H.-P. FU
KODAI MATH. J.
33 (2010), 211–216

MINIMAL SUBMANIFOLDS WITH FLAT NORMAL BUNDLE

Hai-Ping Fu

Abstract

Let M^n $(n \le 7)$ be an n-dimensional complete immersed super stable minimal submanifold in an $(n+p)$-dimensional Euclidean space $\mathbf{R}^{n+p}$ with flat normal bundle. We prove that if the second fundamental form A of M satisfies $\int_M |A|^3 < \infty$, then M is an affine n-dimensional plane.

1. Introduction

Let M^n be an n-minimal submanifold in $\mathbf{R}^{n+p}$. Denote by $|A|$ the norm of the second fundamental form of M.

When $p = 1$, M is said to be stable if

$$0 \le \int_M (|\nabla f|^2 - |A|^2 f^2), \quad \forall f \in C_0^\infty(M). \tag{1.1}$$

Let us recall that the well-known Bernstein's theorem asserts that an entire minimal graph $M^n \subset R^{n+1}$ must be linear if $n \le 7$. Moreover, the dimension restriction is necessary as indicated by the examples of Bombieri, De Giorgi and Giusti. Because of the stability of minimal entire graphs, one is naturally led to the generalization of the classical Bernstein theorem to the question of asking whether all stable minimal hypersurfaces in R^{n+1} are hyperplanes when $n \le 7$.

It is known that a complete stable minimal surface in $\mathbf{R}^3$ must be a plane, which was proved by do Carmo and Peng, and Fischer-Cobrie and Schoen independently [2, 4]. Do Carmo and Peng [3] showed that if M is a stable complete minimal hypersurface in $\mathbf{R}^{n+1}$ and

$$\lim_{R\to\infty} \frac{1}{R^{2+2q}} \int_{B(2R)\setminus B(R)} |A|^2 = 0, \quad q < \sqrt{\frac{2}{n}},$$

Mathematics Subject Classification (2000). 53C42.
Keywords. Minimal submanifolds, flat normal bundle.
Supported by NSFC(10671087) and NSFJP(2008GZS0060).
Received June 23, 2009; revised September 29, 2009.

then M is a hyperplane. Shen and Zhu [7] showed that if M^n is a complete stable minimal hypersurface in $\mathbf{R}^{n+1}$ with finite total curvature, that is,

$$\int_M |A|^n < +\infty,$$

then M is a hyperplane.

When $p \geq 1$, Spruck [9] proved that for a variation vector field $E = f\upsilon$, the second variation of $Vol(M_t)$ satisfies

$$\frac{d^2\ Vol(M_t)}{dt^2} \geq \int_M (|\nabla f|^2 - |A|^2 f^2),$$

where υ is the unit normal vector field and $f \in C_0^\infty(M)$. Motivated by this, Wang introduced the concept of super stability for minimal submanifolds [10]. M is said to be super stable if

$$0 \leq \int_M (|\nabla f|^2 - |A|^2 f^2), \quad \forall f \in C_0^\infty(M). \tag{1.2}$$

When $p = 1$, the definition of super stability is exactly the same as that of stability and the normal bundle is trivially flat. Wang [10] proved that a complete super stable minimal submainfold in $\mathbf{R}^{n+p}$ with finite total curvature is an affine plane. Because the normal bundle becomes complicated in higher codimension, we consider the simplest case when the normal bundle is flat. Recently Smoczyk, Wang, and Xin [8] proved a Bernstein type theorem for minimal submanifolds in R^{n+p} with flat normal bundle under a certain growth condition. Seo [6] showed that if M is a complete super stable minimal submainfold in $\mathbf{R}^{n+p}$ with flat normal bundle and $\int_M |A|^2 < +\infty$, then M is an affine plane.

Now we study super stable minimal submanifolds in $\mathbf{R}^{n+p}$ with flat normal bundle. Our main results in this paper are stated as follows.

Theorem 1.1. *Let M^n $(n \leq 7)$ be a super stable complete immersed minimal submaifold in $\mathbf{R}^{n+p}$ with flat normal bundle. If*

$$\lim_{R\to\infty} \frac{1}{R^{1+2q}} \int_{B(2R)\setminus B(R)} |A|^3 = 0, \quad q < \sqrt{\frac{2}{n}},$$

then M is an affine n-dimensional plane.

Corollary 1.2. *Let M^n $(n \leq 7)$ be a super stable complete immersed minimal submaifold in $\mathbf{R}^{n+p}$ with flat normal bundle. If*

$$\int_M |A|^3 < +\infty,$$

then M is an affine n-dimensional plane.

Remark 1.3. When $n = 3$ and $p = 1$, Li and Wei proved Theorem 1.1 and Corollary 1.2 in [5].

2. Proof of the theorems

We follow the notations of Chern-do Carmo-Kobayashi [1].

Let M^n be an n-minimal submanifold in $\mathbf{R}^{n+p}$. We choose an orthonormal frame $e_1, e_2, \dots, e_{n+p}$ in R^{n+p} such that, restricted to M, the vectors $e_1, e_2, \dots, e_n$ are tangent to M. And we shall denote the second fundamental form by h_{ij}^{α}. Then we have $|A|^2 = \sum (h_{ij}^{\alpha})^2$ and

$$2|A|\Delta|A| + 2|\nabla|A|\,|^2 = \Delta|A|^2 = 2\sum (h_{ijk}^{\alpha})^2 + 2\sum (h_{ij}^{\alpha})\Delta h_{ij}^{\alpha}. \tag{2.1}$$

By Chern-do Carmo-Kobayashi ([1], (2.23)), we have

$$\sum (h_{ij}^{\alpha})\Delta h_{ij}^{\alpha} = -\sum (h_{ik}^{\alpha}h_{jk}^{\beta} - h_{jk}^{\alpha}h_{ik}^{\beta})(h_{il}^{\alpha}h_{jl}^{\beta} - h_{jl}^{\alpha}h_{il}^{\beta}) - \sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ij}^{\beta}h_{kl}^{\beta}.$$

Since M has flat normal bundle, we have $h_{ik}^{\alpha}h_{jk}^{\beta} - h_{jk}^{\alpha}h_{ik}^{\beta} = 0$. Therefore, we obtain

$$\sum (h_{ij}^{\alpha})\Delta h_{ij}^{\alpha} = -\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ij}^{\beta}h_{kl}^{\beta}.$$

For each α, let H_{α} denote the symmetric matrix (h_{ij}^{α}), and set $S_{\alpha\beta} = \sum h_{ij}^{\alpha}h_{ij}^{\beta}$. Then the $(p \times p)$ matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1}, \dots, e_{n+p}$. Thus we have

$$\sum (h_{ij}^{\alpha})\Delta h_{ij}^{\alpha} = -\sum S_{\alpha\alpha}^2 = -\sum_{\alpha}\left(\sum_{i,j}(h_{ij}^{\alpha})^2\right)^2. \tag{2.2}$$

Moreover

$$|A|^4 = (|A|^2)^2 = \left(\sum_{\alpha}\sum_{i,j}(h_{ij}^{\alpha})^2\right)^2 \geq \sum_{\alpha}\left(\sum_{i,j}(h_{ij}^{\alpha})^2\right)^2. \tag{2.3}$$

Hence from (2.1), (2.2) and (2.3) we have

$$2|A|\Delta|A| + 2|\nabla|A|\,|^2 \geq 2\sum (h_{ijk}^{\alpha})^2 - 2|A|^4$$

Since $\sum (h_{ijk}^{\alpha})^2 = |\nabla A|^2$, we get

$$|A|\Delta|A| + |\nabla|A|\,|^2 \geq |\nabla A|^2 - |A|^4. \tag{2.4}$$

From (2.4) and curvature estimate by Y. Xin ([11], Lemma 3.1), we obtain

$$|A|\Delta|A| + |A|^4 \geq \frac{2}{n}|\nabla|A|\,|^2. \tag{2.5}$$

Proof of Theorem 1.1. Let $q \geq 0$ and $f \in C_0^\infty(M)$. Multiplying (2.5) by $|A|^{2q}f^2$ and integrating over M, we obtain

$$\begin{aligned}\frac{2}{n}\int_M |\nabla|A||^2|A|^{2q}f^2 &\leq \int_M |A|^{4+2q}f^2 + \int_M |A|^{2q+1}f^2\Delta|A| \\ &= \int_M |A|^{4+2q}f^2 - 2\int_M |A|^{2q+1}f\langle\nabla f, \nabla|A|\rangle \\ &\quad - (2q+1)\int_M |A|^{2q}f^2|\nabla|A||^2,\end{aligned}$$

which gives

$$\left(\frac{2}{n} + 2q + 1\right)\int_M |\nabla|A||^2|A|^{2q}f^2 \leq \int_M |A|^{4+2q}f^2 - 2\int_M |A|^{2q+1}f\langle\nabla f, \nabla|A|\rangle. \tag{2.6}$$

Using the Cauchy-Schwarz inequality, we can rewrite (2.6) as

$$\left(\frac{2}{n} + 2q + 1 - \varepsilon\right)\int_M |\nabla|A||^2|A|^{2q}f^2 \leq \int_M |A|^{4+2q}f^2 + \frac{1}{\varepsilon}\int_M |A|^{2(q+1)}|\nabla f|^2, \tag{2.7}$$

for some positive constant ε.

On the other hand, replacing f by $|A|^{(1+q)}f$ in the super stability inequality (1.2), we have

$$\begin{aligned}&(1+q)(1+q+\varepsilon)\int_M |\nabla|A||^2|A|^{2q}f^2 \\ &\quad \geq \int_M |A|^{4+2q}f^2 - \left(1 + \frac{1+q}{\varepsilon}\right)\int_M |A|^{2(q+1)}|\nabla f|^2.\end{aligned} \tag{2.8}$$

Subtracting $(2.8) \times \left(\frac{2}{n} + 2q + 1 - \varepsilon\right)$ from $(2.7) \times (1+q)(1+q+\varepsilon)$, it yields that

$$\begin{aligned}&\left[\frac{2}{n} - q^2 - (2+q)\varepsilon\right]\int_M |A|^{4+2q}f^2 \\ &\quad \leq \frac{1+q+\varepsilon}{\varepsilon}\left(\frac{2}{n} + 3q + 2 - \varepsilon\right)\int_M |A|^{2(q+1)}|\nabla f|^2.\end{aligned} \tag{2.9}$$

Taking $q < \sqrt{\frac{2}{n}}$, it is easy to see that $\frac{2}{n} - q^2 > 0$, and then we can choose $\varepsilon > 0$ sufficiently small so that $\frac{2}{n} - q^2 - (2+q)\varepsilon > 0$. It follows from (2.9) that for $q < \sqrt{\frac{2}{n}}$ the following inequality holds:

$$\int_M |A|^{4+2q}f^2 \leq C_1\int_M |A|^{2(q+1)}|\nabla f|^2. \tag{2.10}$$

where C_1 is a constant that depends on n, ε and q.

Before going on our estimates, let us recall the Young's inequality:

$$ab \le \frac{\beta^s a^s}{s} + \frac{\beta^{-t} b^t}{t}, \quad \frac{1}{s} + \frac{1}{t} = 1, \tag{2.11}$$

where $\beta > 0$ is arbitrary and $1 < s < \infty$, $1 < t < \infty$. Let r, $0 < r < 2 + 2q$, be a number yet to be determined. By using (2.11), we obtain

$$\begin{aligned} |A|^{2+2q}|\nabla f|^2 &= f^2\left(|A|^{2+2q}\frac{|\nabla f|^2}{f^2}\right) = f^2\left(|A|^{2+2q-r}|A|^r\frac{|\nabla f|^2}{f^2}\right) \\ &\le f^2\left(\frac{\beta^s}{s}|A|^{s(2+2q-r)} + \frac{\beta^{-t}b^t}{t}\left(|A|^r\frac{|\nabla f|^2}{f^2}\right)^t\right). \end{aligned} \tag{2.12}$$

We now choose r to satisfy the following equations:

$$s(2+2q-r) = 4+2q, \quad rt = 3, \quad \frac{1}{s} + \frac{1}{t} = 1.$$

This is indeed possible, and the solution is

$$r = \frac{6}{1+2q}, \quad s = 1 + \frac{2}{2q-1}, \quad t = \frac{1}{2} + q, \quad \frac{1}{2} < q < \sqrt{\frac{2}{n}}.$$

By use of these values and the fact that β may be made small, from (2.10) and (2.12) we obtain

$$\int_M |A|^{4+2q} f^2 \le C_2 \int_M |A|^3 \frac{|\nabla f|^{1+2q}}{f^{2q-1}}, \tag{2.13}$$

where C_2 is a constant that depends on n, ε, β and q. Now we use the arbitrariness of f to replace f by $f^{1/2+q}$ in (2.13) and obtain

$$\int_M |A|^{4+2q} f^{1+2q} \le C_3 \int_M |A|^3 |\nabla f|^{1+2q}. \tag{2.14}$$

Let f be a smooth function on $[0, \infty)$ such that $f \ge 0$, $f = 1$ on $[0, R]$ and $f = 0$ in $[2R, \infty)$ with $|f'| \le \frac{2}{R}$. Then considering $f \circ r$, where r is the function in the definition of $B(R)$, we have from (2.14)

$$\int_{B(R)} |A|^{4+2q} \le \frac{4C_3}{R^{1+2q}} \int_{B(2R)\setminus B(R)} |A|^3. \tag{2.15}$$

Let $R \to +\infty$, by assumption that $\lim_{R\to\infty} \frac{1}{R^{1+2q}} \int_{B(2R)\setminus B(R)} |A|^3 = 0$, from (2.15) we conclude $|A| = 0$, i.e., M is an affine plane. □

Acknowledgement. The author would like to thank the referee for some suggestions.

References

[1] S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, Functional analysis and related fields, Springer-Verlag, 1970, 59–75.

[2] M. do Carmo and C. K. Peng, Stable complete minimal surfaces in R^3 are planes, Bull. Amer. Math. Soc. **1** (1979), 903–906.

[3] M. do Carmo and C. K. Peng, Stable complete minimal hypersurfaces, Proceeding of the 1980 Beijing Symposium on Differential Geometry and Differential Equations **3**, Beijing, 1980, Science Press, Beijing, 1982, 1349–1358.

[4] D. Fischer-Colbrie and R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Comm. Pure. Appl. Math. **33** (1980), 199–211.

[5] H. Z. Li and G. X. Wei, Stable complete minimal hypersurfaces in $\mathbf{R}^4$, Mathemática Contemporânea **28** (2005), 183–188.

[6] K. Seo, Rigidity of minimal submanifolds with flat normal bundle, Commun. Korean Math. Soc. **23** (2008), 421–426.

[7] Y. B. Shen and X. H. Zhu, On the stable complete minimal hypersurfaces in $\mathbf{R}^{n+1}$, Amer. J. Math. **120** (1998), 103–116.

[8] K. Smoczyk, G. Wang and Y. Xin, Bernstein type theorems with flat normal bundle, Calc. Var. PDE. **26** (2006), 57–67.

[9] J. Spruck, Remarks on the stability of minimal submanifolds of $\mathbf{R}^n$, Math. Z. **144** (1975), 169–174.

[10] Q. L. Wang, On minimal submanifolds in an Euclidean space, Math. Nachr. **261/262** (2003), 176–180.

[11] Y. Xin, Bernstein type theorems without graphic condition, Asian J. Math. **9** (2005), 31–44.

Hai-Ping Fu
Department of Mathematics
Nanchang University
Nanchang 330047
P.R. China
E-mail: mathfu@126.com