A. KODAMA AND S. SHIMIZU
KODAI MATH. J.
33 (2010), 182–191

ADDENDUM TO OUR CHARACTERIZATION OF THE UNIT POLYDISC

Akio Kodama and Satoru Shimizu

Abstract

In 2008, we obtained an intrinsic characterization of the unit polydisc Δ^n in $\mathbf{C}^n$ from the viewpoint of the holomorphic automorphism group. In connection with this, A. V. Isaev investigated the structure of a complex manifold M with the property that every isotropy subgroup of the holomorphic automorhism group of M is compact, and obtained the same characterization of Δ^n as ours among the class of all such manifolds. In this paper, we establish some extensions of these results. In particular, Isaev's characterization of the unit polydisc Δ^n is extended to that of any bounded symmetric domain in $\mathbf{C}^n$.

1. Introduction

This is a continuation of our previous paper [8], and we retain the terminology and notation there.

Let M be a connected complex manifold and $\mathrm{Aut}(M)$ the group of all biholomorphic automorphisms of M. Then, equipped with the compact-open topology, $\mathrm{Aut}(M)$ is a topological group acting continuously on M. It should be remarked here that $\mathrm{Aut}(M)$ does not have the structure of a Lie group, in general; this often causes difficulties in studying various problems related to $\mathrm{Aut}(M)$.

In 1907, it was shown by Poincaré [10] that the Riemann mapping theorem does not hold in the higher dimensional case. In fact, he proved that *there exists no biholomorphic mapping from the unit polydisc* Δ^2 *onto the unit ball* B^2 *in* $\mathbf{C}^2$ by comparing carefully the topological structures of the isotropy subgroups of $\mathrm{Aut}(\Delta^2)$ and $\mathrm{Aut}(B^2)$ at the origin o of $\mathbf{C}^2$. In view of this fact, for a given complex manifold M, it seems to be an interesting problem to bring out some complex analytic nature of M under some topological conditions on $\mathrm{Aut}(M)$. Taking this into account, we asked the following question in [8]: *Let M and N be connected complex manifolds and assume that their holomorphic automorphism*

2000 *Mathematics Subject Classification*. Primary 32M05; Secondary 32Q28.

Keywords and phrases. Holomorphic automorphism groups, Reinhardt domains, Torus actions, Unit polydisc.

Received September 15, 2009; revised November 10, 2009.

groups $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are isomorphic as topological groups. Then is M biholomorphically equivalent to N?* And, as our main result, we obtained the following intrinsic characterization of the unit polydisc Δ^n from the viewpoint of the holomorphic automorphism group:

Theorem A ([8, Theorem]). *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then M is biholomorphically equivalent to* Δ^n.

Later, related to this theorem, Isaev [6] investigated the structure of a complex manifold M with the property that every isotropy subgroup of the $\mathrm{Aut}(M)$-action is compact, and showed the following:

Theorem B ([6, Theorem 1.2]). *Let M be a connected complex manifold of dimension n satisfying the following two conditions*:

(1) *The isotropy subgroup of* $\mathrm{Aut}(M)$ *at every point of M is compact.*

(2) $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\Delta^n)$ *as topological groups.*

Then M is biholomorphically equivalent to Δ^n.

The main purpose of this paper is to establish the following extensions of Theorems A and B, which were announced at the 17th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications in Ho Chi Minh City, Vietnam, August 2009:

Theorem 1. *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that there exists a topological subgroup G of* $\mathrm{Aut}(M)$ *that is isomorphic to the identity component of* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then M is biholomorphically equivalent to* Δ^n.

This theorem will be proved in Section 2 by modifying the proof of Theorem A.

Let W be an arbitrary domain in $\mathbf{C}^n$. Then it is well-known that W admits a smooth envelope of holomorphy (cf. [9]). Hence, as an immediate consequence of this theorem, we obtain the following:

Corollary 1. *Let M be a connected Stein manifold of dimension n or a domain in* $\mathbf{C}^n$. *Assume that there exists a topological subgroup G of* $\mathrm{Aut}(M)$ *that is isomorphic to the identity component of* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then M is biholomorphically equivalent to* Δ^n.

A bounded domain D in $\mathbf{C}^n$ is called *symmetric* if, for each point $p \in D$, there exists an element $s_p \in \mathrm{Aut}(D)$ such that $s_p \circ s_p = \mathrm{id}_D$, $s_p \neq \mathrm{id}_D$ and p is an isolated fixed point of s_p. Clearly, the unit polydisc Δ^n as well as the unit ball

B^n in $\mathbf{C}^n$ is a typical example of bounded symmetric domains. As a natural generalization of Theorem B, we can prove the following theorem in Section 3:

THEOREM 2. *Let M be a connected complex manifold of dimension n and let D be a bounded symmetric domain in $\mathbf{C}^n$. Assume that there exists a topological subgroup G of* $\mathrm{Aut}(M)$ *satisfying the following two conditions*:

(1) *The isotropy subgroup of G at every point of M is compact.*

(2) *G is isomorphic to the identity component of* $\mathrm{Aut}(D)$ *as topological groups.*

Then M is biholomorphically equivalent to D.

Recall that the isotropy subgroup of $\mathrm{Aut}(M)$ at every point of M is compact, provided that M is hyperbolic in the sense of Kobayashi [7]. Hence we have the following:

COROLLARY 2. *Let M be a connected hyperbolic manifold of dimension n and let D be a bounded symmetric domain in $\mathbf{C}^n$. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D)$ *as topological groups. Then M is biholomorphically equivalent to D.*

Finally, it should be remarked that, for a given connected complex manifold M, the following conditions (A) and (B) are mutually independent (for the detail, see Section 4):

(A) M is holomorphically separable and admits a smooth envelope of holomorphy.

(B) The isotropy subgroup of $\mathrm{Aut}(M)$ at every point of M is compact.

In this sense, our Theorems 1 and 2 may be considered as characterizations of model domains from different viewpoints.

Acknowledgment. The authors would like to thank Professor Chifune Kai for his useful comments and informations on the Harish-Chandra realizations of Hermitian symmetric spaces of the non-compact type. Furthermore, the authors are grateful to the referee for comments leading to improvements of the present paper. The authors are partially supported by the Grant-in-Aid for Scientific Research (C) No. 21540169 and (C) No. 18540154, the Ministry of Education, Science, Sports and Culture, Japan.

2. Proof of Theorem 1

Our proof is based on the argument developed in our previous paper [8]. Although there are some overlaps with that paper, we carry out the proof for the sake of completeness and self-containedness.

Let us start with fixing a coordinate system $z = (z_1, \dots, z_n)$ in $\mathbf{C}^n$ and setting

$$\Delta_j = \{z_j \in \mathbf{C} \mid |z_j| < 1\} \quad (1 \le j \le n) \quad \text{and} \quad \Delta^n = \Delta_1 \times \cdots \times \Delta_n.$$

Recall that $\mathrm{Aut}(\Delta_j)$ is a connected, real simple Lie group of dimension 3 with trivial center. Let $\mathrm{Aut}^o(\Delta^n)$ be the identity component of $\mathrm{Aut}(\Delta^n)$. Then we know that $\mathrm{Aut}^o(\Delta^n)$ can be identified with the direct product of $\mathrm{Aut}(\Delta_j)$: $\mathrm{Aut}^o(\Delta^n) = \mathrm{Aut}(\Delta_1) \times \cdots \times \mathrm{Aut}(\Delta_n)$. Let $\mathfrak{g}(\Delta_j)$ and $\mathfrak{g}(\Delta^n)$, respectively, denote the real Lie algebras consisting of all complete holomorphic vector fields on Δ_j and on Δ^n. Then it is well-known that these Lie algebras are canonically identified with the Lie algebras of $\mathrm{Aut}(\Delta_j)$ and $\mathrm{Aut}(\Delta^n)$, respectively. Therefore we have

$$(2.1)\quad \mathfrak{g}(\Delta^n) = \mathfrak{g}(\Delta_1) \oplus \cdots \oplus \mathfrak{g}(\Delta_n), \quad [\mathfrak{g}(\Delta_i), \mathfrak{g}(\Delta_j)] = \{0\} \quad \text{for } 1 \le i, j \le n,\ i \ne j.$$

Moreover, we see that $\mathfrak{g}(\Delta_j)$ contains the holomorphic vector fields

$$H_j := \sqrt{-1} z_j \partial/\partial z_j \quad \text{and} \quad V_j := (1 - z_j^2)\partial/\partial z_j$$

induced by the one-parameter subgroups

$$z_j \mapsto (\exp \sqrt{-1} t) z_j \quad \text{and} \quad z_j \mapsto \frac{(\cosh t) z_j + \sinh t}{(\sinh t) z_j + \cosh t}$$

$(t \in \mathbf{R})$ of $\mathrm{Aut}(\Delta_j)$, respectively. Then, putting $W_j = [H_j, V_j]$, we have

$$(2.2)\quad \mathfrak{g}(\Delta_j) = \mathbf{R}\{H_j, V_j, W_j\} \quad \text{and} \quad [H_j, [H_j, V_j]] = -V_j, \quad [W_j, V_j] = 4H_j$$

for $1 \le j \le n$. These bracket relations will be very important in our proof.

As in Theorem 1 in the introduction, let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy and assume that there exists a topological group isomorphism $\Phi : \mathrm{Aut}^o(\Delta^n) \to G$, where G is the given topological subgroup of $\mathrm{Aut}(M)$. Since Δ^n is a Reinhardt domain in $\mathbf{C}^n$, the n-dimensional torus T^n acts naturally on Δ^n as a connected Lie transformation group, so that, via the isomorphism Φ, T^n now acts effectively and continuously on M by biholomorphic transformations. Hence this action is necessarily real analytic by a classical result of Bochner and Montgomery [3]. Therefore, by a well-known fact due to Barrett, Bedford and Dadok [1], we may assume that M is a Reinhardt domain D in $\mathbf{C}^n$ and that there exists a topological group isomorphism $\Phi : \mathrm{Aut}^o(\Delta^n) \to G \subset \mathrm{Aut}(D)$ such that $\Phi(T(\Delta^n)) = T(D)$, where $T(\Delta^n)$ and $T(D)$, respectively, denote the subgroups of $\mathrm{Aut}(\Delta^n)$ and of $\mathrm{Aut}(D)$ induced by the restrictions of the standard T^n-action on $\mathbf{C}^n$ to Δ^n and to D.

Now, the group G can be turned into a Lie group by transferring the Lie group structure from $\mathrm{Aut}^o(\Delta^n)$ by means of Φ. Since the Lie group G endowed with the compact-open topology acts continuously on D by biholomorphic transformations, the action is real analytic with respect to the Lie group structure induced from $\mathrm{Aut}^o(\Delta^n)$ (cf. [3]). Thus G is now a Lie transformation group of D acting effectively on D by biholomorphic transformations; accordingly, the Lie algebra of G can be identified with the Lie algebra $\mathfrak{g}$ consisting of all holomorphic vector fields on D induced by one-parameter subgroups of G (so-called

G-vector fields on D). We thus obtain the Lie algebra isomorphism $d\Phi : \mathfrak{g}(\Delta^n) \to \mathfrak{g}$ induced by Φ. From now on, for the sake of simplicity, let us put

$$G_j = \Phi(\mathrm{Aut}(\Delta_j)), \quad \mathfrak{g}_j = d\Phi(\mathfrak{g}(\Delta_j)) \quad \text{and}$$
$$I_j = d\Phi(H_j), \quad X_j = d\Phi(V_j), \quad Y_j = d\Phi(W_j)$$

for $1 \le j \le n$. Then $G = G_1 \times \cdots \times G_n$ and, by (2.1) and (2.2), we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\} \quad \text{for } 1 \le i, j \le n,\ i \ne j; \tag{2.3}$$

$$\mathfrak{g}_j = \mathbf{R}\{I_j, X_j, Y_j\} \quad \text{and} \quad [I_j, [I_j, X_j]] = -X_j, \quad [Y_j, X_j] = 4I_j \tag{2.4}$$

for every $1 \le j \le n$.

Put $D^* = D \cap (\mathbf{C}^*)^n$ and, for a point $z \in D$, let $(\mathfrak{g}_j)_z$ denote the subspace of the tangent space to D at z that consist of the values of the elements of $\mathfrak{g}_j$ at z. Then, using the bracket relations (2.3) and (2.4), one can verify the following assertion:

1) *For every point $z_o \in D^*$, there exist a local holomorphic coordinate system $(U, w_1, \ldots, w_n)$ on D^*, centered at z_o, and a nowhere dense real analytic subset $\mathscr{A}$ of U such that $(\mathfrak{g}_j)_p = \mathbf{C}\{(\partial/\partial w_j)_p\}$ for $p \in U \backslash \mathscr{A}$ and $1 \le j \le n$.*

Therefore, if we choose a point $p \in U \backslash \mathscr{A}$ and consider the orbits

$$D_p := G \cdot p \quad \text{and} \quad S_j := G_j \cdot p \ \ (1 \le j \le n)$$

of G and of G_j passing through p, then the assertion 1) together with (2.3) guarantees us that every S_j is a complex submanifold of D and D_p is an open subset of D. Hence D_p is a Reinhardt domain in $\mathbf{C}^n$, because G is connected and contains the torus $T(D) = T^n$. More precisely, in exactly the same way as in the proof of [8, Theorem], it can be shown that

2) *every S_j is biholomorphically equivalent to the unit disc Δ_j*;

3) *D_p is biholomorphically equivalent to the unit polydisc Δ^n*; and

4) *D is a bounded domain in $\mathbf{C}^n$ and D_p is an open dense subset of D.*

Thus the proof of Theorem 1 is now reduced to showing that D_p is also closed in D. If G is a closed subgroup of $\mathrm{Aut}(D)$, then G acts properly on D, as seen in the proof of [8; Theorem]. Consequently, the orbit $D_p = G \cdot p$ has to be closed in D in this case. Here, whether or not G is closed in $\mathrm{Aut}(D)$, we want to verify the closedness of D_p in D. To this end, assume the contrary that there exists a boundary point $q \in \partial D_p$ in D. Let d_D denote the Kobayashi distance on D and let $K(x; r) = \{y \in D \mid d_D(x, y) < r\}$ be the Kobayashi ball of radius $r > 0$ with center $x \in D$. Since d_D induces the standard topology of D (cf. [2], [12]) and p is an interior point of D_p, one can pick a small $r > 0$ in such a way that $K(p; r) \subset D_p$. For such an $r > 0$, choose a point $x_o \in D_p \cap K(q; r)$ arbitrarily and let g_o be an element of G such that $x_o = g_o \cdot p$. Then, since d_D is invariant under the action of $G \subset \mathrm{Aut}(D)$, we have

$$d_D(g_o^{-1} \cdot q, p) = d_D(q, g_o \cdot p) = d_D(q, x_o) < r,$$

which means that $g_o^{-1} \cdot q \in K(p; r) \subset D_p$ and hence $q \in g_o \cdot D_p = D_p$, a contradiction to $q \in \partial D_p$. Therefore D_p is, in fact, closed in D and accordingly $D = D_p$ is biholomorphically equivalent to Δ^n; completing the proof of Theorem 1. □

3. Proof of Theorem 2

We shall use several fundamental facts on symmetric spaces without proofs. For the details, the reader may consult, for instance, Helgason's book [4].

Let M be a connected complex manifold of dimension n and let D be a bounded symmetric domain in $\mathbf{C}^n$. Let $\mathbf{G}$ be the identity component of $\mathrm{Aut}(D)$ and let $\mathfrak{G}$ be its Lie algebra. Fix a point $o \in D$ once and for all and let $\mathbf{K}$ be the isotropy subgroup of $\mathbf{G}$ at o. Then $\mathbf{G}$ is a semi-simple Lie group with trivial center that acts transitively on D and $\mathbf{K}$ is a maximal compact subgroup of $\mathbf{G}$. Note that, since a maximal compact subgroup of a connected Lie group is always connected, $\mathbf{K}$ is a connected Lie subgroup of $\mathbf{G}$. Moreover, D can now be represented as the coset space $D = \mathbf{G}/\mathbf{K}$. Consider here the involutive automorphism $\sigma : g \mapsto s_o g s_o$ of $\mathbf{G}$, where s_o denotes the symmetry of D with respect to o, and put $s = d\sigma$, the involutive automorphism of $\mathfrak{G}$ induced by σ. Let $\mathfrak{K}$ and $\mathfrak{P}$ be the eigenspaces of s for the eigenvalues $+1$ and -1, respectively. Then $\mathfrak{K}$ coincides with the Lie algebra of $\mathbf{K}$ and we have

$$\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}, \quad [\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{K}, \quad [\mathfrak{K}, \mathfrak{P}] \subset \mathfrak{P} \quad \text{and} \quad [\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{K}. \tag{3.1}$$

As usual, we identify $\mathfrak{P}$ with the tangent space $T_o(D)$ to D at o; accordingly, $\mathfrak{P} = T_o(D)$ has the complex structure J_o^D induced by the standard complex structure tensor J^D on D. Thus $\mathfrak{P}$ can be regarded as a complex vector space. Moreover, under the identification $T_o(D) = \mathfrak{P}$, the linear isotropy group $\mathbf{K}^*$ of $\mathbf{G}$ at o is just the group $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K})$, where $\mathrm{Ad}_{\mathbf{G}}$ is the adjoint representation of $\mathbf{G}$. We will often use this fact in the proof.

Assume now that there exists a topological group isomorphism $\Phi : \mathbf{G} \to G$, where G is the given topological subgroup of $\mathrm{Aut}(M)$ in Theorem 2. Since $\mathbf{G}$ is a Lie group, G has a unique Lie group structure with respect to which $\Phi : \mathbf{G} \to G$ is a Lie group isomorphism. Thus, by the same reasoning as in the proof of Theorem 1, G becomes a Lie transformation group of M acting effectively on M by biholomorphic transformations. We denote by $\mathfrak{g}$ the Lie algebra of G and by $d\Phi : \mathfrak{G} \to \mathfrak{g}$ the Lie algebra isomorphism induced by Φ.

Fix a point $p \in M$ arbitrarily and denote by K the isotropy subgroup of G at p. Then, by our assumption, K is a compact subgroup of G. Here, along the same line as in [6], we shall show that G acts transitively on M; accordingly, M can be written in the form $M = G/K$. To this end, choose a maximal compact subgroup $\hat{K}$ of G containing K. Then, since any two maximal compact subgroups of G are always conjugate under an inner automorphism of G, one can find an element $g_o \in G$ such that $\hat{K} = g_o \Phi(\mathbf{K}) g_o^{-1}$. Moreover, notice that the orbit $G \cdot p = G/K$ of G passing through p is a real analytic submanifold of M. Thus

$$2n \geq \dim G/K \geq \dim G/\hat{K} = \dim \mathbf{G}/\mathbf{K} = 2n,$$

from which we have $K = \hat{K}$, $\dim G/K = 2n$ and hence the orbit $G \cdot p = G/K$ is open in M. Since this is true for any point $q \in M$ with $q \neq p$ and since M is connected, we conclude that $M = G/K$, as desired. Therefore, by replacing Φ by $g_o \Phi(\cdot) g_o^{-1}$ if necessary, one may assume that $\hat{K} = \Phi(\mathbf{K})$; consequently, Φ induces a real analytic diffeomorphism, say again,

$$\Phi : D = \mathbf{G}/\mathbf{K} \to G/K = M. \tag{3.2}$$

Put $\mathfrak{k} = d\Phi(\mathfrak{K})$ and $\mathfrak{p} = d\Phi(\mathfrak{P})$. Then $\mathfrak{k}$ is the Lie subalgebra of $\mathfrak{g}$ corresponding to K and we have the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with the same properties as in (3.1). Let J^M be the G-invariant complex structure tensor on M and let J_p^M be the complex structure on $T_p(M) = \mathfrak{p}$ induced by J^M. Then, since J_p^M commutes with each element in the linear isotropy group K^* of G at p, so does with $\mathrm{Ad}_G(k)$ for all $k \in K$, where Ad_G is the adjoint representation of G.

In order to complete the proof of Theorem 2, we need to prove that, after a slight modification if necessary, the diffeomorphism Φ in (3.2) gives rise to a biholomorphic equivalence between D and M. For this purpose, by using the fact that $d\Phi$ gives a linear isomorphism from $\mathfrak{P}$ onto $\mathfrak{p}$, let us define the endomorphism J_o^* of $\mathfrak{P}$ by the formula

$$d\Phi(J_o^* X) = J_p^M(d\Phi(X)) \quad \text{for all } X \in \mathfrak{P}. \tag{3.3}$$

Then $J_o^* \circ J_o^* = -I$ and moreover, since

$$d\Phi(\mathrm{Ad}_{\mathbf{G}}(k)X) = \mathrm{Ad}_G(\Phi(k))\, d\Phi(X) \quad \text{for all } k \in \mathbf{K} \text{ and all } X \in \mathfrak{P},$$

it can be easily seen that J_o^* commutes with $\mathrm{Ad}_{\mathbf{G}}(k)$ for all $k \in \mathbf{K}$. Therefore $D = \mathbf{G}/\mathbf{K}$ admits a unique almost complex structure tensor J^* which coincides with J_o^* at o and is invariant under the action of $\mathbf{G}$. The proof is now divided into two cases as follows:

Case 1. *D is irreducible.* In this case, $\mathbf{G}$ is a simple Lie group and $\mathbf{K}$ is a maximal compact subgroup of $\mathbf{G}$ with one-dimensional center isomorphic to the circle group S^1. By definition of the irreducibility, $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K})$ now acts irreducibly on $\mathfrak{P}$. Hence, Schur's lemma implies that $J_o^* = cI$ with some constant $c \in \mathbf{C}$; accordingly $J_o^* = \pm\sqrt{-1}I = \pm J_o^D$ and $J^* = \pm J^D$, because $(J_o^*)^2 = -I$. Moreover, we would like to assert here the following: one may assume, without loss of generality, that D is invariant under the complex conjugation $\psi : z \to \bar{z}$ of $\mathbf{C}^n$ with respect to $\mathbf{R}^n$. Indeed, in the case where D is one of the four classical domains, it is well-known that D can be realized as a subdomain $\tilde{D}$ in some complex matrix space (cf. [5]). Then, a glance at $\tilde{D}$ tells us that it is invariant under the complex conjugation ψ. On the other hand, in the case where D is an exceptional bounded symmetric domain, it is shown in Roos [11; Section 3] that its Harish-Chandra realization $\tilde{D}$ has an explicit algebraic and geometric description using exceptional Jordan triple systems; from which it follows at once that $\tilde{D}$ is invariant under the complex conjugation ψ, as asserted. Thus, taking the diffeomorphism $\Phi \circ \psi$ instead of Φ in (3.2) if necessary, we may assume that $J^* = J^D$. This combined with (3.3) yields that $\Phi : D \to M$ is holomorphic;

consequently, it gives a biholomorphic equivalence between D and M, as required.

Case 2. *D is reducible*. In this case, D can be uniquely (up to an order) decomposed into the direct product

$$D = D_1 \times \cdots \times D_r, \tag{3.4}$$

where the factors D_i are irreducible bounded symmetric domains in $\mathbf{C}^{n_i}$ with $n_1 + \cdots + n_r = n$. Here, as in Case 1, one may assume that each D_i is invariant under the complex conjugation. Let $\mathbf{G}$ and $\mathbf{G}_i$ be the identity components of $\mathrm{Aut}(D)$ and of $\mathrm{Aut}(D_i)$. And, writing $o = (o_1, \dots, o_r)$ with $o_i \in D_i$ according to the decomposition (3.4), we denote by $\mathbf{K}$ and $\mathbf{K}_i$ the isotropy subgroups of $\mathbf{G}$ and of $\mathbf{G}_i$ at o and at o_i, respectively. Then, as mentioned in Case 1, each $\mathbf{G}_i$ is a simple Lie group with $\mathbf{K}_i$ as a maximal compact subgroup of it and D_i is a homogeneous space of $\mathbf{G}_i$. Moreover, we have $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ and $\mathbf{K} = \mathbf{K}_1 \times \cdots \times \mathbf{K}_r$, so that D can be expressed as

$$D = \mathbf{G}/\mathbf{K} = \mathbf{G}_1/\mathbf{K}_1 \times \cdots \times \mathbf{G}_r/\mathbf{K}_r. \tag{3.5}$$

Let $\mathfrak{G}_i$ be the Lie algebra of $\mathbf{G}_i$. Let σ_i be the involutive automorphism $g \mapsto s_{o_i} g s_{o_i}$ of $\mathbf{G}_i$ and put $s_i = d\sigma_i$. Then, denoting by $\mathfrak{K}_i$ and $\mathfrak{P}_i$, respectively, the eigenspaces of s_i for the eigenvalues $+1$ and -1, we obtain the direct sum decomposition $\mathfrak{G}_i = \mathfrak{K}_i \oplus \mathfrak{P}_i$ as in (3.1). As before, we identify $\mathfrak{P}_i = T_{o_i}(D_i)$ and we denote also by J^{D_i} the standard complex structure tensor on D_i. Let J_o^* be the complex structure on $\mathfrak{P} = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_r$ defined by (3.3). Then, since J_o^* commutes with $\mathrm{Ad}_{\mathbf{G}}(k)$ for all $k \in \mathbf{K}$ and since $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K}_i)$ acts irreducibly on $\mathfrak{P}_i$ and trivially on $\mathfrak{P}_j$ for $j \neq i$, it follows that each $\mathfrak{P}_i$ is invariant under J_o^*. Thus J_o^* is decomposed $J_o^* = J_{o_1}^* \times \cdots \times J_{o_r}^*$, where each $J_{o_i}^*$ is the restriction of J_o^* to $\mathfrak{P}_i$. Therefore, letting J_i^* be the unique $\mathbf{G}_i$-invariant almost complex structure tensor on D_i which coincides with $J_{o_i}^*$ at o_i, we have $J^* = J_1^* \times \cdots \times J_r^*$. Moreover, since $\mathrm{Ad}_{\mathbf{G}_i}(\mathbf{K}_i)$ acts now irreducibly on $\mathfrak{P}_i$, Schur's lemma again implies that $J_i^* = \pm J^{D_i}$ for each $1 \leq i \leq r$. Finally, consider a real analytic diffeomorphism $\hat{\Phi} : D = D_1 \times \cdots \times D_r \to M$ given by

$$\hat{\Phi}(u) = \Phi(\gamma_1(u_1), \dots, \gamma_r(u_r)) \quad \text{for } u = (u_1, \dots, u_r) \in D_1 \times \cdots \times D_r = D,$$

where $\gamma_i(u_i) = u_i$ or $\gamma_i(u_i) = \bar{u}_i$, the complex conjugation in $\mathbf{C}^{n_i}$, for $1 \leq i \leq r$ and Φ is the diffeomorphism appearing in (3.2). Then, replacing Φ by a suitable $\hat{\Phi}$ if necessary, we have $J^* = J^D$. This means that $\Phi : D \to M$ is holomorphic. Therefore, we have shown that Φ gives a biholomorphic equivalence between D and M; thereby completing the proof of Theorem 2. □

4. A concluding remark

In this section, we would like to illustrate that the conditions (A) and (B) stated in the introduction are mutually independent, in general, with concrete examples as follows:

Example 1. Consider the two-dimensional complex Euclidean space $\mathbf{C}^2$, for instance. Then, the condition (A) is trivially satisfied for $\mathbf{C}^2$. On the other hand, notice that the isotropy subgroup $\mathrm{Aut}_o(\mathbf{C}^2)$ of $\mathrm{Aut}(\mathbf{C}^2)$ at the origin o of $\mathbf{C}^2$ contains the biholomorphic mappings $\varphi_v : \mathbf{C}^2 \to \mathbf{C}^2$ defined by

$$\varphi_v(z, w) = (z, w \exp(vz)), \quad (z, w) \in \mathbf{C}^2 \text{ for } v = 1, 2, \dots.$$

Clearly this says that $\mathrm{Aut}_o(\mathbf{C}^2)$ is not to be compact; hence, the condition (B) is not satisfied for $\mathbf{C}^2$.

Example 2. Take an arbitrary compact connected hyperbolic manifold X of dimension $\geqq 2$ and consider the manifold M obtained from X by deletion of one point, say $M = X \backslash \{p\}$ $(p \in X)$. Then, being a complex submanifold of the hyperbolic manifold X, M is also hyperbolic. Accordingly, the condition (B) is automatically satisfied for M. However, we assert that M is not holomorphically separable and does not admit a smooth envelope of holomorphy. To verify this, note that any holomorphic function on M can be holomorphically extended to X and hence it must be constant, because X is a compact connected complex manifold of dimension $\geqq 2$. Thus, M is never holomorphically separable. Moreover, assume that there exists a smooth envelope of holomorphy of M. Then, since every Stein manifold can be realized as a closed complex submanifold of some $\mathbf{C}^N$, we have a holomorphic imbedding $F : M \to \mathbf{C}^N$. But, since any holomorphic function on M is now constant as mentioned above, F must be also constant. Clearly, this is a contradiction. Therefore the condition (A) is not satisfied for this manifold M.

References

[1] D. E. Barrett, E. Bedford and J. Dadok, T^n-actions on holomorphically separable complex manifolds, Math. Z. **202** (1989), 65–82.

[2] T. Barth, The Kobayashi distance induces the standard topology, Proc. Amer. Math. Soc. **35** (1972), 439–441.

[3] S. Bochner and D. Montgomery, Groups of differentiable and real or complex analytic transformations, Ann. of Math. **46** (1945), 685–694.

[4] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, New York, London, Toronto, Sydney, San Francisco, 1978.

[5] L. K. Hua, Harmonic analysis of functions of several complex variables in the classical domains, Translations of math. monographs **6**, Amer. Math. Soc., Providence, 1963.

[6] A. V. Isaev, A remark on a theorem by Kodama and Shimizu, J. Geom. Anal. **18** (2008), 795–799.

[7] S. Kobayashi, Hyperbolic complex spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1998.

[8] A. Kodama and S. Shimizu, An intrinsic characterization of the unit polydisc, Michigan Math. J. **56** (2008), 173–181.

[9] R. Narasimhan, Several complex variables, Univ. of Chicago Press, Chicago and London, 1971.

[10] H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme, Rend. Circ. Mat. Palermo **23** (1907), 185–220.

[11] G. Roos, Exceptional symmetric domains, Contemp. Math. **468** (2008), 157–189.
[12] H. L. Royden, Remarks on the Kobayashi metric, Proc. Maryland Conference on Several Complex Variables, Lecture notes math. **185**, Springer-Verlag, Berlin, Heidelberg, New York, 1971, 125–137.

Akio Kodama
Division of Mathematical and Physical Sciences
Graduate School of Natural Science and Technology
Kanazawa University
Kanazawa, 920-1192
Japan
E-mail: kodama@kenroku.kanazawa-u.ac.jp

Satoru Shimizu
Mathematical Institute
Tohoku University
Sendai, 980-8578
Japan
E-mail: shimizu@math.tohoku.ac.jp