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QUOTIENT CURVES OF SMOOTH PLANE CURVES WITH AUTOMORPHISMS

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Abstract

We obtain several results of quotient curves of smooth plane curves with automorphisms. Such automorphisms can be divided into two types (type I and type II). The quotient curves of smooth plane curves with automorphisms of type I are extremal curves in the sense of Castelnuovo's bound. We also show some partial result on automorphisms of type II and give examples.

1. Introduction and preliminaries

We consider the following problem:

Problem. Let C be a smooth plane curve over $\mathbf{C}$ with an automorphism σ. Examine the quotient curve $C/\langle\sigma\rangle$.

Previously we completely classified double coverings between smooth plane curves ([HKO, Theorem 2.1]). It is a special case of this problem.

In this article we obtain a concrete description of quotient curves of smooth plane curves under some assumption on their automorphisms. As a corollary, we completely determine quotient curves obtained from involutions of smooth plane curves.

Notation and Conventions

For an irreducible curve C, $g(C)$ denotes the geometric genus of its normalization.

A g_d^r is a linear system of degree d and dimension r on a smooth curve. A 1-dimensional linear system is called a *pencil*. For a smooth curve C, its *gonality* is defined as the minimum degree of pencils and denoted by $\mathrm{gon}(C)$.

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For a divisor D on a normal projective variety, $|D|$ denotes the complete linear system associated to D and $\Phi_{|D|}$ is the rational map associated to $|D|$. For two divisors D and D', $D \sim D'$ denotes that they are linearly equivalent.

For a non-negative integer n, Σ_n denotes the Hirzebruch surface with index n. The Picard group of Σ_n is generated by two divisors Δ_n and Γ_n with $\Delta_n^2 = -n$, $\Delta_n\Gamma_n = 1$ and $\Gamma_n^2 = 0$, where Δ_n (resp. Γ_n) is the minimal section (resp. the class of fiber) of Σ_n.

For a real number x, $[x]$ denotes the greatest integer not greater than x.

We quote a classical result on curves due to Castelnuovo for later use.

THEOREM 1.1 (Castelnuovo bound, [ACGH, p. 116]). *The maximum of the geometric genus of a non-degenerate irreducible (possibly singular) curve of degree d in $\mathbf{P}^r$ is given by*

$$\pi_0(d,r) = \binom{m}{2}(r-1) + m\varepsilon,$$

where $m := \left[\dfrac{d-1}{r-1}\right]$ *and* $\varepsilon := d - 1 - m(r-1)$.

A curve is said to be *extremal* if the genus attains the maximum. Any extremal curve is smooth.

2. On automorphisms of type I

Let C be a smooth plane curve of degree $d \geq 4$. Assume that C has an automorphism σ of order $n \geq 2$. Let $\pi : C \to B = C/\langle\sigma\rangle$ denote the cyclic covering induced by σ. Note that g_d^2 on C is unique (cf. [S, Proposition 3.13]). Hence σ is extended to an automorphism $\tilde{\sigma}$ of $\mathbf{P}^2$. We may assume that $\tilde{\sigma}$ is given by a $(3,3)$ diagonal matrix, which is one of the following type:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \eta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & & \\ & \eta^k & \\ & & \eta^l \end{pmatrix},$$

where η is a primitive n-th root of unity and k, l are coprime integer with $1 \leq k < l < n$. In this article we shall say that σ is *of type I* (resp. *of type II*) if $\tilde{\sigma}$ is given by a matrix of the former type (resp. the latter type).

Remark 2.1. (1) Any involution (automorphism of order 2) is of type I.

(2) If an automorphism has at least 4 fixed points, then it is of type I. Equivalently, an automorphism of type II has at most 3 fixed points, since a matrix of the latter form fixes only 3 points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$.

Our main result is the following theorem.

Theorem 2.2. *Let C be a smooth plane curve of degree $d \geq 4$ with an automorphism σ of order $n \geq 2$. Let $\pi : C \to B = C/\langle\sigma\rangle$ denote the cyclic covering induced by σ, f the number of fixed points of σ. If σ is of type I, then the following hold*:

(1) $d \equiv 0$ *or* $1 \pmod{n}$.

(2) *The quotient curve B is isomorphic to an extremal curve of degree d in $\mathbf{P}^{n+1}$ and lies on a cone over a rational curve, i.e., the image of Σ_n under the morphism $\varphi = \Phi_{|\Delta_n + n\Gamma_n|}$. Furthermore, the strict transform of B under φ is isomorphic to B and linearly equivalent to $\left[\frac{d}{n}\right]\Delta_n + d\Gamma_n$. In particular,* $\mathrm{gon}(B) = \left[\frac{d}{n}\right]$ *holds.*

(3) $f = \begin{cases} d & (\textit{if } d \equiv 0 \pmod{n}) \\ d+1 & (\textit{if } d \equiv 1 \pmod{n}). \end{cases}$

Conversely, let n and d be positive integers with $n \geq 2$ and $d \equiv 0$ or $1 \pmod{n}$. If B is a smooth curve as in (2), *then there exists a smooth plane curve C with an automorphism σ of order n of type I that induces a cyclic covering $\pi : C \to B$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} C & \hookrightarrow & \mathbf{P}^2 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ B & \hookrightarrow & S, \end{array}$$

where $S = \mathbf{P}^2/\langle\tilde{\sigma}\rangle$. This surface S is naturally identified with a weighted projective space $\mathbf{P}(1,1,n)$. Then $\tilde{\pi} : \mathbf{P}^2 \to S$ is given by $\tilde{\pi}((X:Y:Z)) = (X, Y, Z^n)$, where $(X:Y:Z)$ is a homogeneous coordinate of $\mathbf{P}^2$. We identify S with its image in $\mathbf{P}^{n+1}$ under the embedding

$$S = \mathbf{P}(1,1,n) \hookrightarrow \mathbf{P}^{n+1}([s,t,u] \mapsto (s^n : s^{n-1}t : \cdots : t^n : u)),$$

where $[s,t,u]$ is the equivalence class of $(s,t,u) \in \mathbf{A}^3$. Then S is a cone over a rational normal curve with the vertex $Q_0 = (0:0:\cdots:0:1)$. It is the image of Σ_n under the morphism $\varphi = \Phi_{|\Delta_n + n\Gamma_n|}$. Let $P_0 = (0:0:1)$ be the unique point of the fiber of Q_0 under $\tilde{\pi}$, ψ the blow-up of $\mathbf{P}^2$ at P_0. Then we obtain the following commutative diagram:

$$\begin{array}{ccccc} C & \hookrightarrow & \mathbf{P}^2 & \xleftarrow{\psi} & \Sigma_1 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} & & \downarrow{\scriptstyle\varpi} \\ B & \hookrightarrow & S & \xleftarrow[\varphi]{} & \Sigma_n. \end{array}$$

We identify C (resp. B) with its strict transform under ψ (resp. φ). Note that $\varpi^*\Delta_n = n\Delta_1$, $\varpi^*\Gamma_n \sim \Gamma_1$. Suppose that B is linearly equivalent to $a\Delta_n + b\Gamma_n$. Then $C = \varpi^*B \sim na\Delta_1 + b\Gamma_1$. On the other hand, we have

$$C \sim \begin{cases} (d-1)\Delta_1 + d\Gamma_1 & \text{(if } C \text{ passes through } P_0 = (0:0:1)), \\ d\Delta_1 + d\Gamma_1 & \text{(if } C \text{ does not pass through } P_0 = (0:0:1)). \end{cases}$$

It follows that $d \equiv 0$ or $1 \pmod{n}$, $a = \left[\dfrac{d}{n}\right]$ and $b = d$. Thus we have $B \sim \left[\dfrac{d}{n}\right]\Delta_n + d\Gamma_n$. In particular $\deg B = B(\Delta_n + n\Gamma_n) = d$. Next we check that $B \subset \mathbf{P}^{n+1}$ is extremal. First we assume that $d \equiv 1 \pmod{n}$, i.e., $d = ne + 1$ for some $e \in \mathbf{N}$. Then

$$\pi_0(d, n+1) = \binom{e}{2} n = \frac{1}{2} ne(e-1).$$

On the other hand, we have

$$K_{\Sigma_n} \sim -2\Delta_n - (n+2)\Gamma_n, \quad B \sim e\Delta_n + d\Gamma_n,$$

where K_{Σ_n} is the canonical divisor of Σ_n. Using the adjunction formula we have

$$\begin{aligned} 2g(B) - 2 = B(B + K_{\Sigma_n}) &= (e\Delta_n + d\Gamma_n)((e-2)\Delta_n + (d-n-2)\Gamma_n) \\ &= (e-2) + e(d-n-2) \\ &= e(d-n-1) - 2, \end{aligned}$$

which implies that

$$g(B) = \frac{1}{2} e(d-n-1) = \frac{1}{2} ne(e-1) = \pi_0(d, n+1).$$

Thus B is extremal. The proof is similar when $d \equiv 0 \pmod{n}$.

Finally, we show the assertion for f. Let $P_1, P_2, \ldots, P_f$ be the fixed points of σ, R the ramification divisor of π. Then clearly $R \geq (n-1)\sum_{i=1}^{f} P_i$. On the other hand, if P is a ramification point of π, then P is fixed under σ^j for some $1 \leq j < n$. Hence P is fixed under σ, since σ is given by a matrix

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \eta \end{pmatrix},$$

where η is a primitive n-th root of unity. Thus we have $R = (n-1)\sum_{i=1}^{f} P_i$.

First we assume that $d = ne + 1$ again. By using the Riemann-Hurwitz formula, we obtain that

$$\begin{aligned} (n-1)f = 2g(C) - 2 - n(2g(B) - 2) &= d(d-3) - n\{e(d-n-1) - 2\} \\ &= (ne+1)(ne-2) - n\{ne(e-1) - 2\} \\ &= (n-1)(ne+2), \end{aligned}$$

which implies that $f = ne + 2 = d + 1$. Similarly we obtain that $f = d$ if $d \equiv 0 \pmod{n}$.

Conversely, let n and d be positive integers with $n \geq 2$ and $d \equiv 0$ or $1 \pmod{n}$, B a smooth curve as in (2) in the theorem. First we assume that $d \equiv 1 \pmod{n}$, i.e., $d = ne + 1$ for some integer e. Then B is linearly equivalent to $e\Delta_n + d\Gamma_n$ on Σ_n.

Note that the linear system $|\Delta_n + n\Gamma_n|$ on Σ_n is $(n+1)$-dimensional and

$$B(\Delta_n + n\Gamma_n) = (e\Delta_n + d\Gamma_n)(\Delta_n + n\Gamma_n) = d.$$

Hence $|(\Delta_n + n\Gamma_n)|_B|$ is an $(n+1)$-dimensional linear system on B of degree d. We denote it by g_d^{n+1}. There exists a point P in B such that $g_d^{n+1} = |ng_e^1 + P|$, where $g_e^1 = |\Gamma_n|_B|$. Let $D = \sum_{i=1}^{e} Q_i$ be an effective divisor in g_e^1 with $Q_i \neq Q_j$ for $i \neq j$, x a meromorphic function on B whose polar divisor is D. Since the g_d^{n+1} is very ample, there exists a meromorphic function y on B with polar divisor $nD + P$ such that x and y generate the function field of B. Moreover, we may assume that $x(P) = 0$ and the supports of zero divisors of x and y are disjoint. These assumptions are not essential but technical.

Let $\Psi : B \to \mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ be a projection defined by $\Psi((x, y)) = x$. Then, $y(x)$ is an e-valued meromorphic function of x except for the ramification points of Ψ. Hence we have

$$\prod_{i=1}^{e} (y - y_i(x)) = 0,$$

where $y_1(x), y_2(x), \ldots, y_e(x)$ are the branches of $y(x)$. Then B has a plane model defined by an equation of the following form:

$$y^e + a_1(x)y^{e-1} + \cdots + a_j(x)y^{e-j} + \cdots + a_e(x) = 0.$$

where $a_j(x)$ $(1 \leq j \leq e)$ is a rational function of x.

Since $\Psi^{-1}(\infty) = \{Q_1, Q_2, \ldots, Q_e\}$, Q_i is not a ramification point of Ψ, whence we can take $t = x^{-1}$ as a local coordinate of B at Q_i. We may assume that $y_i(x)$ is the branch of y at Q_i. Noting that $y_i(x)$ has a pole of order n at Q_i, we have

$$y_i(x) = c_{i0}t^{-n} + c_{i1}t^{-(n-1)} \cdots = c_{i0}x^n + c_{i1}x^{(n-1)} \cdots \quad (c_{i0} \neq 0)$$

in a neighborhood of Q_i. Since $a_j(x)$ is a symmetric polynomial of the $y_i(x)$'s, $a_e(x)$ has a pole of order $ne = d - 1$ at $x = \infty$ and $a_j(x)$ $(1 \leq j \leq e - 1)$ has a pole of order at most jn at $x = \infty$.

Let v be the order of meromorphic function x at P $(1 \leq v \leq e)$. Then we can take a local coordinate s of B at P with $s^v = x$. Let $y_1, \ldots, y_v$ be the branches of y at P. Since y has a simple pole at P, we have

$$y_i(s) = c_i' s^{-1} + \cdots \quad (c_i' = \epsilon^{i-1} c_1' \neq 0 \ (\epsilon = e^{2\pi\sqrt{-1}/v})).$$

Hence

$$y_1(x) \cdots y_v(x) = c_1' \cdots c_v' s^{-v} + \cdots = c_1' \cdots c_v' x^{-1} + \cdots.$$

By the choice of y, $y_j(x)|_{x=0} \neq 0, \infty$ for $j = v+1, \dots, e$, whence we have

$$y_1(x) \cdots y_e(x) = c'' x^{-1} + \cdots \quad (c'' \neq 0)$$

near $x = 0$. Therefore, $a_e(x)$ has a simple pole at $x = 0$. Similarly, $a_j(x)$ $(1 \leq j \leq e-1)$ has a pole of order at most 1 at $x = 0$.

Since $y_i(x)$ has no pole in $\mathbf{C} - \{0\}$, $xa_j(x)$ is a polynomial of x with $\deg xa_e(x) = d$ and $\deg xa_j(x) \leq jn + 1$ $(1 \leq j \leq e-1)$, respectively. Furthermore, we may assume that $a_e(x)$ has no multiple root (after replacing y to $y - c$ for a suitable $c \in \mathbf{C}$ if necessary).

Let C be the plane curve defined by

$$y^{d-1} + a_1(x) y^{d-1-n} + \cdots + a_j(x) y^{d-1-jn} + \cdots + a_e(x) = 0.$$

Then, C has an automorphism $\sigma : (x, y) \mapsto (x, \eta y)$, where η is a primitive n-th root of unity and σ induces a cyclic covering $\pi : C \to B$ $((x, y) \mapsto (x, y^n))$. Let $x_1, x_2, \dots, x_d$ be the zeros of $xa_e(x)$. Then, the points $(x, y) = (x_1, 0), (x_2, 0), \dots, (x_d, 0)$ are fixed points of σ. Substituting $x = X/Z$, $y = Y/Z$ to the above equation, we have

$$X\left(Y^{d-1} + Z^n a_1\left(\frac{X}{Z}\right) Y^{d-1-n} + \cdots + Z^{d-1} a_e\left(\frac{X}{Z}\right)\right) = 0,$$

Thus, the point $(X, Y, Z) = (0, 1, 0)$ is a smooth point of C, whence it is a fixed point of σ. Thus, the number of fixed points of π is at least $d+1$. Since $g(B) = \frac{1}{2n}(d-1)(d-n-1)$, using the Riemann-Hurwitz formula, we have

$$\begin{aligned} 2g(C) - 2 &\geq n(2g(B) - 2) + (n-1)(d+1) \\ &= (d-1)(d-n-1) - 2n + (n-1)(d+1) \\ &= d^2 - 3d. \end{aligned}$$

On the other hand, since C is a plane curve of degree d, $g(C) \leq \frac{1}{2}(d-1)(d-2)$. It follows that $g(C) = \frac{1}{2}(d-1)(d-2)$, i.e., C is a smooth plane curve of degree d.

In case $d \equiv 0 \pmod{n}$, we can prove the existence of a desired smooth plane curve in a similar way and it is easier than the above case. □

In particular, we can completely determine quotient curves obtained from involutions of smooth plane curves, since any involution is of type I. Thus Theorem 2.2 is an improvement of our previous work [HKO].

3. On automorphisms of prime order of type II

In this section we show a partial result on automorphisms of type II and give several examples.

PROPOSITION 3.1. *Let C be a smooth plane curve of degree $d \geq 4$, σ an automorphism of prime order $p \geq 3$ of type II and f the number of fixed points of σ. Then one of the following holds*:

(1) $f = 0$ *and* $d \equiv 0 \pmod p$.

(2) $f = 2$ *and* $d \equiv 1$ *or* $2 \pmod p$.

(3) $f = 3$, $d^2 - 3d + 3 \equiv 0 \pmod p$ *and* $p \equiv 1 \pmod 6$ *or* $p = 3$.

In particular, we obtain some restriction on the order of automorphisms of smooth plane curves from the above proposition and Theorem 2.2.

COROLLARY 3.2. *If a smooth plane curve of degree $d \geq 4$ has an automorphism of prime order p with $p \not\equiv 1 \pmod 6$, then $d \equiv 0, 1$ or $2 \pmod p$ holds.*

Proof of Proposition 3.1. Let $\pi : C \to B = C/\langle\sigma\rangle$ be the cyclic covering induced by σ, R the ramification divisor of π and $\tilde{\sigma}$ the automorphism of $\mathbf{P}^2$ such that $\tilde{\sigma}|_C = \sigma$. Then $\tilde{\sigma}$ is given by a matrix

$$\begin{pmatrix} 1 & & \\ & \eta^k & \\ & & \eta^l \end{pmatrix},$$

where η is a primitive p-th root of unity, k and l are coprime integers with $1 \leq k < l < p$. Hence $0 \leq f \leq 3$ (see Remark 2.1) and $\deg R = (p-1)f$. By using the Riemann-Hurwitz formula, we have

$$d(d-3) = 2g(C) - 2 = p(2g(B) - 2) + (p-1)f.$$

It follows that

$$(*) \qquad d(d-3) + f \equiv 0 \pmod p.$$

First we exclude the case where $f = 1$ by reduction to absurdity. Suppose that $f = 1$. We may assume that $(1:0:0)$ is the unique fixed point of σ. Note that the line $l : x = 0$ is invariant under $\tilde{\sigma}$. Hence it cuts out an effective divisor D on C of degree d that is invariant under σ. Then D is the sum of divisors of the form $\sum_{i=1}^{p} \sigma^i(P)$. Thus p divides d, since the line l does not pass through $(1:0:0)$. It contradicts the equation $(*)$.

Next suppose that $f = 0$. Then we obtain that $d \equiv 0 \pmod p$ similarly. If $f = 2$ then $(d-1)(d-2) \equiv 0 \pmod p$ by $(*)$, which implies the conclusion.

Finally suppose that $f = 3$. Then $d^2 - 3d + 3 \equiv 0 \pmod p$ holds by $(*)$. Assume that $p \geq 5$ and put $a := d - 2$. Then, by Fermat's little theorem, it is easy to show that $d^2 - 3d + 3 = a^2 + a + 1 \equiv 0 \pmod p$ has a solution if and only if $p \equiv 1 \pmod 6$. □

In the end we show the existence of curves in each case of Proposition 3.1 by several examples.

Example 3.3. For each condition in Proposition 3.1, there exists a smooth plane curve with an automorphism of type II satisfying the condition.

(1) $f = 0$, $d \equiv 0 \pmod{p}$. The Fermat curve $x^d + y^d + z^d = 0$ of degree d has an automorphism σ induced by the matrix

$$\begin{pmatrix} 1 & & \\ & \eta & \\ & & \eta^2 \end{pmatrix} \quad (\eta \text{ is a primitive } p\text{-th root of unity with } p|d).$$

This automorphism σ has no fixed point.

(2) $f = 2$, $d \equiv 1 \pmod{p}$. The smooth plane curve defined by the equation

$$x^d + xy^{d-1} + xz^{d-1} + y^2z^{d-2} = 0$$

has an automorphism σ induced by the matrix

$$\begin{pmatrix} 1 & & \\ & \eta & \\ & & \eta^2 \end{pmatrix} \quad (\eta \text{ is a primitive } p\text{-th root of unity with } d \equiv 1 \pmod{p}).$$

Then σ fixes two points $(0 : 1 : 0)$ and $(0 : 0 : 1)$.

(3) $f = 2$, $d \equiv 2 \pmod{p}$. The smooth plane curve defined by the equation

$$x^{d-1}z + xz^{d-1} + y^d = 0$$

has an automorphism σ induced by the matrix

$$\begin{pmatrix} 1 & & \\ & \eta & \\ & & \eta^2 \end{pmatrix} \quad (\eta \text{ is a primitive } p\text{-th root of unity with } d \equiv 2 \pmod{p}).$$

Then σ fixes two points $(1 : 0 : 0)$ and $(0 : 0 : 1)$.

(4) $f = 3$, $d^2 - 3d + 3 \equiv 0 \pmod{p}$ and $p \equiv 1 \pmod{6}$. Then the smooth plane curve defined by the equation

$$x^{d-1}y + y^{d-1}z + z^{d-1}x = 0$$

has an automorphism σ induced by the matrix

$$\begin{pmatrix} 1 & & \\ & \eta & \\ & & \eta^k \end{pmatrix},$$

where η is a primitive p-th root of unity and k is a positive integer such that $d \equiv 2 - k \pmod{p}$. Then σ fixes three points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$. For example, if $d = 4$ and $p = 7$, then we can take $k = 5$ and the curve defined above is the Klein quartic $x^3y + y^3z + z^3x = 0$. It is well-known that this curve has an automorphism of order 7, since its automorphism group has order 168.

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