \tilde{\lambda}_j)\Psi.$$

From the above two equations, we obtain $\Psi = 0$ on $\mathscr{H}_j$, for $1 \leq j \leq 2n-2$.

Next, we look at the interior set, $\mathrm{Int}(\mathscr{G} - \mathscr{H})$ of $\mathscr{G} - \mathscr{H}$, where $\mathscr{H} = \bigcup\{\mathscr{H}_j \mid 1 \leq j \leq 2n-2\}$. From Lemma 4.2, each $\lambda \in \mathrm{Spec}_A(D)$ is the solution of

$$(\lambda - \tilde{\lambda})(h - \lambda - \tilde{\lambda}) = 0.$$

Hence, M has at most five distinct principal curvatures: α (with multiplicity 1); λ_1 (with multiplicity $2m_1$); λ_2 (with multiplicity $2m_2$); λ_3, $\lambda_4 = \tilde{\lambda}_3$ (both with multiplicity m_3), where $n - 1 = m_1 + m_2 + m_3$; λ_1, λ_2 are the solutions of $\lambda - \tilde{\lambda} = 0$ and $h - \lambda_3 - \tilde{\lambda}_3 = 0$.

By making use of Lemma 3.1(3), the equations

$$\lambda - \tilde{\lambda} = 0 \quad \text{and} \quad h - \lambda - \tilde{\lambda} = 0$$

can be rewritten as

$$\lambda^2 - \alpha\lambda - c = 0 \tag{11}$$

and

$$2\lambda^2 - 2h\lambda + (h\alpha + 2c) = 0, \tag{12}$$

respectively. Since λ_1 and λ_2 are the solutions of the equation (11), we can see that both λ_1, λ_2 are locally constant and these principal curvatures satisfy the following relationship

$$\lambda_1 + \lambda_2 = \alpha, \quad \lambda_1 \lambda_2 + c = 0.$$

Similarly, since λ_3 and λ_4 are the solutions of the equation (12), we also get

$$\lambda_3 + \tilde{\lambda}_3 = h, \quad 2\lambda_3\tilde{\lambda}_3 = h\alpha + 2c.$$

Moreover, we have

$$\begin{aligned} h &= \alpha + 2m_1\lambda_1 + 2m_2\lambda_2 + m_3(\lambda_3 + \tilde{\lambda}_3) \\ &= (2m_1 + 1)\lambda_1 - (2m_2 + 1)\frac{c}{\lambda_1} + m_3(\lambda_3 + \tilde{\lambda}_3). \end{aligned}$$

Since $h = \lambda_3 + \tilde{\lambda}_3$, we obtain

$$(2m_1 + 1)\lambda_1 - (2m_2 + 1)\frac{c}{\lambda_1} + (m_3 - 1)h = 0. \tag{13}$$

Now, we consider two cases: (i) $m_3 \neq 1$ and (ii) $m_3 = 1$.

Case (i): $m_3 \neq 1$. The equation (13) shows that h is locally constant and hence from Lemma 4.2, we see that all $\lambda \in \mathrm{Spec}_A(D)$ are also locally constant on $\mathrm{Int}(\mathscr{G} - \mathscr{H})$. From these observations, together with (6) and (9), give $\Psi = 0$ on $\mathrm{Int}(\mathscr{G} - \mathscr{H})$.

Case (ii): $m_3 = 1$. In this case, the equation (13) reduces to $(2m_1 + 1)\lambda_1 - (2m_2 + 1)(c/\lambda_1) = 0$. Therefore, we obtain

$$\lambda_1^2 = \frac{2m_2 + 1}{2m_1 + 1}c. \tag{14}$$

This implies that $c > 0$ (for convenience, we assume $c = 1$). Next, by using Lemma 3.1, the scalar curvature ρ (:= trace S) is given by

$$\begin{aligned} \rho &= 4n^2 - 4 + h^2 - \langle A, A \rangle \\ &= 4n^2 - 4 + (\lambda_3 + \tilde{\lambda}_3)^2 - \alpha^2 - 2m_1\lambda_1^2 - 2m_2\lambda_2^2 - \lambda_3^2 - \tilde{\lambda}_3^2 \\ &= 4n^2 - 2 + h\alpha - \alpha^2 - 2m_1\lambda_1^2 - 2m_2\frac{1}{\lambda_1^2}. \end{aligned}$$

Let $v = \langle S\xi, \xi \rangle = 2n - 2 + h\alpha - \alpha^2$. Then by the above equation, (14) and the fact that $n - 1 = m_1 + m_2 + 1$

$$\begin{aligned} \rho - v &= 4n^2 - 2n - 2m_1\lambda_1^2 - 2m_2\frac{1}{\lambda_1^2} \\ &= 4n^2 - 4n + 2 + \lambda_1^2 + \frac{1}{\lambda_1^2}. \end{aligned} \tag{15}$$

On the other hand, we have

$$\rho - v = 2m_1\sigma_1 + 2m_2\sigma_2 + \sigma_3 + \tilde{\sigma}_3$$

where $\tilde{\sigma}_3 = 2n + 1 + h\tilde{\lambda}_3 - \tilde{\lambda}_3^2$. It follows from (6) and the above equation that

$$d(\rho - v) = (\rho - v)\Psi.$$

By (15), we can see that $\rho - v$ is locally a positive constant, together with the above equation, yield $\Psi = 0$ on $\mathrm{Int}(\mathscr{G} - \mathscr{H})$. Hence, by the continuity of Ψ, we conclude that $\Psi = 0$ on $\mathscr{G}$. □

5. Proof of Theorem 1.2

Note that on the interior set $\mathrm{Int}(M - \mathscr{G})$ of $M - \mathscr{G}$, the Ricci tensor S is of the form

$$SX = aX + v\eta(X)\xi$$

for any $X \in \Gamma(TM)$, with $a = 0$ and $v = (2n - 2)c + h\alpha - \alpha^2$. This shows that each connected component of $\mathrm{Int}(M - \mathscr{G})$ is congruent to an open part of a pseudo-Einstein real hypersurface (for precise definition of pseudo-Einstein real hypersurfaces, see [3] and [9]) and according to [10, Theorem 6.12], the Ricci tensor S of a pseudo-Einstein real hypersurface in $M_n(c)$ is η-parallel. Thus, we get $\Psi = 0$ on $\mathrm{Int}(M - \mathscr{G})$. Moreover, by the results in Section 4 and the continuity of Ψ, we obtain that Ψ is identically zero on the whole of M, i.e., the Ricci tensor S is η-parallel. Hence, our statement follows from Theorem 2.1.

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