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KODAI MATH. J.
33 (2010), 233–239

THE CLASSES OF BOUNDED HARMONIC FUNCTIONS AND HARMONIC FUNCTIONS WITH FINITE DIRICHLET INTEGRALS ON HYPERBOLIC RIEMANN SURFACES

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Dedicated to Professor Yoichi Imayoshi on his sixtieth birthday

Introduction

For an open Riemann surface R, we denote by $HP(R)$, $HB(R)$ and $HD(R)$ the class of differences of positive harmonic functions on R, bounded harmonic functions on R, and harmonic functions with finite Drichlet integrals on R, respectively. And denote by $HP_+(R)$, $HB_+(R)$ and $HD_+(R)$ the class of positive harmonic functions on R, bounded and positive harmonic functions on R, and positive harmonic functions with finite Drichlet integrals on R, respectively. Note that $HX(R) = HX_+(R) - HX_+(R)$, $X = P, B, D$. It is easily seen that $HB(R) \subset HP(R)$ and $HD(R) \subset HP(R)$ (cf. [9]).

We say that an open Riemann surface R is *parabolic* (resp. *hyperbolic*) if R does not admit (resp. admits) Green's functions on R. It is well-known that, if R is parabolic, then $HP(R)$, $HB(R)$ and $HD(R)$ consist of constant functions (cf. [9]).

Hereafter, we consider only hyperbolic Riemann surfaces R. Let $\Delta = \Delta^{R,M}$ and $\Delta_1 = \Delta_1^{R,M}$ the *Martin boundary* of R and the *minimal Martin boundary* of R, respectively. We refer to [2] for details about the Martin boundary.

In [4] (resp. [5]) we gave necessary and sufficient conditions in terms of Martin boundary in order that the converse $HB(R) \supset HP(R)$ (resp. $HD(R) \supset HP(R)$) of the above respective inclusion relations hold.

Though the inclusion relation between $HB(R)$ and $HD(R)$ does not generally hold, it seems to be an interesting problem to give a necessary and sufficient condition in order that $HB(R)$ coincides with $HD(R)$. The purpose of this article is to prove the following

Main Theorem. *Suppose that R is hyperbolic. Then the followings are equivalent by pairs*:

Received May 19, 2009; revised November 10, 2009.

(i) $HB(R) = HD(R)$;

(ii) *there exists a null set N of Δ with respect to the harmonic measure such that $\Delta_1 \backslash N$ consists of finitely many points with positive harmonic measures whose Martin functions have finite Dirichlet integrals*;

(iii) $\dim HB(R) = \dim HD(R) < \infty$,

where $\dim HX(R)$ *is the dimension of the linear space* $HX(R)$, $X = B, D$.

Finally the author would like to express his deepest gratitude to Prof. S. Segawa for his valuable comment and at the same time to a referee for his helpful advice. He told the author that Prof. M. Nakai [8] gave an alternative proof for Main Theorem.

1. Preliminaries

In this section we state several propositions in order to prove Main Theorem in Introduction in the next section.

Let z_0 be a specified point of R, which serves as a reference point. Denote by $\omega_z(\cdot)$ the *harmonic measure* on Δ with respect to $z \in R$. We also denote by $k_\zeta(z)$ $((\zeta, z) \in (R \cup \Delta) \times R)$ the *Martin function* on R with pole at ζ.

First we give a characterization for boundedness of Martin function.

PROPOSITION 1 (cf. [2, Hilfssatz 13.3]). *Let ζ belong to Δ_1. Then the Martin function $k_\zeta(\cdot)$ with pole at ζ is bounded on R if and only if the harmonic measure $\omega_\cdot(\{\zeta\})$ of the singleton $\{\zeta\}$ is positive.*

Next we review for fundamental properties concerning $HD(R)$.

DEFINITION 1. Fix $z_0 \in R$. For $h, u \in HD(R)$, set

$$D(h, u) := \int_R (\operatorname{grad} h(z), \operatorname{grad} u(z))\, dv(z),$$

where $\operatorname{grad} h(z)$ is the gradient of h at z, $(\operatorname{grad} h(z), \operatorname{grad} u(z))$ is the usual inner product of two vectors $\operatorname{grad} h(z)$ and $\operatorname{grad} u(z)$ in $\mathbf{R}^2$ and v is the area element on R.

For $h \in HD(R)$, denote by $D(h) := D(h, h)$. We call it the Dirichlet integral of h. By the discussion in [1, p. 400] we have the following

PROPOSITION 2 (cf. [1]). *$HD(R)$ is a Hilbert space with the inner product $D(\cdot, \cdot)$ in the above definition, two functions being identified if their difference is a constant function.*

PROPOSITION 3 (cf. [3]). *$h \in HD(R)$ if and only if h has the minimal fine limit $h^*(\zeta)$ at almost every point $\zeta (\in \Delta_1)$ with respect to the harmonic measure ω_{z_0} such*

that $h(z) = \int_{\Delta_1} h^*(\zeta)\, d\omega_z(\zeta)$, $\int_{\Delta_1} |h^*(\zeta)|^2\, d\omega_{z_0}(\zeta) < \infty$, *and the following property holds*:

$$\int_{\Delta_1}\int_{\Delta_1} (h^*(\zeta) - h^*(\xi))^2 \theta_{z_0}(\zeta, \xi)\, d\omega_{z_0}(\zeta)\, d\omega_{z_0}(\xi) < \infty,$$

where $\theta_{z_0}(\zeta, \xi)$ *is the Naïm kernel on* $(R \cup \Delta \backslash \{z_0\}) \times (R \cup \Delta \backslash \{z_0\})$ (cf. [6]).

Then, moreover,

$$D(h) = q \int_{\Delta_1}\int_{\Delta_1} (h^*(\zeta) - h^*(\xi))^2 \theta_{z_0}(\zeta, \xi)\, d\omega_{z_0}(\zeta)\, d\omega_{z_0}(\xi),$$

where q *is the absolute constasnt.*

Denote by $MHB_+(R)$ (resp. $MHD_+(R)$) the class of all finite limit functions of monotone increasing sequences of $HB_+(R)$ (resp. $HD_+(R)$). Set $MHX(R) = MHX_+(R) - MHX_+(R)$ $(X = B, D)$. The class $MHB(R)$ is called the class of quasi-bounded functions on R. By [2, Folgesatz 13.1, Satz 13.4 and Satz 14.2] we have the following

LEMMA 1 (cf. [2]). *It holds that*

$$MHB(R) = \Big\{ h \,|\, h \text{ has the minimal fine limit } h^*(\zeta) \text{ at almost every point } \zeta(\in \Delta_1) \text{ with respect to } \omega_{z_0} \text{ with } u(z) = \int_{\Delta_1^M} u^*(\zeta)\, d\omega_z(\zeta) \Big\}.$$

From Proposition 3 and Lemma 1 the next lemma is easily deduced.

LEMMA 2. $MHD(R) \subset MHB(R)$.

By the above lemma we have the following

PROPOSITION 4. *Suppose that* $\dim HB(R) < \infty$. *Then* $HD(R) \subset HB(R)$.

Proof. Suppose that $\dim HB(R) < \infty$. By [5, Theorem 2], $HB(R) = MHB(R)$. Hence, it follows from Lemma 2 that $HD(R) \subset MHD(R) \subset MHB(R) = HB(R)$.

2. Proof of Main Theorem

Suppose that (i) holds. Further we suppose that there exists a point $\zeta \in \Delta$ such that $\omega_{z_0}(U_\rho(\zeta)) > 0$ for any positive ρ and $\omega_{z_0}(\{\zeta\}) = 0$, where $U_\rho(\zeta)$ is the disc with center ζ and radius ρ with respect to the standard metric on $R \cup \Delta$. Hence, there exists a monotone decreasing sequence $\{\rho_n\}_{n=1}^{\infty}$ with

$\lim_{n\to\infty}\rho_n = 0$, $\omega_{z_0}(U_{\rho_n}(\zeta)\setminus U_{\rho_{n+1}}(\zeta)) > 0$ $(n\in\mathbf{N})$ and $\lim_{n\to\infty}\omega_{z_0}(U_{\rho_n}(\zeta)) = 0$. Set $u_n(z) = \omega_z(U_{\rho_n}(\zeta))$ $(z\in R, n\in\mathbf{N})$. Since $HB(R) = HD(R)$, $D(u_n) < +\infty$. First we show that $\{D(u_n)\}_{n=1}^{\infty}$ is bounded. Suppose that $\{D(u_n)\}_{n=1}^{\infty}$ is unbounded. Set $V_n = U_{\rho_n}(\zeta)\setminus U_{\rho_{n+1}}(\zeta)$ $(n\in\mathbf{N})$. Then $U_{\rho_n}(\zeta) = \bigcup_{\tau=n}^{\infty} V_\tau$ $(n\in\mathbf{N})$ and $V_0 = \Delta\setminus U_{\rho_1}(\zeta)$. By Proposition 3 we have

$$\begin{aligned}\frac{1}{2}D(u_n) &= q\int_{U_{\rho_n}(\zeta)}\int_{\Delta\setminus U_{\rho_n}(\zeta)}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &= q\sum_{\tau=n}^{\infty}\sum_{\sigma=0}^{n-1}\int_{V_\tau}\int_{V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi).\end{aligned}$$

Hence there exists a subsequence $\{D(u_{n_\nu})\}_{\nu=1}^{\infty}$ of $\{D(u_n)\}_{n=1}^{\infty}$ with

$$\begin{aligned}&\int_{\bigcup_{\tau=n_\nu}^{n_{\nu+1}-1}V_\tau}\int_{\bigcup_{\sigma=0}^{n_\nu-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &\quad= \sum_{\tau=n_\nu}^{n_{\nu+1}-1}\sum_{\sigma=0}^{n_\nu-1}\int_{V_\tau}\int_{V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi) \ge \nu^5.\end{aligned}$$

Set $u = \sum_{\nu=1}^{\infty} u_{n_\nu}/\nu^2$. It is easily seen that $u\in HB(R)$. On the other hand, we have

$$\begin{aligned}\frac{1}{2}D(u) &= q\sum_{\mu=1}^{\infty}\sum_{\nu=0}^{\mu-1}\left(\sum_{j=\nu+1}^{\mu}\frac{1}{j^2}\right)^2\int_{\bigcup_{\tau=n_\mu}^{n_{\mu+1}-1}V_\tau}\int_{\bigcup_{\sigma=n_\nu}^{n_{\nu+1}-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\quad(n_0=0)\\ &\ge q\sum_{\mu=1}^{l}\sum_{\nu=0}^{\mu-1}\left(\sum_{j=\nu+1}^{\mu}\frac{1}{j^2}\right)^2\int_{\bigcup_{\tau=n_\mu}^{n_{\mu+1}-1}V_\tau}\int_{\bigcup_{\sigma=n_\nu}^{n_{\nu+1}-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &\ge q\sum_{\nu=0}^{l-1}\left(\sum_{j=\nu+1}^{l}\frac{1}{j^2}\right)^2\int_{\bigcup_{\tau=n_l}^{n_{l+1}-1}V_\tau}\int_{\bigcup_{\sigma=n_\nu}^{n_{\nu+1}-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &\ge \frac{q}{l^4}\sum_{\nu=0}^{l-1}\int_{\bigcup_{\tau=n_l}^{n_{l+1}-1}V_\tau}\int_{\bigcup_{\sigma=n_\nu}^{n_{\nu+1}-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &= \frac{q}{l^4}\int_{\bigcup_{\tau=n_l}^{n_{l+1}-1}V_\tau}\int_{\bigcup_{\sigma=0}^{n_l-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &\ge \frac{q}{l^4}l^5 = ql\end{aligned}$$

for every $l\in\mathbf{N}$. Hence $u\notin HD(R)$. This is a contradiction.

By definition of u_n we find that $\{u_n\}_{n=1}^{\infty}$ converges to 0 locally uniformlly on R. Taking sufficiently large intger m_0 and replacing $\{u_n\}_{n=1}^{\infty}$ with $\{u_n\}_{n=m_0}^{\infty}$, we may suppose that $u_1 < 1$ on R. This implies that $\omega_{z_0}(V_0) > 0$. Since $\{u_n\}_{n=1}^{\infty}$

converges to 0 locally uniformlly on R and $\{D(u_n)\}_{n=1}^{\infty}$ is bounded, by [9, the discussion in the proof of Theorem in p. 149] we find that, for every $v \in HD(R)$,

$$D(u_n, v) \to 0 \quad (n \to \infty).$$

By Mazur's Theorem (cf. [10, Theorem 2 (p. 120)]), for every v, there exist an integer n_v and non-negative sequences $\{\alpha_{v,j}\}_{j=1}^{n_v}$ such that $\sum_{j=1}^{n_v} \alpha_{v,j} = 1$ and $D(\sum_{j=1}^{n_v} \alpha_{v,j} u_j) < v^{-2}$. On the other hand, since $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j\}_{v=1}^{\infty}$ is bounded, we can take a subsequence of $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j\}_{v=1}^{\infty}$ such that $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j(z_0)\}_{v=1}^{\infty}$ converges to a constant α. Hence, by [3, Theorems 4.1 and 4.2], $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j\}_{v=1}^{\infty}$ converges to α in $L^2(\Delta, \omega_{z_0})$, where $L^2(\Delta, \omega_{z_0})$ is the set of square integrable functions on Δ with respect to ω_{z_0} and hence, by [3, the result in the first paragraph of section 12], $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j\}_{v=1}^{\infty}$ converges to α locally uniformlly on R. Hence, by [3, Theorem 4.3], and the facts that $\sum_{j=1}^{n_v} \alpha_{v,j} = 1$ and that $\omega_{z_0}(V_0) > 0$, we find that $\alpha = 0$.

Set $w_v = \sum_{j=1}^{n_v} \alpha_{v,j} u_j$. Take a subsequence $\{w_{v_\lambda}\}_{\lambda=1}^{\infty}$ of $\{w_v\}_{v=1}^{\infty}$ with $w_{v_\lambda}(z_0) < 1/\lambda^2$. Set $s = \sum_{\lambda=1}^{\infty} w_{v_\lambda}$. By [3, Theorem 4.2] s is well-defined. We find that $s \in HD(R)$ and that s is unbounded on any neigborhood of ζ, that is, $s \in HD(R) \backslash HB(R)$. This is a conradiction.

Hence, if $\zeta (\in \Delta)$ satisfies that $\omega_{z_0}(U_\rho(\zeta)) > 0$ for every positive $\rho, \omega_{z_0}(\{\zeta\}) > 0$. It follows from this fact that there exists a subset N of Δ such that $\omega_{z_0}(N) = 0$ and that $\Delta_1 \backslash N$ consists of at most countably many points with positive harmonic measure. To see this set

$$N = \{\zeta \in \Delta \,|\, \text{there exists a positive } \rho_\zeta \text{ with } \omega_{z_0}(U_{\rho_\zeta}(\zeta)) = 0\}$$

and set $F = \Delta \backslash N$. Clearly $F \cup N = \Delta$, $F \cap N = \emptyset$ and $\omega_{z_0}(\{\zeta\}) > 0$ for every $\zeta \in F$. Hence F is an at most countable subset of Δ_1 because $\omega_{z_0}(\Delta) = 1$ and $\omega_{z_0}(\Delta \backslash \Delta_1) = 0$. Hence it is sufficient to prove that $\omega_{z_0}(N) = 0$. Set $O = \bigcup_{\zeta \in N} U_{\rho_\zeta}(\zeta)$. Clearly O is an open subset of $R \cup \Delta$ and $O \cap \Delta = N$. By the Lindelöf theorem there exists a sequence $\{\xi_n\}_{n=1}^{\infty}$ of N with $O = \bigcup_{n=1}^{\infty} U_{\rho_{\xi_n}}(\xi_n)$. Hence $\omega_{z_0}(N) \le \omega_{z_0}(O) \le \sum_{n=1}^{\infty} \omega_{z_0}(U_{\rho_{\xi_n}}(\xi_n)) = 0$, and hence, $\omega_{z_0}(N) = 0$.

Suppose that $\sharp(\Delta_1 \backslash N) = \aleph_0$, where $\sharp(\Delta_1 \backslash N)$ is the cardinal number of $\Delta_1 \backslash N$. Set $\Delta_1 \backslash N = \{\zeta_n\}_{n=1}^{\infty}$.

Set $u_n(z) = \omega_z(\{\zeta_j\}_{j=n}^{\infty})$ $(z \in R)$. Since $HB(R) = HD(R)$, $D(u_n) < \infty$. First we show that $\{D(u_n)\}_{n=1}^{\infty}$ is bounded. Suppose that $\{D(u_n)\}_{n=1}^{\infty}$ is unbounded. By Proposition 3 we have

$$\frac{1}{2} D(u_n) = q \sum_{\tau=n}^{\infty} \sum_{\sigma=1}^{n-1} \theta_{z_0}(\zeta_\tau, \zeta_\sigma) \omega_{z_0}(\{\zeta_\tau\}) \omega_{z_0}(\{\zeta_\sigma\}).$$

Hence there exists a subsequence $\{D(u_{n_v})\}_{v=1}^{\infty}$ of $\{D(u_n)\}_{n=1}^{\infty}$ with

$$\sum_{\tau=n_v}^{n_{v+1}-1} \sum_{\sigma=1}^{n_v-1} \theta_{z_0}(\zeta_\tau, \zeta_\sigma) \omega_{z_0}(\{\zeta_\tau\}) \omega_{z_0}(\{\zeta_\sigma\}) \ge v^5.$$

Set $u=\sum_{\nu=1}^{\infty}u_{n_\nu}/\nu^2$. It is easily seen that $u\in HB(R)$. On the other hand, for any integer $l(\geq 2)$, we have

$$\begin{aligned}\frac{1}{2}D(u)&=q\sum_{\mu=2}^{\infty}\sum_{\nu=1}^{\mu-1}\sum_{\tau=n_\mu}^{n_{\mu+1}-1}\sum_{\sigma=n_\nu}^{n_{\nu+1}-1}\left(\sum_{j=\nu+1}^{\mu}\frac{1}{j^2}\right)^2\theta_{z_0}(\zeta_\tau,\zeta_\sigma)\omega_{z_0}(\{\zeta_\tau\})\omega_{z_0}(\{\zeta_\sigma\})\\&\geq q\sum_{\mu=2}^{l}\sum_{\nu=1}^{\mu-1}\sum_{\tau=n_\mu}^{n_{\mu+1}-1}\sum_{\sigma=n_\nu}^{n_{\nu+1}-1}\left(\sum_{j=\nu+1}^{\mu}\frac{1}{j^2}\right)^2\theta_{z_0}(\zeta_\tau,\zeta_\sigma)\omega_{z_0}(\{\zeta_\tau\})\omega_{z_0}(\{\zeta_\sigma\})\\&\geq\frac{q}{l^4}\sum_{\mu=2}^{l}\sum_{\nu=1}^{\mu-1}\sum_{\tau=n_\mu}^{n_{\mu+1}-1}\sum_{\sigma=n_\nu}^{n_{\nu+1}-1}\theta_{z_0}(\zeta_\tau,\zeta_\sigma)\omega_{z_0}(\{\zeta_\tau\})\omega_{z_0}(\{\zeta_\sigma\})\\&\geq\frac{q}{l^4}l^5=ql.\end{aligned}$$

Hence $u\notin HD(R)$. This is a contradiction.

By definition of u_n we find that $\{u_n\}_{n=1}^{\infty}$ converges to 0 locally uniformlly on R. Replacing $\{u_n\}_{n=1}^{\infty}$ with $\{u_n\}_{n=2}^{\infty}$, we may suppose that $u_1<1$ on R. This implies that $\omega_{z_0}(\Delta\backslash\{\zeta_n\}_{n=1}^{\infty})>0$. Since $\{u_n\}_{n=1}^{\infty}$ converges to 0 locally uniformlly on R and $\{D(u_n)\}_{n=1}^{\infty}$ is bounded, by [9, the discussion in the proof of Theorem in p. 149] we find that, for every $v\in HD(R)$,

$$D(u_n,v)\to 0\quad(n\to\infty).$$

By Mazur's Theorem (cf. [10, Theorem 2 (p. 120)]), for every v, there exist an integer n_v and non-negative sequences $\{\alpha_{v,j}\}_{j=1}^{n_v}$ such that $\sum_{j=1}^{n_v}\alpha_{v,j}=1$ and $D(\sum_{j=1}^{n_v}\alpha_{v,j}u_j)<v^{-2}$. On the other hand, since $\{\sum_{j=1}^{n_v}\alpha_{v,j}u_j\}_{v=1}^{\infty}$ is bounded, we can take a subsequence of $\{\sum_{j=1}^{n_v}\alpha_{v,j}u_j\}_{v=1}^{\infty}$ such that $\{\sum_{j=1}^{n_v}\alpha_{v,j}u_j(z_0)\}_{v=1}^{\infty}$ converges to a constant α. Hence, by [3, Theorems 4.1 and 4.2], $\{\sum_{j=1}^{n_v}\alpha_{v,j}u_j\}_{v=1}^{\infty}$ converges to α in $L^2(\Delta,\omega_{z_0})$, where $L^2(\Delta,\omega_{z_0})$ is the set of square integrable functions on Δ with respect to ω_{z_0} and hence, by [3, the result in the first paragraph of section 12], $\{\sum_{j=1}^{n_v}\alpha_{v,j}u_j\}_{v=1}^{\infty}$ converges to α locally uniformlly on R. Hence, by [3, Theorem 4.3], and the facts that $\sum_{j=1}^{n_v}\alpha_{v,j}=1$ and that $\omega_{z_0}(\Delta\backslash\{\zeta_n\}_{n=1}^{\infty})>0$, we find that $\alpha=0$.

Set $w_v=\sum_{j=1}^{n_v}\alpha_{v,j}u_j$. Take a subsequence $\{w_{v_\lambda}\}_{\lambda=1}^{\infty}$ of $\{w_v\}_{v=1}^{\infty}$ with $w_{v_\lambda}(z_0)<1/\lambda^2$. Set $s=\sum_{\lambda=1}^{\infty}w_{v_\lambda}$. By [3, Theorem 4.2] s is well-defined. Clearly $s\in HD(R)$. Let ξ_0 be an accumlating point of $\{\zeta_j\}_{j=1}^{\infty}$. We find that s is unbounded on any neighborhood of ξ_0. Hence $s\in HD(R)\backslash HB(R)$. This is a contradiction. Hence $\sharp\{\zeta_n\}_{n\geq 1}<\infty$.

Hence, setting $N=\Delta_1\backslash\{\zeta\in\Delta_1:\omega_{z_0}(\zeta)>0\}$, by Proposition 1, we find that $\omega_{z_0}(N)=0$, $\sharp(\Delta_1\backslash N)<\infty$, and $k_\zeta\in HB(R)\cap HD(R)$ for all $\zeta\in\Delta_1\backslash N$. Therefore we have (ii).

Suppose that (ii) holds. Hence, there exists a null set N of Δ with respect to the harmonic measure such that $\Delta_1\backslash N$ consists of finitely many points and the Martin function k_ζ on R with pole at a point ζ of $\Delta_1\backslash N$ is a bounded and positive harmonic function with a finite Dirichlet integral. Put $\sharp(\Delta_1\backslash N)=m$.

$\Delta_1 \backslash N = \{\zeta_1, \ldots, \zeta_m\}$. Take any $h \in HB(R)$ (resp. $h \in HD(R)$). Then there exist $h_\iota \in HP_+(R)$ $(\iota = 1, 2)$ with $h = h_1 - h_2$ on R. By the Martin reprensentation theorem there exist the positive measures μ_1 and μ_2 such that

$$h_\iota(z) = \int_{\Delta_1} k_\zeta(z)\, d\mu_\iota(\zeta) = \sum_{j=1}^{m} k_{\zeta_j}(z)\mu_\iota(\{\zeta_j\}) \quad (\iota = 1, 2).$$

Hence $h(z) = h_1(z) - h_2(z) = \sum_{j=1}^{m} k_{\zeta_j}(z)(\mu_1(\{\zeta_j\}) - \mu_2(\{\zeta_j\}))$. Since $k_{\zeta_j} \in HD(R)$ (resp. $k_{\zeta_j} \in HB(R)$) $(j = 1, \ldots, m)$, $h \in HD(R)$ (resp. $h \in HB(R)$). Hence, $HB(R) \subset HD(R)$ (resp. $HD(R) \subset HB(R)$), and hence, $HB(R) = HD(R)$. Hence $\dim HB(R) = \dim HD(R) < \infty$. Therefore we have (iii).

Suppose that (iii) holds. Since $\dim HB(R) < \infty$, by Proposition 4, we find that $HD(R) \subset HB(R)$. Since $HD(R)$ is a linear subspace of the linear space $HB(R)$, by the assertion (iii), we find that $HB(R) = HD(R)$. Therefore we have (i).

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