K. ŌSHIMA AND H. ŌSHIMA
KODAI MATH. J.
32 (2009), 530–546

HOMOTOPY GROUPS OF THE SPACES OF SELF-MAPS OF LIE GROUPS II

KATSUMI ŌSHIMA AND HIDEAKI ŌSHIMA

Abstract

We compute the homotopy groups of the spaces of self-maps of SU(3) and Sp(2).

1. Introduction

The present paper is a continuation of [3] and is devoted to the computation of $\pi_n\,\mathrm{map}_*(G,G)$, the n-th homotopy group of the space of pointed self-maps of G, for $G = \mathrm{SU}(3), \mathrm{Sp}(2)$ and $9 \le n \le 11$. We computed $\pi_n\,\mathrm{map}_*(G,G)$ for $G = \mathrm{SU}(3), \mathrm{Sp}(2)$ and $0 \le n \le 8$ in [3, 5]. Our main result is given by the following theorem.

THEOREM 1.1.

n	$\pi_n\,\mathrm{map}_*(\mathrm{SU}(3),\mathrm{SU}(3))$	$\pi_n\,\mathrm{map}_*(\mathrm{Sp}(2),\mathrm{Sp}(2))$
9	$\mathbf{Z}_8 \oplus \mathbf{Z}_2^3 \oplus \mathbf{Z}_3^3 \oplus \mathbf{Z}_5^2 \oplus \mathbf{Z}_7$	$\mathbf{Z}_2^6$
10	$\mathbf{Z}_4 \oplus \mathbf{Z}_2^2 \oplus \mathbf{Z}_9 \oplus \mathbf{Z}_3^3 \oplus \mathbf{Z}_5$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2^5 \oplus \mathbf{Z}_5$
11	$\mathbf{Z}_8 \oplus \mathbf{Z}_4^2 \oplus \mathbf{Z}_2^2 \oplus \mathbf{Z}_9 \oplus \mathbf{Z}_3^4 \oplus \mathbf{Z}_7^2$	$\mathbf{Z}_{32} \oplus \mathbf{Z}_8^2 \oplus \mathbf{Z}_2^2 \oplus \mathbf{Z}_{27} \oplus \mathbf{Z}_5^2 \oplus \mathbf{Z}_7^2$

Here $\mathbf{Z}_n^r$ denotes the direct sum of r copies of $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$.

In §2, we state our main theorem (Theorem 2.1), explain how to deduce Theorem 1.1 from the main theorem, and give a diagram being useful for computations. We prove Theorem 2.1 in §3 and §4.

2. Methods

We use notations of [3, 9] freely. Also we use results in [9] about $\pi_{n+k}(\mathrm{S}^n)$ for $k \le 19$ without particular comments. We denote by $\#a$ the order of an element a of a group, and by $\mathrm{Indet}\{\alpha,\beta,\gamma\}$ the indeterminacy of the Toda bracket $\{\alpha,\beta,\gamma\}$ [9].

Received December 24, 2008; revised August 6, 2009.

Our main theorem is

THEOREM 2.1. (1) $[C_{\eta_{12}}, \mathrm{SU}(3)]_{(2)} = \mathbf{Z}_8\{\overline{[\sigma''']}\} \oplus \mathbf{Z}_2\{(\Sigma^9 q_3)^*[\nu_5^2]\nu_{11}\}$ *and*

$$4\overline{[\sigma''']} = (\Sigma^9 q_3)^* i_*\mu'.$$

(2) $[C_{\eta_{13}}, \mathrm{SU}(3)]_{(2)} = \mathbf{Z}_4\{\underline{(\Sigma^{10} q_3)^*}[2\iota_5]\nu_5\sigma_8\}$.
(3) $[C_{\eta_{14}}, \mathrm{SU}(3)]_{(2)} = \mathbf{Z}_8\{\overline{i_*\mu'}\} \oplus \mathbf{Z}_4\{[\nu_5^2]\Sigma\overline{\nu_{10}}\} \oplus \mathbf{Z}_2\{(\Sigma^{11} q_3)^*[\nu_5\bar{\nu}_8]\}$ *and*

$$2\overline{i_*\mu'} = (\Sigma^{11} q_3)^*[2\iota_5]\zeta_5, \quad p_*\overline{i_*\mu'} = \pm(\Sigma^{11} q_3)^*\zeta_5.$$

(4) $[C_{\Sigma^9\omega}, \mathrm{Sp}(2)] = \mathbf{Z}_2^4\{\underline{(\Sigma^9 q_3)^*\overline{[\sigma'\eta_{14}]}}\eta_{15}, \underline{(\Sigma^9 q_3)^*}[\nu_7]\nu_{10}^2, i_*\overline{\mu_3}, i_*\eta_3\overline{\varepsilon_4}\}$.
(5) $[C_{\Sigma^{10}\omega}, \mathrm{Sp}(2)]_{(2,3)} = \mathbf{Z}_8\{\overline{[\nu_7]}\nu_{10}\} \oplus \mathbf{Z}_2^2\{\underline{2\overline{[\nu_7]}}\nu_{10} - (\Sigma^{10} q_3)^*[\nu_7]\sigma_{10}, i_*\mu_3\overline{\eta_{12}}\}$.
(6) $[C_{\Sigma^{11}\omega}, \mathrm{Sp}(2)]_{(2,3)} = \mathbf{Z}_8^2\{(\Sigma^{11} q_3)^*[\zeta_7], 2[2\sigma']\} \oplus \mathbf{Z}_2\{(\Sigma^{11} q_3)^* i_*\bar{\varepsilon}_3\} \oplus$
$\mathbf{Z}_{27}\{\overline{i_*\alpha_3(3)}\}$.

We prove (1), (2), (3), (4), (5), (6) of Theorem 2.1 in §3.1, §3.2, §3.3, §4.1, §4.2, §4.3, respectively.

Theorem 1.1 follows from Theorem 2.1, [6] ([3, Table 1, Table 4] and Table 6 below), [9] ($\pi_m(\mathrm{S}^n)$ for $m \leq 21$ and $n = 3, 5, 7$) and the following four facts.

(i) There is the canonical isomorphism $\pi_n \operatorname{map}_*(G, G) \cong [\Sigma^n G, G]$.

(ii) It follows from [1] that $\Sigma^3\,\mathrm{SU}(3) \simeq C_{\eta_6} \vee \mathrm{S}^{11}$ and $\Sigma^2\,\mathrm{Sp}(2) \simeq C_{\Sigma^2\omega} \vee \mathrm{S}^{12}$ and hence

$$[\Sigma^n\,\mathrm{SU}(3), \mathrm{SU}(3)] \cong [C_{\eta_{n+3}}, \mathrm{SU}(3)] \oplus \pi_{n+8}(\mathrm{SU}(3)) \quad \text{for } n \geq 3 \quad ([3, \text{ Lemma 3.2}]),$$

$$[\Sigma^n\,\mathrm{Sp}(2), \mathrm{Sp}(2)] \cong [C_{\Sigma^n\omega}, \mathrm{Sp}(2)] \oplus \pi_{n+10}(\mathrm{Sp}(2)) \quad \text{for } n \geq 2 \quad ([3, \text{ Lemma 4.1}]).$$

(iii) If p is an odd prime, then $\mathrm{SU}(3)_{(p)} \simeq (\mathrm{S}^3 \times \mathrm{S}^5)_{(p)}$ and so

$$\begin{aligned}[\Sigma^n\,\mathrm{SU}(3), \mathrm{SU}(3)]_{(p)} \cong{}& \pi_{n+3}(\mathrm{S}^3)_{(p)} \oplus \pi_{n+3}(\mathrm{S}^5)_{(p)} \oplus \pi_{n+5}(\mathrm{S}^3)_{(p)} \oplus \pi_{n+5}(\mathrm{S}^5)_{(p)} \\ &\oplus \pi_{n+8}(\mathrm{S}^3)_{(p)} \oplus \pi_{n+8}(\mathrm{S}^5)_{(p)} \quad \text{for } n \geq 1.\end{aligned}$$

(iv) If p is a prime ≥ 5, then $\mathrm{Sp}(2)_{(p)} \simeq (\mathrm{S}^3 \times \mathrm{S}^7)_{(p)}$ and so

$$\begin{aligned}[\Sigma^n\,\mathrm{Sp}(2), \mathrm{Sp}(2)]_{(p)} \cong{}& \pi_{n+3}(\mathrm{S}^3)_{(p)} \oplus \pi_{n+3}(\mathrm{S}^7)_{(p)} \oplus \pi_{n+7}(\mathrm{S}^3)_{(p)} \oplus \pi_{n+7}(\mathrm{S}^7)_{(p)} \\ &\oplus \pi_{n+10}(\mathrm{S}^3)_{(p)} \oplus \pi_{n+10}(\mathrm{S}^7)_{(p)} \quad \text{for } n \geq 1.\end{aligned}$$

n	19	20	21
$\pi_n(\mathrm{Sp}(2))$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{32} \oplus \mathbf{Z}_2$

Table 6

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and

$$\cdots \xleftarrow{\Sigma j} \Sigma Z \xleftarrow{\Sigma f} \Sigma Y \xleftarrow{q} C_f \xleftarrow{j} Z \xleftarrow{f} Y$$

a cofibre sequence. In order to compute the homotopy set $[\Sigma^n C_f, E] = [C_{\Sigma^n f}, E]$, we will use some part of the following commutative diagram with exact rows and columns.

(2.1)

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
\cdots & \xrightarrow{(\Sigma^2 j)^*} & [\Sigma^2 Z, B] & \xrightarrow{(\Sigma^2 f)^*} & [\Sigma^2 Y, B] & \xrightarrow{(\Sigma q)^*} & [\Sigma C_f, B] & \xrightarrow{(\Sigma j)^*} & [\Sigma Z, B] & \xrightarrow{(\Sigma f)^*} & [\Sigma Y, B] \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
\cdots & \xrightarrow{(\Sigma j)^*} & [\Sigma Z, F] & \xrightarrow{(\Sigma f)^*} & [\Sigma Y, F] & \xrightarrow{q^*} & [C_f, F] & \xrightarrow{j^*} & [Z, F] & \xrightarrow{f^*} & [Y, F] \\
 & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\cdots & \xrightarrow{(\Sigma j)^*} & [\Sigma Z, E] & \xrightarrow{(\Sigma f)^*} & [\Sigma Y, E] & \xrightarrow{q^*} & [C_f, E] & \xrightarrow{j^*} & [Z, E] & \xrightarrow{f^*} & [Y, E] \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
\cdots & \xrightarrow{(\Sigma j)^*} & [\Sigma Z, B] & \xrightarrow{(\Sigma f)^*} & [\Sigma Y, B] & \xrightarrow{q^*} & [C_f, B] & \xrightarrow{j^*} & [Z, B] & \xrightarrow{f^*} & [Y, B]
\end{array}
$$

3. SU(3)

The purpose of this section is to prove (1), (2) and (3) of Theorem 2.1. We use the following exact sequence:

(3.1)
$$\pi_{n+4}(\mathrm{SU}(3))_{(2)} \xrightarrow{\eta_{n+4}^*} \pi_{n+5}(\mathrm{SU}(3))_{(2)} \xrightarrow{(\Sigma^n q_3)^*} [C_{\eta_{n+3}}, \mathrm{SU}(3)]_{(2)}$$
$$\xrightarrow{(\Sigma^n i')^*} \pi_{n+3}(\mathrm{SU}(3))_{(2)} \xrightarrow{\eta_{n+3}^*} \pi_{n+4}(\mathrm{SU}(3))_{(2)}$$

3.1. Proof of Theorem 2.1 (1). By (3.1) and [6] ([3, Table 1]), we have the following exact sequence:

$$\mathbf{Z}_2\{i_*\varepsilon'\} \xrightarrow{\eta_{13}^*} \mathbf{Z}_4\{[\nu_5^2]\nu_{11}\} \oplus \mathbf{Z}_2\{i_*\mu'\} \xrightarrow{(\Sigma^9 q_3)^*} [C_{\eta_{12}}, \mathrm{SU}(3)]_{(2)}$$
$$\xrightarrow{(\Sigma^9 i')^*} \mathbf{Z}_4\{[\sigma''']\} \xrightarrow{\eta_{12}^*} \cdots$$

Lemma 3.1. (1) ([3, Lemma 3.4(1)]). $\eta_{12}^*[\sigma'''] = 0$.
(2) $i_*\nu'\varepsilon_6 = i_*\varepsilon'\eta_{13} = 2[\nu_5^2]\nu_{11} = i_*\varepsilon_3\nu_{11}$.
(3) ([2, Proposition 3.7(4)]). $[C_{\eta_{12}}, S^5]_{(2)} = \mathbf{Z}_4\{p_*\overline{[\sigma''']}\} \oplus \mathbf{Z}_2\{(\Sigma^9 q_3)^*\nu_5^3\}$ *and*
$$2 \cdot p_*\overline{[\sigma''']} \equiv (\Sigma^9 q_3)^*\mu_5 \pmod{(\Sigma^9 q_3)^*\nu_5^3}.$$
(4) ([2, Proposition 3.7(5)]). $[C_{\eta_{13}}, S^6]_{(2)} = \mathbf{Z}_4\{\Sigma p_*\overline{[\sigma''']}\}$ *and* $2 \cdot \Sigma p_*\overline{[\sigma''']} = (\Sigma^{10} q_3)^*\mu_6$.

Before the proof of this lemma, we prove Theorem 2.1 (1) from the lemma.

Consider (2.1) for the fibration $SU(3) \xrightarrow{\hat{i}} G_2 \xrightarrow{\hat{p}} S^6$ and the cofibration $S^4 \xrightarrow{\eta_3} S^3 \xrightarrow{i'} C_{\eta_3}$, that is, the following commutative diagram with exact rows and columns, where G_2 is the exceptional Lie group of rank 2.

$$\begin{array}{ccccccccc}
\pi_{14}(G_2) & \xrightarrow{\eta_{14}^*} & \pi_{15}(G_2) & \xrightarrow{(\Sigma^{10}q_3)^*} & [C_{\eta_{13}}, G_2] & \xrightarrow{(\Sigma^{10}i')^*} & \pi_{13}(G_2) & \xrightarrow{\eta_{13}^*} & \pi_{14}(G_2) \\
\downarrow \hat{p}^* & & \downarrow \hat{p}^* & & \downarrow \hat{p}^* & & \downarrow \hat{p}^* & & \downarrow \hat{p}^* \\
\pi_{14}(S^6) & \xrightarrow{\eta_{14}^*} & \pi_{15}(S^6) & \xrightarrow{(\Sigma^{10}q_3)^*} & [C_{\eta_{13}}, S^6] & \xrightarrow{(\Sigma^{10}i')^*} & \pi_{13}(S^6) & \xrightarrow{\eta_{13}^*} & \pi_{14}(S^6) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
\pi_{13}(SU(3)) & \xrightarrow{\eta_{13}^*} & \pi_{14}(SU(3)) & \xrightarrow{(\Sigma^9 q_3)^*} & [C_{\eta_{12}}, SU(3)] & \xrightarrow{(\Sigma^9 i')^*} & \pi_{12}(SU(3)) & \xrightarrow{\eta_{12}^*} & \pi_{13}(SU(3))
\end{array}$$

Since $\pi_{13}(G_2) = 0$ and the first η_{14}^* of the diagram is surjective by [4], we have $[C_{\eta_{13}}, G_2] = 0$. Hence the third and the fourth ∂ of the diagram are injective. Moreover, by [9, Lemma 6.3, p. 64] and Lemma 3.1(1),(2),(4), the above diagram induces the following commutative diagram with exact rows, where three ∂'s are injective.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}_2\{(\Sigma^{10}q_3)^*\mu_6\} & \longrightarrow & \mathbf{Z}_4\{\Sigma p_*\overline{[\sigma''']}\} & \xrightarrow{(\Sigma^{10}i')^*} & \mathbf{Z}_2\{2\sigma''\} & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & \mathbf{Z}_2^2\{(\Sigma^9 q_3)^*[\nu_5^2]\nu_{11}, (\Sigma^9 q_3)^* i_*\mu'\} & \longrightarrow & [C_{\eta_{12}}, SU(3)]_{(2)} & \xrightarrow{(\Sigma^9 i')^*} & \mathbf{Z}_4\{[\sigma''']\} & \longrightarrow & 0
\end{array}$$

We have $2\overline{[\sigma''']} - \partial\Sigma p_*\overline{[\sigma''']} \in \mathrm{Image}((\Sigma^9 q_3)^*) \cong \mathbf{Z}_2^2$ and so

$$4\overline{[\sigma''']} = 2\partial\Sigma p_*\overline{[\sigma''']} = \partial(2\Sigma p_*\overline{[\sigma''']}) = \partial(\Sigma^{10}q_3)^*\mu_6.$$

Since $\partial(\Sigma^{10}q_3)^*\mu_6$ is of order 2, the order of $\overline{[\sigma''']}$ is 8. Recall from [6] ([3, Table 1]) that $\pi_{14}(SU(3))_{(2)} = \mathbf{Z}_4\{[\nu_5^2]\nu_{11}\} \oplus \mathbf{Z}_2\{i_*\mu'\}$ and $2[\nu_5^2]\nu_{11} = i_*\varepsilon_3\nu_{11}$. Hence we can write $\partial\mu_6 = 2a \cdot [\nu_5^2]\nu_{11} + b \cdot i_*\mu' = a \cdot i_*\varepsilon'\eta_{13} + b \cdot i_*\mu'$ $(a, b \in \{0, 1\})$. If $b = 0$, then $a = 1$ and $\partial\mu_6 = 2[\nu_5^2]\nu_{11}$ and hence $\partial(\Sigma^{10}q_3)^*\mu_6 = (\Sigma^9 q_3)^*\partial\mu_6 = 0$, and this is a contradiction. Hence $b = 1$, that is, $\partial\mu_6 = 2a \cdot [\nu_5^2]\nu_{11} + i_*\mu'$. Therefore

$$4\overline{[\sigma''']} = \partial(\Sigma^{10}q_3)^*\mu_6 = (\Sigma^9 q_3)^*\partial\mu_6 = (\Sigma^9 q_3)^* i_*\mu'.$$

Thus we obtain Theorem 2.1 (1).

Proof of Lemma 3.1. We refer (1) to [3].

(2) The first equality follows from the equality $\varepsilon'\eta_{13} = \nu'\varepsilon_6$ [9, (7.12)]. The last equality is in [6, (4.1)]. In order to prove the second equality, we consider the following homotopy exact sequence of the fibration $S^3 \xrightarrow{i} SU(3) \xrightarrow{p} S^5$.

$$\text{(3.2)}\quad \mathbf{Z}_4\{[2\iota_5]\nu_5\sigma_8\} \xrightarrow{p_*} \mathbf{Z}_8\{\nu_5\sigma_8\} \oplus \mathbf{Z}_2\{\eta_5\mu_6\} \xrightarrow{\partial} \mathbf{Z}_4\{\mu'\} \oplus \mathbf{Z}_2^2\{\varepsilon_3\nu_{11}, \varepsilon'\eta_{13}\}$$
$$\xrightarrow{i_*} \mathbf{Z}_4\{[\nu_5^2]\nu_{11}\} \oplus \mathbf{Z}_2\{i_*\mu'\}$$

The boundary homomorphism ∂ satisfies $\partial(\iota_5) = \eta_3$ and so $\partial(\eta_5\mu_6) = \eta_3^2\mu_5 = 2\mu'$. By [6, (4.4)], we have $2\iota_5 \circ (\nu_5\sigma_8) = 2(\nu_5\sigma_8)$ and so $p_*[2\iota_5]\nu_5\sigma_8 = 2(\nu_5\sigma_8)$ and hence the order of $\partial(\nu_5\sigma_8)$ is 2. Thus if we write $\partial(\nu_5\sigma_8) = x \cdot \mu' + y \cdot \varepsilon_3\nu_{11} + z \cdot \varepsilon'\eta_{13}$ for $0 \le x \le 3$ and $y, z \in \{0, 1\}$, then $x = 0$ or 2. Since $i_*\varepsilon_3\nu_{11} = 2[\nu_5^2]\nu_{11}$, we have $0 = i_*\partial(\nu_5\sigma_8) = 2y[\nu_5^2]\nu_{11} + z \cdot i_*\varepsilon'\eta_{13}$. If $z = 0$, then $y = 0$ and so $0 \neq \partial(\nu_5\sigma_8) = x \cdot \mu'$ and hence $x = 2$, and therefore $\partial(\nu_5\sigma_8 + \eta_5\mu_6) = 0$. This is impossible. Thus $z = 1$ and

$$i_*\varepsilon'\eta_{13} = 2y[\nu_5^2]\nu_{11}. \tag{3.3}$$

On the other hand, it follows from [9, Proposition 1.4, Lemma 5.4, (5.4)] that

$$\{[\nu_5^2], \eta_{11}, 2\iota_{12}\} \circ \eta_{13} = -([\nu_5^2] \circ \{\eta_{11}, 2\iota_{12}, \eta_{12}\}) = 2[\nu_5^2]\nu_{11} \neq 0$$

and hence $i_*\varepsilon'\eta_{13} = 2[\nu_5^2]\nu_{11}$, since $\pi_{13}(\mathrm{SU}(3)) \circ \eta_{13}$ is generated by $i_*\varepsilon'\eta_{13}$. Hence $y = 1$ by (3.3) and so $i_*\varepsilon'\eta_{13} = 2[\nu_5^2]\nu_{11}$.

We refer (3) to [2].

While (4) was announced in [2, Proposition 3.7], we will prove it because our notations are different from theirs. We have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
\mathbf{Z}_2\{\varepsilon_5\} & \xrightarrow{\eta_{13}^*} & \mathbf{Z}_2^3\{\nu_5^3, \mu_5, \eta_5\varepsilon_6\} & \xrightarrow{(\Sigma^9 q_3)^*} & \mathbf{Z}_2\{q^*\nu_5^3\} \oplus \mathbf{Z}_4\{p_*\overline{[\sigma''']}\} & \xrightarrow{(\Sigma^9 i')^*} & \mathbf{Z}_2\{\sigma'''\} & \xrightarrow{\eta_{12}^*} & 0 \\
\downarrow \Sigma & & \cong \downarrow \Sigma & & \downarrow \Sigma & & \cong \downarrow \Sigma & & \\
\mathbf{Z}_8\{\bar{\nu}_6\} \oplus \mathbf{Z}_2\{\varepsilon_6\} & \xrightarrow{\eta_{14}^*} & \mathbf{Z}_2^3\{\nu_6^3, \mu_6, \eta_6\varepsilon_7\} & \xrightarrow{(\Sigma^{10} q_3)^*} & [C_{\eta_{13}}, \mathrm{S}^6]_{(2)} & \xrightarrow{(\Sigma^{10} i')^*} & \mathbf{Z}_2\{2\sigma''\} & \xrightarrow{\eta_{13}^*} & 0
\end{array}$$

We have $\eta_{14}^*\bar{\nu}_6 = \nu_6^3$ and $\eta_{14}^*\varepsilon_6 = \eta_6\varepsilon_7$ by [9, Lemma 6.3, (7.5)]. Hence $2\Sigma p_*\overline{[\sigma''']} = (\Sigma^{10}q_3)^*\mu_6$ by (2) and the second exact row of the above diagram. Therefore $\#\Sigma p_*\overline{[\sigma''']} = 4$ and $[C_{\eta_{13}}, \mathrm{S}^6]_{(2)} = \mathbf{Z}_4\{\Sigma p_*\overline{[\sigma''']}\}$ and $2\Sigma p_*\overline{[\sigma''']} = (\Sigma^{10}q_3)^*\mu_6$. □

3.2. Proof of Theorem 2.1 (2). By (3.1) and [6] ([3, Table 1]), we obtain the following exact sequence:

$$\mathbf{Z}_4\{[\nu_5^2]\nu_{11}\} \oplus \mathbf{Z}_2\{i_*\mu'\} \xrightarrow{\eta_{14}^*} \mathbf{Z}_4\{[2\iota_5]\nu_5\sigma_8\} \xrightarrow{(\Sigma^{10}q_3)^*} [C_{\eta_{13}}, \mathrm{SU}(3)]_{(2)}$$
$$\xrightarrow[(\Sigma^{10}i')^*]{} \mathbf{Z}_2\{i_*\varepsilon'\} \xrightarrow[\eta_{13}^*]{} \mathbf{Z}_4\{[\nu_5^2]\nu_{11}\} \oplus \mathbf{Z}_2\{i_*\mu'\}$$

We have $\eta_{14}^*[\nu_5^2]\nu_{11} = 0$. Since $p_* : \pi_{15}(\mathrm{SU}(3))_{(2)} \to \pi_{15}(\mathrm{S}^5)_{(2)} = \mathbf{Z}_8\{\nu_5\sigma_8\} \oplus \mathbf{Z}_2\{\eta_5\mu_6\}$ is injective and $p_*\eta_{14}^*i_*\mu' = p_*i_*\mu'\eta_{14} = 0$, it follows that $\eta_{14}^*i_*\mu' = 0$ and hence that the above $(\Sigma^{10}q_3)^*$ is injective. By Lemma 3.1 (1), $\eta_{13}^*i_*\varepsilon' = 2[\nu_5^2]\nu_{11}$. Hence the above $(\Sigma^{10}q_3)^*$ is surjective. Thus $(\Sigma^{10}q_3)^* : \mathbf{Z}_4\{[2\iota_5]\nu_5\sigma_8\} \cong [C_{\eta_{13}}, \mathrm{SU}(3)]_{(2)}$ is an isomorphism.

3.3. Proof of Theorem 2.1 (3). Consider (2.1) for the fibration $\mathrm{S}^3 \to \mathrm{SU}(3) \to \mathrm{S}^5$ and the cofibration $\mathrm{S}^4 \xrightarrow{\eta_3} \mathrm{S}^3 \to C_{\eta_3}$, that is, the following commutative diagram with exact rows and columns.

(3.4)

$$
\begin{array}{ccccc}
\mathbf{Z}_2^2\{\nu'\mu_6, \nu'\eta_6\varepsilon_7\} & \xrightarrow{\eta_{15}^*} & \mathbf{Z}_2\{\nu'\eta_6\mu_7\} & \xrightarrow{(\Sigma^{11}q_3)^*} & [C_{\eta_{14}}, \mathrm{S}^3]_{(2)} \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathbf{Z}_4\{[2\iota_5]\nu_5\sigma_8\} & \xrightarrow{\eta_{15}^*} & \mathbf{Z}_4\{[2\iota_5]\zeta_5\} \oplus \mathbf{Z}_2\{[\nu_5\bar{\nu}_8]\} & \xrightarrow{(\Sigma^{11}q_3)^*} & [C_{\eta_{14}}, \mathrm{SU}(3)]_{(2)} \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
\mathbf{Z}_8\{\nu_5\sigma_8\} \oplus \mathbf{Z}_2\{\eta_5\mu_6\} & \xrightarrow{\eta_{15}^*} & \mathbf{Z}_8\{\zeta_5\} \oplus \mathbf{Z}_2^2\{\nu_5\bar{\nu}_8, \nu_5\varepsilon_8\} & \xrightarrow{(\Sigma^{11}q_3)^*} & [C_{\eta_{14}}, \mathrm{S}^5]_{(2)} \\
 & \xrightarrow{(\Sigma^{11}i')^*} & \mathbf{Z}_4\{\mu'\} \oplus \mathbf{Z}_2^2\{\varepsilon_3\nu_{11}, \nu'\varepsilon_6\} & \xrightarrow{\eta_{14}^*} & \mathbf{Z}_2^2\{\nu'\mu_6, \nu'\eta_6\varepsilon_7\} \\
 & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
 & \xrightarrow{(\Sigma^{11}i')^*} & \mathbf{Z}_4\{[\nu_5^2]\nu_{11}\} \oplus \mathbf{Z}_2\{i_*\mu'\} & \xrightarrow{\eta_{14}^*} & \mathbf{Z}_4\{[2\iota_5]\nu_5\sigma_8\} \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
 & \xrightarrow{(\Sigma^{11}i')^*} & \mathbf{Z}_2^3\{\nu_5^3, \mu_5, \eta_5\varepsilon_6\} & \xrightarrow{\eta_{14}^*} & \mathbf{Z}_8\{\nu_5\sigma_8\} \oplus \mathbf{Z}_2\{\eta_5\mu_6\}
\end{array}
$$

LEMMA 3.2. (1) ([2, Proposition 3.3(6)])

$$\Sigma : [C_{\eta_{13}}, \mathrm{S}^{10}]_{(2)} = \mathbf{Z}_8\{\overline{\nu_{10}}\} \cong [C_{\eta_{14}}, \mathrm{S}^{11}]_{(2)}.$$

(2) $[C_{\eta_{14}}, \mathrm{S}^5]_{(2)} = \mathbf{Z}_4\{(\Sigma^{11}q_3)^*\zeta_5\} \oplus \mathbf{Z}_2^2\{(\Sigma^{11}q_3)^*\nu_5\bar{\nu}_8, \nu_5^2\Sigma\overline{\nu_{10}}\}$.
(3) $[C_{\eta_{14}}, \mathrm{S}^3]_{(2)} = \mathbf{Z}_2^2\{\overline{2\mu'}, \overline{\varepsilon_3\nu_{11}}\}$.
(4) *The following sequence is exact*:

$$0 \longrightarrow \mathbf{Z}_4\{[2\iota_5]\zeta_5\} \oplus \mathbf{Z}_2\{[\nu_5\bar{\nu}_8]\} \xrightarrow{(\Sigma^{11}q_3)^*} [C_{\eta_{14}}, \mathrm{SU}(3)]_{(2)}$$
$$\xrightarrow{(\Sigma^{11}i')^*} \mathbf{Z}_4\{[\nu_5^2]\nu_{11}\} \oplus \mathbf{Z}_2\{i_*\mu'\} \longrightarrow 0$$

(5) *The following sequence is exact*:

$$0 \to \mathbf{Z}_2^2\{\overline{2\mu'}, \overline{\varepsilon_3\nu_{11}}\} \xrightarrow{i_*} [C_{\eta_{14}}, \mathrm{SU}(3)]_{(2)} \xrightarrow{p_*} [C_{\eta_{14}}, \mathrm{S}^5]_{(2)} \to 0$$

(6) *There exists* $\overline{i_*\mu'}$ *such that* $2 \cdot \overline{i_*\mu'} = (\Sigma^{11}q_3)^*[2\iota_5]\zeta_5$ *and* $p_*\overline{i_*\mu'} = \pm(\Sigma^{11}q_3)^*\zeta_5$.

Before the proof of the lemma, we prove Theorem 2.1(3) from the lemma. By (3.4) and Lemma 3.2 (2),(4),(5), we see that $\mathbf{Z}_2\{(\Sigma^{11}q_3)^*[\nu_5\bar{\nu}_8]\}$ is a direct summand of $[C_{\eta_{14}}, \mathrm{SU}(3)]_{(2)}$ and that the order of $[\nu_5^2]\Sigma\overline{\nu_{10}}$ is 4, since the order of $[\nu_5^2]$ is 4 by [6]. Then Theorem 2.1(3) is easily obtained from Lemma 3.2(4),(6). □

Proof of Lemma 3.2. (1) Since $[C_{\eta_{13}}, S^{10}]$ is stable and $\pi_{14}(S^{10}) = \pi_{15}(S^{10}) = 0$, we obtain (1).

(2) In the third row of (3.4) we have $\nu_5\sigma_8\eta_{15} = \nu_5\varepsilon_8$ by [9, p. 152], $\eta_5\mu_6\eta_{15} = 4\zeta_5$ by [7, Proposition (2.2)] and [9, (7.7), (7.14)], $\nu_5^3\eta_{14} = 0$ by [9, Proposition 5.2], and $\eta_5\varepsilon_6\eta_{14} = 4(\nu_5\sigma_8)$ by [9, Lemma 6.6, (7.5), (7.10)]. Hence we have the following exact sequence:

$$(3.5)\quad 0 \to \mathbf{Z}_4\{(\Sigma^{11}q_3)^*\zeta_5\} \oplus \mathbf{Z}_2\{(\Sigma^{11}q_3)^*\nu_5\bar{\nu}_8\} \to [C_{\eta_{14}}, S^5]_{(2)} \xrightarrow{(\Sigma^{11}i')^*} \mathbf{Z}_2\{\nu_5^3\} \to 0$$

Since $(\Sigma^{11}i')^*\nu_5^2\Sigma\overline{\nu_{10}} = \nu_5^3$, the order of $\nu_5^2\Sigma\overline{\nu_{10}}$ is 2. Hence (3.5) splits and (2) is obtained.

(3) In the first row of (3.4) we have $\nu'\mu_6\eta_{15} = \nu'\eta_6\mu_7$, $\mu'\eta_{14} = \nu'\mu_6$ by [7, Proposition (2.2)] and $\varepsilon_3\nu_{11}\eta_{14} = 0$, $\nu'\varepsilon_6\eta_{14} = \nu'\eta_6\varepsilon_7$ by [9]. Hence $(\Sigma^{11}i')^*: [C_{\eta_{14}}, S^3]_{(2)} \cong \mathbf{Z}_2^2\{2\mu', \varepsilon_3\nu_{11}\}$. Thus we obtain (3).

(4) By the same proof of [6, (4.4)], we have $2\iota_5 \circ (\nu_5\sigma_8\eta_{15}) = 2(\nu_5\sigma_8\eta_{15})$ and hence

$$p_*\eta_{15}^*([2\iota_5]\nu_5\sigma_8) = 2\iota_5 \circ (\nu_5\sigma_8\eta_{15}) = 2(\nu_5\sigma_8\eta_{15}) = 0.$$

Since $2\iota_5 \circ \zeta_5 = 2\zeta_5$ by [6, (4.4)], the second p_* of (3.4) is injective, and hence $\eta_{15}^*([2\iota_5]\nu_5\sigma_8) = 0$. Also $p_*([2\iota_5]\nu_5\sigma_8) = 2(\nu_5\sigma_8)$ by [6, (4.4)], and $\eta_{14}^*([\nu_5^2]\nu_{11}) = 0$. Thus we obtain (4).

(5) Since the orders of $[C_{\eta_{14}}, S^3]_{(2)}$, $[C_{\eta_{14}}, SU(3)]_{(2)}$ and $[C_{\eta_{14}}, S^5]_{(2)}$ are respectively 4, 64 and 16 by (3), (4) and (2), we obtain (5).

(6) By (2) and (4), we can write

$$(3.6)\qquad p_*\overline{i_*\mu'} = x \cdot (\Sigma^{11}q_3)^*\zeta_5 + y \cdot (\Sigma^{11}q_3)^*\nu_5\bar{\nu}_8 + z \cdot \nu_5^2\Sigma\overline{\nu_{10}},$$

$$(3.7)\qquad 2 \cdot \overline{i_*\mu'} = A \cdot (\Sigma^{11}q_3)^*[2\iota_5]\zeta_5 + B \cdot (\Sigma^{11}q_3)^*[\nu_5\bar{\nu}_8],$$

where $x, A \in \{0,1,2,3\}$ and $y, z, B \in \{0,1\}$.

By applying $(\Sigma^{11}i')^*$ to (3.6), we have $0 = z \cdot \nu_5^3$. Hence $z = 0$.

We will show that $x = 1$ or 3. If $x = 0$, then $\overline{i_*\mu'} - y \cdot (\Sigma^{11}q_3)^*[\nu_5\bar{\nu}_8] \in \mathrm{Ker}(p_*) = \mathrm{Image}(i_*)$ and hence

$$i_*\mu' = (\Sigma^{11}i')^*(\overline{i_*\mu'} - y(\Sigma^{11}q_3)^*[\nu_5\bar{\nu}_8]) \in i_*(\Sigma^{11}i')^*[C_{\eta_{14}}, S^3]_{(2)} = \mathbf{Z}_2\{2[\nu_5^2]\nu_{11}\}$$

by (3) and [6, (4.1)]. This is impossible. Hence $x \neq 0$. If $x = 2$, then

$$p_*\overline{i_*\mu'} = p_*(\Sigma^{11}q_3)^*([2\iota_5]\zeta_5 + y[\nu_5\bar{\nu}_8]),$$

since $2\iota_5 \circ \zeta_5 = 2\zeta_5$ by [6, (4.4)]. Now $\overline{i_*\mu'} - (\Sigma^{11}q_3)^*([2\iota_5]\zeta_5 + y[\nu_5\bar{\nu}_8]) \in \mathrm{Ker}(p_*) = \mathrm{Image}(i_*)$. Hence $\overline{i_*\mu'} - (\Sigma^{11}q_3)^*([2\iota_5]\zeta_5 + y[\nu_5\bar{\nu}_8]) \equiv 0 \pmod{i_*\overline{2\mu'}, i_*\overline{\varepsilon_3\nu_{11}}}$ by (3). Then $i_*\mu' = (\Sigma^{11}i')^*(\overline{i_*\mu'} - (\Sigma^{11}q_3)^*([2\iota_5]\zeta_5 + y[\nu_5\bar{\nu}_8])) \equiv 0 \pmod{2i_*\mu', 2[\nu_5^2]\nu_{11}}$ by (4) and [6, (4.1)]. This is impossible. Hence $x \neq 2$. Therefore $x = 1$ or 3 as desired.

By applying p_* to (3.7), we have $2p_*\overline{i_*\mu'} = 2A(\Sigma^{11}q_3)^*\zeta_5 + B(\Sigma^{11}q_3)^*\nu_5\bar{\nu}_8$. The left term of this equality is $2x \cdot (\Sigma^{11}q_3)^*\zeta_5$ by (3.6). Hence $A \equiv x \pmod 2$ and $B = 0$. Rewrite $w \cdot \overline{i_*\mu'} + y(\Sigma^{11}q_3)^*[\nu_5\bar{\nu}_9]$ as $\overline{i_*\mu'}$, where w is 1 or -1 according as A is 1 or 3. Then $2 \cdot \overline{i_*\mu'} = (\Sigma^{11}q_3)^*[2\iota_5]\zeta_5$ and $p_*\overline{i_*\mu'} =$

$-wx(\Sigma^{11}q_3)^*\zeta_5 = \pm(\Sigma^{11}q_3)^*\zeta_5$. Thus we obtain (6). This completes the proof of Lemma 3.2. □

4. Sp(2)

The purpose of this section is to prove (4), (5) and (6) of Theorem 2.1. Recall that $\omega = \nu' + \alpha_1(3)$ and so $\Sigma^n\omega = 2\nu_{n+3} + \alpha_1(n+3)$ for $n \geq 2$.

4.1. Proof of Theorem 2.1 (4). By [6] ([3, Table 4]) and [9], we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & & & [C_{\Sigma^9\omega}, S^3] & \xrightarrow{(\Sigma^9 i')^*} & \mathbf{Z}_2^2\{\mu_3, \eta_3\varepsilon_4\} & \xrightarrow{(\Sigma^9\omega)^*} & 0 \\
 & & & & \downarrow{\scriptstyle i_*} & & \cong\downarrow{\scriptstyle i_*} & & \\
0 & \longrightarrow & \mathbf{Z}_2^2\{[\sigma'\eta_{14}]\eta_{15}, [\nu_7]\nu_{10}^2\} & \xrightarrow{(\Sigma^9 q_3)^*} & [C_{\Sigma^9\omega}, \mathrm{Sp}(2)] & \xrightarrow{(\Sigma^9 i')^*} & \mathbf{Z}_2^2\{i_*\mu_3, i_*\eta_3\varepsilon_4\} & \xrightarrow{(\Sigma^9\omega)^*} & 0 \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\
0 & \longrightarrow & \mathbf{Z}_2^4\{\sigma'\eta_{14}^2, \nu_7^3, \mu_7, \eta_7\varepsilon_8\} & \underset{\cong}{\xrightarrow{(\Sigma^9 q_3)^*}} & [C_{\Sigma^9\omega}, S^7] & \longrightarrow & 0 & &
\end{array}$$

Commutativity of this diagram implies that the first $(\Sigma^9 q_3)^*$ has a left inverse. Hence the second row splits and so (4) is obtained.

4.2. Proof of Theorem 2.1 (5). In the following exact sequence

$$\pi_{17}(\mathrm{Sp}(2)) \xrightarrow{(\Sigma^{10}q_3)^*} [C_{\Sigma^{10}\omega}, \mathrm{Sp}(2)] \xrightarrow{(\Sigma^{10}i')^*} \pi_{13}(\mathrm{Sp}(2))$$

we have $\pi_{17}(\mathrm{Sp}(2))_{(3)} = \pi_{13}(\mathrm{Sp}(2))_{(3)} = 0$ by [6] ([3, Table 4]). Hence

$$[C_{\Sigma^{10}\omega}, \mathrm{Sp}(2)]_{(2,3)} = [C_{\Sigma^{10}\omega}, \mathrm{Sp}(2)]_{(2)}. \tag{4.1}$$

Therefore it suffices to compute $[C_{\Sigma^{10}\omega}, \mathrm{Sp}(2)]_{(2)}$.

LEMMA 4.1. (1) *We have the following commutative diagram with exact rows and columns from* (2.1).

(4.2)

$$\begin{array}{ccccccccc}
 & & & & [C_{\Sigma^{11}\omega}, S^7]_{(2)} & \xrightarrow{(\Sigma^{11} i')^*} & \mathbf{Z}_8\{\sigma'\} & \xrightarrow{(\Sigma^{11}\omega)^*} & \cdots \\
 & & & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \xrightarrow{(\Sigma^{11}\omega)^*} & \mathbf{Z}_2\{\varepsilon_3\nu_{11}^2\} & \xrightarrow{(\Sigma^{10} q_3)^*} & [C_{\Sigma^{10}\omega}, S^3]_{(2)} & \xrightarrow{(\Sigma^{10} i')^*} & \mathbf{Z}_4\{\varepsilon'\} \oplus \mathbf{Z}_2\{\eta_3\mu_4\} & \xrightarrow{(\Sigma^{10}\omega)^*} & 0 \\
 & & \downarrow{\scriptstyle i_*=0} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \\
\mathbf{Z}_{16}\{[2\sigma']\} & \xrightarrow{(\Sigma^{11}\omega)^*} & \mathbf{Z}_8\{[\nu_7]\sigma_{10}\} & \xrightarrow{(\Sigma^{10} q_3)^*} & [C_{\Sigma^{10}\omega}, \mathrm{Sp}(2)]_{(2)} & \xrightarrow{(\Sigma^{10} i')^*} & \mathbf{Z}_4\{[\nu_7]\nu_{10}\} \oplus \mathbf{Z}_2\{i_*\eta_3\mu_4\} & \xrightarrow{(\Sigma^{10}\omega)^*} & 0 \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\
\mathbf{Z}_8\{\sigma'\} & \xrightarrow{(\Sigma^{11}\omega)^*} & \mathbf{Z}_8\{\nu_7\sigma_{10}\} \oplus \mathbf{Z}_2\{\eta_7\mu_8\} & \xrightarrow{(\Sigma^{10} q_3)^*} & [C_{\Sigma^{10}\omega}, S^7]_{(2)} & \xrightarrow{(\Sigma^{10} i')^*} & \mathbf{Z}_2\{\nu_7^2\} & \xrightarrow{(\Sigma^{10}\omega)^*} & 0
\end{array}$$

We have

(4.3) $(\Sigma^{10}q_3)^*(\mathbf{Z}_8\{\nu_7\sigma_{10}\} \oplus \mathbf{Z}_2\{\eta_7\mu_8\}) = \mathbf{Z}_2^2\{(\Sigma^{10}q_3)^*\nu_7\sigma_{10}, (\Sigma^{10}q_3)^*\eta_7\mu_8\},$

(4.4) $(\Sigma^{11}i')^*[C_{\Sigma^{11}\omega}, \mathrm{S}^7]_{(2)} = \mathbf{Z}_2\{4\sigma'\},$

(4.5) $(\Sigma^{10}q_3)^*\mathbf{Z}_8\{[\nu_7]\sigma_{10}\} = \mathbf{Z}_4\{(\Sigma^{10}q_3)^*[\nu_7]\sigma_{10}\},$

(4.6) $[C_{\Sigma^{10}\omega}, \mathrm{S}^7]_{(2)} = \mathbf{Z}_2^3\{(\Sigma^{10}q_3)^*\nu_7\sigma_{10}, (\Sigma^{10}q_3)^*\eta_7\mu_8, \nu_7\Sigma^5\overline{\nu_5}\}.$

(2) *The order of* $i_*\overline{\varepsilon'}$ *is* 4.

(3) $\mathbf{Z}_2\{i_*\mu_3\overline{\eta_{12}}\}$ *is a direct summand of* $[C_{\Sigma^{10}\omega}, \mathrm{Sp}(2)]_{(2)}$ *and* $(\Sigma^{10}i')^*i_*\mu_3\overline{\eta_{12}} = i_*\eta_3\mu_4$.

Before the proof of Lemma 4.1 we prove Theorem 2.1 (5) from Lemma 4.1. By Lemma 4.1 (3), $[C_{\Sigma^{10}\omega}, \mathrm{Sp}(2)]_{(2)} = \mathbf{Z}_2\{i_*\mu_3\overline{\eta_{12}}\} \oplus L$, where L is the subgroup generated by $\overline{[\nu_7]\nu_{10}}$ and $(\Sigma^{10}q_3)^*[\nu_7]\sigma_{10}$. Hence it suffices for Theorem 2.1 (5) to prove that $L = \mathbf{Z}_8\{\overline{[\nu_7]\nu_{10}}\} \oplus \mathbf{Z}_2\{2\cdot\overline{[\nu_7]\nu_{10}} - (\Sigma^{10}q_3)^*[\nu_7]\sigma_{10}\}$.

By (4.5) and the third row of (4.2), we can write

(4.7) $4\cdot\overline{[\nu_7]\nu_{10}} = x\cdot(\Sigma^{10}q_3)^*[\nu_7]\sigma_{10} \quad (x \in \{0,1,2,3\}).$

Since $i_*\varepsilon' = 2[\nu_7]\nu_{10}$ by [6, (5.1)], it follows that $i_*\overline{\varepsilon'} - 2\cdot\overline{[\nu_7]\nu_{10}} \in \mathrm{Ker}(\Sigma^{10}i')^* = \mathrm{Image}(\Sigma^{10}q_3)^*$ and hence from (4.5) that $i_*\overline{\varepsilon'} - 2\cdot\overline{[\nu_7]\nu_{10}} \in \mathbf{Z}_4\{(\Sigma^{10}q_3)^*[\nu_7]\nu_{10}\}$, that is,

(4.8) $i_*\overline{\varepsilon'} - 2\cdot\overline{[\nu_7]\nu_{10}} = y\cdot(\Sigma^{10}q_3)^*[\nu_7]\sigma_{10} \quad (y \in \{0,1,2,3\}).$

By Lemma 4.1 (2), (4.7) and (4.8), we have

$$0 \neq 2\cdot i_*\overline{\varepsilon'} = (x+2y)(\Sigma^{10}q_3)^*[\nu_7]\nu_{10}, \quad 0 = 4\cdot i_*\overline{\varepsilon'} = 2(x+2y)(\Sigma^{10}q_3)^*[\nu_7]\nu_{10}.$$

Hence $x + 2y \equiv 2 \pmod 4$ and so $2\cdot i_*\overline{\varepsilon'} = 2(\Sigma^{10}q_3)^*[\nu_7]\sigma_{10}$ and $x = 0$ or 2. To induce a contradiction, assume $x = 0$. Then $y = 1$ or 3 and $\#\overline{[\nu_7]\nu_{10}} = 4$ by (4.7). On the other hand, it follows from (4.8) that $0 = p_*i_*\overline{\varepsilon'} = 2\cdot p_*\overline{[\nu_7]\nu_{10}} + y(\Sigma^{10}q_3)^*\nu_7\sigma_{10}$. Hence the order of $p_*\overline{[\nu_7]\nu_{10}} \in [C_{\Sigma^{10}\omega}, \mathrm{S}^7]_{(2)}$ is 4. This contradicts (4.6). Hence $x = 2$ so that $\#\overline{[\nu_7]\nu_{10}} = 8$ and $4\overline{[\nu_7]\nu_{10}} = 2(\Sigma^{10}q_3)^*[\nu_7]\sigma_{10}$ by (4.7). Therefore

$$L = \mathbf{Z}_8\{\overline{[\nu_7]\nu_{10}}\} \oplus \mathbf{Z}_2\{2\cdot\overline{[\nu_7]\nu_{10}} - (\Sigma^{10}q_3)^*[\nu_7]\sigma_{10}\}$$

as desired. □

Proof of Lemma 4.1. Since $\Sigma^n\omega \equiv 2\nu_{n+3} \pmod{\alpha_1(n+3)}$ for $n \geq 2$ and since $\pi_{17}(\mathrm{S}^3)_{(2)} \cong \pi_{16}(\mathrm{S}^3)_{(2)} \cong \pi_{13}(\mathrm{S}^7) \cong \mathbf{Z}_2$ and $\pi_{16}(\mathrm{Sp}(2)) \cong \mathbf{Z}_2^2$, it follows that the homomorphisms $(\Sigma^{10}\omega)^* : \pi_{13}(X)_{(2)} \to \pi_{16}(X)_{(2)}$ for $X = \mathrm{S}^3, \mathrm{S}^7, \mathrm{Sp}(2)$ and $(\Sigma^{11}\omega)^* : \pi_{14}(\mathrm{S}^3)_{(2)} \to \pi_{17}(\mathrm{S}^3)_{(2)}$ are trivial. Hence the second row of (4.2) is exact and the second and the third $(\Sigma^{10}i')^*$ are surjective. We have $(\Sigma^{11}\omega)^*\sigma' = 2(\sigma'\nu_{14}) = 2k\cdot\nu_7\sigma_{10}$ for some odd integer k by [9, (7.19)]. Hence we obtain (4.3), (4.4) and (4.5). To prove (4.6), consider the following commutative diagram with exact rows.

(4.9)

$$
\begin{array}{ccccccccc}
0 & \xrightarrow{(\Sigma^{10}\omega)^*} & \mathbf{Z}_2\{\nu'\eta_6\mu_7\} & \xrightarrow{(\Sigma^9 q_3)^*} & [C_{\Sigma^9\omega}, \mathrm{S}^3]_{(2)} & \xrightarrow{(\Sigma^9 i')^*} & \mathbf{Z}_2^2 & \xrightarrow{(\Sigma^9\omega)^*} & 0 \\
 & & \partial \big\uparrow & & \partial \big\uparrow & & \partial \big\uparrow & & \\
\mathbf{Z}_8\{\sigma'\} & \xrightarrow{(\Sigma^{11}\omega)^*} & \mathbf{Z}_8\{\nu_7\sigma_{10}\} \oplus \mathbf{Z}_2\{\eta_7\mu_8\} & \xrightarrow{(\Sigma^{10} q_3)^*} & [C_{\Sigma^{10}\omega}, \mathrm{S}^7]_{(2)} & \xrightarrow{(\Sigma^{10} i')^*} & \mathbf{Z}_2\{\nu_7^2\} & \xrightarrow{(\Sigma^{10}\omega)^*} & 0 \\
 & & \Sigma^2 \big\uparrow \cong & & \Sigma^2 \big\uparrow & & \Sigma^2 \big\uparrow \cong & & \\
0 & \xrightarrow{(\Sigma^{9}\omega)^*} & \mathbf{Z}_8\{\nu_5\sigma_8\} \oplus \mathbf{Z}_2\{\eta_5\mu_6\} & \xrightarrow{(\Sigma^8 q_3)^*} & [C_{\Sigma^8\omega}, \mathrm{S}^5]_{(2)} & \xrightarrow{(\Sigma^8 i')^*} & \mathbf{Z}_2\{\nu_5^2\} & \xrightarrow{(\Sigma^8\omega)^*} & 0 \\
 & & & & \nu_{5*} \big\uparrow & & \nu_{5*} \big\uparrow & & \\
 & & & & [C_{\Sigma^8\omega}, \mathrm{S}^8]_{(2)} & \xrightarrow{(\Sigma^8 i')^*} & \mathbf{Z}_8\{\nu_8\} & \xrightarrow{(\Sigma^8\omega)^*} & 0 \\
 & & & & \Sigma^3 \big\uparrow & & \Sigma^3 \big\uparrow \cong & & \\
 & & \mathbf{Z}_2\{\sigma'''\} & \xrightarrow{(\Sigma^5 q_3)^*} & [C_{\Sigma^5\omega}, \mathrm{S}^5]_{(2)} & \xrightarrow{(\Sigma^5 i')^*} & \mathbf{Z}_8\{\nu_5\} & \xrightarrow{(\Sigma^5\omega)^*} & 0 \\
 & & \big\uparrow & & \Sigma \big\uparrow & & \Sigma \big\uparrow & & \\
 & & 0 & \xrightarrow{(\Sigma^4 q_3)^*} & [C_{\Sigma^4\omega}, \mathrm{S}^4]_{(2)} & \xrightarrow{(\Sigma^4 i')^*} & \mathbf{Z}_{(2)}\{\nu_4\} \oplus \mathbf{Z}_4\{\Sigma\nu'\} & \xrightarrow{(\Sigma^4\omega)^*} & \mathbf{Z}_8\{\nu_4^2\}
\end{array}
$$

Since $(\Sigma^4\omega)^*\Sigma\nu' = \Sigma\nu' \circ 2\nu_7 = 0$, we can take $\overline{\Sigma\nu'} \in [C_{\Sigma^4\omega}, \mathrm{S}^4]_{(2)}$ such that

$$\#\overline{\Sigma\nu'} = 4. \tag{4.10}$$

Since $\Sigma^2\nu' = 2\nu_5$ by [9, Lemma 5.4], we have $(\Sigma^5 i')^*\Sigma\overline{\Sigma\nu'} = 2\nu_5 = (\Sigma^5 i')^*(2\overline{\nu_5})$ so that there exists an integer x such that

$$\Sigma\overline{\Sigma\nu'} = 2\overline{\nu_5} + x \cdot (\Sigma^5 q_3)^*\sigma'''$$

and hence

$$\Sigma^6\overline{\Sigma\nu'} = 2\Sigma^5\overline{\nu_5} + 8x(\Sigma^{10} q_3)^*\sigma_{10}, \tag{4.11}$$

since $\Sigma^4\sigma''' = 8\sigma^9$ by [9, Lemma 5.14]. Since $(\Sigma^8 i')^*(\nu_5\Sigma^4\overline{\Sigma\nu'}) = \nu_5\Sigma^5\nu' = 2\nu_5^2 = 0$, we can write

$$\nu_5\Sigma^4\overline{\Sigma\nu'} = a \cdot (\Sigma^8 q_3)^*\nu_5\sigma_8 + b \cdot (\Sigma^8 q_3)^*\eta_5\mu_6 \quad (a \in \mathbf{Z}, b \in \{0,1\}).$$

We have $0 = 4(\nu_5\Sigma^4\overline{\Sigma\nu'}) = 4a \cdot (\Sigma^8 q_3)^*\nu_5\sigma_8$, where the first equality follows from (4.10). Hence $a \equiv 0 \pmod 2$ so that, by (4.3), we have

$$\nu_7\Sigma^6\overline{\Sigma\nu'} = \Sigma^2(\nu_5\Sigma^4\overline{\Sigma\nu'}) = b \cdot (\Sigma^{10} q_3)^*\eta_7\mu_8. \tag{4.12}$$

In (4.9), we have $\partial(\eta_7\mu_8) = \nu'\eta_6\mu_7$ and $\partial\nu_7^2 = 0$, since $\partial\iota_7 \equiv \pm\nu' \pmod{\alpha_1(3)}$. Hence $(\Sigma^8 i')^*\partial\overline{\nu_7^2} = 0$. Therefore

$$\partial([C_{\Sigma^{10}\omega}, \mathrm{S}^7]_{(2)}) = \mathbf{Z}_2\{(\Sigma^9 q_3)^*\nu'\eta_6\mu_7\}. \tag{4.13}$$

Now, by (4.11) and (4.12), we have

$$b \cdot (\Sigma^{10} q_3)^*\eta_7\mu_8 = \nu_7\Sigma^6\overline{\Sigma\nu'} = \nu_7(2\Sigma^5\overline{\nu_5} + 8x(\Sigma^{10} q_3)^*\sigma_{10}) = 2(\nu_7\Sigma^5\overline{\nu_5}).$$

Also $0=\partial(2\nu_7\Sigma^5\overline{\nu_5})=\partial(b\cdot(\Sigma^{10}q_3)^*\eta_7\mu_8)=b\cdot(\Sigma^9 q_3)^*\nu'\eta_6\mu_7$, where the first equality follows from (4.13). Hence $b=0$ so that $2(\nu_7\Sigma^5\overline{\nu_5})=0$. This proves (4.6) and completes the proof of (1).

(2) We have $4\overline{\varepsilon'}=(\Sigma^{10}q_3)^*(\alpha)$ for some $\alpha\in\mathbf{Z}_2\{\varepsilon_3\nu_{11}^2\}$. Then $i_*(4\overline{\varepsilon'})=i_*(\Sigma^{10}q_3)^*(\alpha)=(\Sigma^{10}q_3)^*i_*(\alpha)=0$. Thus $\# i_*\overline{\varepsilon'}$ is 1, 2 or 4. Set $k=\# i_*\overline{\varepsilon'}$. To induce a contradiction, assume k is 1 or 2. Since $i_*(k\overline{\varepsilon'})=0$, we have $k\overline{\varepsilon'}=\partial(\beta)$ for some $\beta\in[C_{\Sigma^{11}\omega},\mathrm{S}^7]_{(2)}$. Then

$$0\neq k\varepsilon'=(\Sigma^{10}i')^*(k\overline{\varepsilon'})=(\Sigma^{10}i')^*\partial(\beta)=\partial(\Sigma^{11}i')^*\beta\in\partial(\mathbf{Z}_2\{4\sigma'\})=0,$$

since $(\Sigma^{11}i')^*[C_{\Sigma^{11}\omega},\mathrm{S}^7]_{(2)}=\mathrm{Ker}(\Sigma^{11}\omega)^*=\mathbf{Z}_2\{4\sigma'\}$. This is a contradiction. Thus $\# i_*\overline{\varepsilon'}=4$ as desired.

(3) By the exact sequence

$$\pi_{17}(\mathrm{S}^{12})=0\longrightarrow[C_{\Sigma^{10}\omega},\mathrm{S}^{12}]\xrightarrow{(\Sigma^{10}i')^*}\pi_{13}(\mathrm{S}^{12})\longrightarrow\pi_{16}(\mathrm{S}^{12})=0$$

we have $[C_{\Sigma^{10}\omega},\mathrm{S}^{12}]=\mathbf{Z}_2\{\overline{\eta_{12}}\}$. Since $(\Sigma^{10}i')^*(\mu_3\overline{\eta_{12}})=\mu_3\eta_{12}=\eta_3\mu_4$ and since $\#(\mu_3\overline{\eta_{12}})=2$, $\mathbf{Z}_2\{\mu_3\overline{\eta_{12}}\}$ is a direct summand of $[C_{\Sigma^{10}\omega},\mathrm{S}^3]_{(2)}$. Thus we obtain (3), since $(\Sigma^{10}i')^*i_*\mu_3\overline{\eta_{12}}=i_*(\Sigma^{10}i')^*\mu_3\overline{\eta_{12}}=i_*\mu_3\eta_{12}$. This completes the proof of Lemma 4.1. □

4.3. Proof of Theorem 2.1 (6). Let

$$\mathrm{S}^6\xrightarrow{\nu'}\mathrm{S}^3\xrightarrow{i_2'}C_{\nu'}\xrightarrow{q_{3,2}}\mathrm{S}^7;\quad \mathrm{S}^6\xrightarrow{\alpha_1(3)}\mathrm{S}^3\xrightarrow{i_3'}C_{\alpha_1(3)}\xrightarrow{q_{3,3}}\mathrm{S}^7$$

be the usual cofibrations. If $n\geq 2$, then $9\Sigma^n\omega=2\nu_{n+3}$ and $4\Sigma^n\omega=\alpha_1(n+3)$ and so we have the following commutative diagram of cofibrations:

$$\begin{array}{ccccc}
\mathrm{S}^{n+6} & \xrightarrow{2\nu_{n+3}} & \mathrm{S}^{n+3} & \longrightarrow & C_{2\nu_{n+3}}\\
\Big\downarrow{\scriptstyle 9\iota_{n+6}} & & \Big\| & & \Big\downarrow{\scriptstyle f}\\
\mathrm{S}^{n+6} & \xrightarrow{\Sigma^n\omega} & \mathrm{S}^{n+3} & \longrightarrow & C_{\Sigma^n\omega}
\end{array};\quad
\begin{array}{ccccc}
\mathrm{S}^{n+6} & \xrightarrow{\alpha_1(n+3)} & \mathrm{S}^{n+3} & \longrightarrow & C_{\alpha_1(n+3)}\\
\Big\downarrow{\scriptstyle 4\iota_{n+6}} & & \Big\| & & \Big\downarrow{\scriptstyle g}\\
\mathrm{S}^{n+6} & \xrightarrow{\Sigma^n\omega} & \mathrm{S}^{n+3} & \longrightarrow & C_{\Sigma^n\omega}
\end{array}$$

Then for any $n\geq 2$ and for any pointed space X we have

$$f^*:[C_{\Sigma^n\omega},X]_{(2)}\cong[C_{2\nu_{n+3}},X]_{(2)};\quad g^*:[C_{\Sigma^n\omega},X]_{(3)}\cong[C_{\alpha_1(n+3)},X]_{(3)}.$$

If $n\geq 5$, then $[C_{\Sigma^n\omega},\mathrm{Sp}(2)]$ is a finite group and so

$$\begin{aligned}[C_{\Sigma^n\omega},\mathrm{Sp}(2)]_{(2,3)}&=[C_{\Sigma^n\omega},\mathrm{Sp}(2)]_{(2)}\oplus[C_{\Sigma^n\omega},\mathrm{Sp}(2)]_{(3)}\\&\cong[C_{2\nu_{n+3}},\mathrm{Sp}(2)]_{(2)}\oplus[C_{\alpha_1(n+3)},\mathrm{Sp}(2)]_{(3)}.\end{aligned}$$

Therefore it suffices to prove the following:

(4.14) $$[C_{2\nu_{14}},\mathrm{Sp}(2)]_{(2)}=\mathbf{Z}_8^2\{(\Sigma^{11}q_{3,2})^*[\zeta_7],\overline{2[2\sigma']}\}\oplus\mathbf{Z}_2\{(\Sigma^{11}q_{3,2})^*i_*\bar{\varepsilon}_3\},$$

(4.15) $$[C_{\alpha_1(14)},\mathrm{Sp}(2)]_{(3)}=\mathbf{Z}_{27}\{\overline{i_*\alpha_3(3)}\}.$$

Lemma 4.2. (1) $[C_{2\nu_{14}}, S^3]_{(2)} = \mathbf{Z}_4\{\overline{\mu'}\} \oplus \mathbf{Z}_2^3\{\varepsilon_3\overline{\nu_{11}}, \varepsilon'\overline{\eta_{13}}, (\Sigma^{11}q_{3,2})^*\bar{\varepsilon}_3\}$.
(2) $[C_{2\nu_{14}}, S^7]_{(2)} = \mathbf{Z}_8\{(\Sigma^{11}q_{3,2})^*\zeta_7\} \oplus \mathbf{Z}_2^2\{(\Sigma^{11}q_{3,2})^*\bar{\nu}_7\nu_{16}, \overline{4\sigma'}\}$.
(3) ([3, Proposition 4.4(1)]) $3\iota_5 \circ \pi_{12}(S^5) = 3\pi_{12}(S^5)$.
(4) ([9]) $\pi_{13}(S^3)_{(3)} = \mathbf{Z}_3\{\alpha_1(3)\alpha_2(6)\}$ *and* $\alpha_2(3)\alpha_1(10) = -\alpha_1(3)\alpha_2(6)$.
(5) ([9]) $\pi_{17}(S^3)_{(3)} = \mathbf{Z}_3\{\alpha_1(3)\alpha_3'(6)\}$ *and* $\alpha_3(3)\alpha_1(14) = \alpha_1(3)\alpha_3'(6)$.

Before the proof of Lemma 4.2, we prove (4.14) and (4.15) from Lemma 4.2.

For $n \geq 2$, to simplify notations, we denote $\Sigma^n q_{3,2} : C_{2\nu_{n+3}} \to S^{n+7}$ and $\Sigma^n q_{3,3} : C_{\alpha_1(n+3)} \to S^{n+7}$ by q, and $\Sigma^n i_2' : S^{n+3} \to C_{2\nu_{n+3}}$ and $\Sigma^n i_3' : S^{n+3} \to C_{\alpha_1(n+3)}$ by i''.

To prove (4.14), we consider (2.1) for the fibration $S^3 \xrightarrow{i} Sp(2) \xrightarrow{p} S^7$ and the cofibration $S^8 \xrightarrow{2\nu_5} S^5 \xrightarrow{i''} C_{2\nu_5}$, that is, the following commutative diagram with exact rows and columns, where $\rightarrowtail$ means a monomorphism.

$$\begin{array}{ccccccc}
 & & [C_{2\nu_{15}}, Sp(2)]_{(2)} & \xrightarrow{i''^*} & \mathbf{Z}_2\{[\sigma'\eta_{14}]\} & \xrightarrow{(2\nu_{15})^*} & 0 \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\
0 & \longrightarrow & [C_{2\nu_{15}}, S^7]_{(2)} & \underset{\cong}{\xrightarrow{i''^*}} & \mathbf{Z}_2^3\{\sigma'\eta_{14}, \bar{\nu}_7, \varepsilon_7\} & \xrightarrow{(2\nu_{15})^*} & 0 \\
\downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
\mathbf{Z}_2\{\bar{\varepsilon}_3\} & \overset{q^*}{\rightarrowtail} & [C_{2\nu_{14}}, S^3]_{(2)} & \xrightarrow{i''^*} & \mathbf{Z}_4\{\mu'\} \oplus \mathbf{Z}_2^2\{\varepsilon_3\nu_{11}, \nu'\varepsilon_6\} & \xrightarrow{(2\nu_{14})^*} & 0 \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow \\
\mathbf{Z}_8\{[\zeta_7]\} \oplus \mathbf{Z}_2\{i_*\bar{\varepsilon}_3\} & \overset{q^*}{\rightarrowtail} & [C_{2\nu_{14}}, Sp(2)]_{(2)} & \xrightarrow{i''^*} & \mathbf{Z}_{16}\{[2\sigma']\} & \xrightarrow{(2\nu_{14})^*} & \mathbf{Z}_8\{[\nu_7]\sigma_{10}\} \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
\mathbf{Z}_8\{\zeta_7\} \oplus \mathbf{Z}_2\{\bar{\nu}_7\nu_{15}\} & \overset{q^*}{\rightarrowtail} & [C_{2\nu_{14}}, S^7]_{(2)} & \xrightarrow{i''^*} & \mathbf{Z}_8\{\sigma'\} & \xrightarrow{(2\nu_{14})^*} & \mathbf{Z}_8\{\nu_7\sigma_{10}\} \oplus \mathbf{Z}_2\{\eta_7\mu_8\}
\end{array}$$

Here we have not used Lemma 4.2 but [6, 9]. Since $p_*(2\nu_{14})^*[2\sigma'] = 4(\sigma'\nu_{14}) = 4(\nu_7\sigma_{10}) = p_*(4[\nu_7]\sigma_{10})$ by [9, (7.19)], we have $(2\nu_{14})^*[2\sigma'] = 4[\nu_7]\sigma_{10}$. Since $\partial(\iota_7) = \nu'$, we have $\partial(\bar{\nu}_7) = \nu'\bar{\nu}_6 = \varepsilon_3\nu_{11}$, $\partial(\bar{\nu}_7\nu_{15}) = \varepsilon_3\nu_{11}^2$ and $\partial(\varepsilon_7) = \nu'\varepsilon_6 = \varepsilon'\eta_{13}$. It then follows from Lemma 4.2(1),(2) that the above diagram induces the following commutative diagram with short exact rows and exact columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{Z}_2\{\bar{\varepsilon}_3\} & \xrightarrow{q^*} & \mathbf{Z}_2\{q^*\bar{\varepsilon}_3\} \oplus \mathbf{Z}_4\{\overline{\mu'}\} & \xrightarrow{i''^*} & \mathbf{Z}_4\{\mu'\} & \to & 0 \\
 & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \\
0 & \to & \mathbf{Z}_8\{[\zeta_7]\} \oplus \mathbf{Z}_2\{i_*\bar{\varepsilon}_3\} & \xrightarrow{q^*} & [C_{2\nu_{14}}, Sp(2)]_{(2)} & \xrightarrow{i''^*} & \mathbf{Z}_8\{2[2\sigma']\} & \to & 0 \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\
0 & \to & \mathbf{Z}_8\{\zeta_7\} \oplus \mathbf{Z}_2\{\bar{\nu}_7\nu_{15}\} & \xrightarrow{q^*} & \mathbf{Z}_8\{q^*\zeta_7\} \oplus \mathbf{Z}_2^2\{q^*\bar{\nu}_7\nu_{15}, \overline{4\sigma'}\} & \xrightarrow{i''^*} & \mathbf{Z}_2\{4\sigma'\} & \to & 0
\end{array}$$

Write $p_*\overline{2[2\sigma']} = x\cdot q^*\zeta_7 + y\cdot q^*\bar{\nu}_7\nu_{15} + z\cdot\overline{4\sigma'}$ $(x, y, z \in \mathbf{Z})$. Then $p_*(\overline{2[2\sigma']} - x\cdot q^*[\zeta_7]) = y\cdot q^*\bar{\nu}_7\nu_{15} + z\cdot\overline{4\sigma'}$ and $2(\overline{2[2\sigma']} - x\cdot q^*[\zeta_7]) \in \mathrm{Ker}(p_*) = \mathrm{Image}(i_*)$. Hence we can write $2\cdot\overline{2[2\sigma']} - 2x\cdot q^*[\zeta_7] = A\cdot i_*\overline{\mu'} + B\cdot i_*q^*\bar{\varepsilon}_3$ for some $A, B \in \mathbf{Z}$. Multiplying by 4, we have $8\cdot\overline{2[2\sigma']} = 0$. Hence the order of $\overline{2[2\sigma']}$ is 8. Therefore the second row of the above diagram splits and we obtain (4.14).

We prove (4.15). By Lemma 4.2(4),(5) and equalities $3\beta_1(5) = -\alpha_1(5)\alpha_2(8)$, $3\beta_1(7) = 0$ ([9, Lemma 13.8, Theorem 13.9]), we have $\alpha_2(7)\alpha_1(14) = 0$ and $\alpha_1(3)\alpha_2(6)\alpha_1(13) = 0$ and $\alpha_1(14)^* : \pi_{17}(\mathrm{S}^3)_{(3)} \cong \pi_{14}(\mathrm{S}^3)_{(3)}$. Hence we have the following commutative diagram with exact rows and columns from (2.1).

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Z}_3\{\alpha_4(3)\} & \xrightarrow[\cong]{q^*} & [C_{\alpha_1(14)}, \mathrm{S}^3]_{(3)} & \xrightarrow{0} & \mathbf{Z}_3\{\alpha_3(3)\} & \xrightarrow[\cong]{\alpha_1(14)^*} & \mathbf{Z}_3\{\alpha_1(3)\alpha_3'(6)\} \\
 & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*\,\cong} & & \\
0 & \longrightarrow & \mathbf{Z}_9 & \xrightarrow{q^*} & [C_{\alpha_1(14)}, \mathrm{Sp}(2)]_{(3)} & \longrightarrow & \mathbf{Z}_3 & \xrightarrow{\alpha_1(14)^*} & 0 \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle 0\,p_*} & & \\
0 & \longrightarrow & \mathbf{Z}_9\{\alpha_3'(7)\} & \xrightarrow{q^*} & [C_{\alpha_1(14)}, \mathrm{S}^7]_{(3)} & \xrightarrow{i''_*} & \mathbf{Z}_3\{\alpha_2(7)\} & \xrightarrow{\alpha_1(14)^*} & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \cong\,\partial} & & \\
\mathbf{Z}_3\{\alpha_3(3)\} & \xrightarrow[\cong]{\alpha_1(14)^*} & \mathbf{Z}_3\{\alpha_1(3)\alpha_3'(6)\} & \xrightarrow[0]{q^*} & [C_{\alpha_1(13)}, \mathrm{S}^3]_{(3)} & \xrightarrow[\cong]{} & \mathbf{Z}_3\{\alpha_1(3)\alpha_2(6)\} & \xrightarrow{\alpha_1(13)^*} & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since $\partial(\alpha_2(7)) = \partial(\iota_7)\alpha_2(6) = \alpha_1(3)\alpha_2(6)$, the third ∂ is an isomorphism so that the second ∂ is surjective. Since $\partial q^*\alpha_3'(7) = q^*\partial\alpha_3'(7) = 0$, there exists $a \in [C_{\alpha_1(14)}, \mathrm{Sp}(2)]_{(3)}$ such that $p_*(a) = q^*\alpha_3'(7)$. The order of a is 9 or 27. In order to induce a contradiction, assume the order of a is 9. By the exactness of the first column of the above diagram, there exists a generator b of $\pi_{18}(\mathrm{Sp}(2))_{(3)} = \mathbf{Z}_9$ such that $p_*b = 3\alpha_3'(7)$. Then $p_*(q^*b - 3a) = 0$. Hence $q^*b - 3a \in \mathrm{Ker}(p_*) = \mathrm{Image}(i_*) \cong \mathbf{Z}_3$ and $0 = 3(q^*b - 3a) = 3q^*b$. This contradicts $\# q^*b = 9$. Therefore the order of a is 27. Hence $[C_{\alpha_1(14)}, \mathrm{Sp}(2)]_{(3)} = \mathbf{Z}_{27}\{\overline{i_*\alpha_3(3)}\}$. This completes the proof of (4.15).

Proof of Lemma 4.2 (1). As seen above, we have the following exact sequence.

$$0 \to \mathbf{Z}_2\{\bar{\varepsilon}_3\} \xrightarrow{q^*} [C_{2\nu_{14}}, \mathrm{S}^3]_{(2)} \xrightarrow{i''_*} \mathbf{Z}_4\{\mu'\} \oplus \mathbf{Z}_2^2\{\varepsilon_3\nu_{11}, \nu'\varepsilon_6\} \to 0$$

We have

$$
\begin{aligned}
4\overline{\mu'} &= 4\iota_3 \circ \overline{\mu'} \quad \text{(since } \mathrm{S}^3 \text{ is an H-space)} \\
&\in \{4\iota_3, \mu', 2\nu_{14}\} \circ q \quad \text{(by [9, Proposition 1.9])}
\end{aligned}
$$

and

$$\mathrm{Indet}\{4\iota_3, \mu', 2\nu_{14}\} = 4\iota_3 \circ \pi_{18}(S^3) + \pi_{15}(S^3) \circ 2\nu_{15} = \mathbf{Z}_{15},$$
$$\Sigma^2\{4\iota_3, \mu', 2\nu_{14}\} \subset \{4\iota_5, \Sigma^2\mu', 2\nu_{16}\}_2 = \{2\iota_5, 2\Sigma^2\mu', 2\nu_{16}\}_2$$
$$= \{2\iota_5, \eta_5^2\mu_7, 2\nu_{16}\}_2 \supset \{2\iota_5 \circ \eta_5^2, \mu_7, 2\nu_{16}\}_2 = \{0\}.$$

Hence

$$\{4\iota_5, \Sigma^2\mu', 2\nu_{16}\}_2 = \mathrm{Indet}\{4\iota_5, \Sigma^2\mu', 2\nu_{16}\}_2 = 4\iota_5 \circ \Sigma^2\pi_{18}(S^3) = \mathbf{Z}_{15},$$
$$\{4\iota_3, \mu', 2\nu_{14}\} = \mathbf{Z}_{15}.$$

Therefore, if we take $\overline{\mu'}$ as a 2-primary element, then $4\iota_3 \circ \overline{\mu'} = 0$, that is, $4\overline{\mu'} = 0$. Since $i''^*(\varepsilon_3\overline{\nu_{11}}) = \varepsilon_3\nu_{11}$ and $i''^*(\varepsilon'\overline{\eta_{13}}) = \varepsilon'\eta_{13} = \nu'\varepsilon_6$, it suffices to prove that $\#\varepsilon_3\overline{\nu_{11}} = \#\varepsilon'\overline{\eta_{13}} = 2$. This is done as follows. Since $\overline{\nu_{11}} = \Sigma\overline{\nu_{10}}$ and $\#\varepsilon_3 = 2$, $\#\varepsilon_3\overline{\nu_{11}} = 2$. Since $i''^* : [C_{2\nu_{14}}, S^{13}] \to \pi_{14}(S^{13})$ is an isomorphism, $\#\overline{\eta_{13}} = 2$ so that $\#\varepsilon'\overline{\eta_{13}} = 2$. □

Proof of Lemma 4.2 (2). We have the following exact sequence.

$$0 \to \mathbf{Z}_8\{\zeta_7\} \oplus \mathbf{Z}_2\{\bar{\nu}_7\nu_{15}\} \xrightarrow{q^*} [C_{2\nu_{14}}, S^7]_{(2)} \xrightarrow{i''^*} \mathbf{Z}_2\{4\sigma'\} \to 0$$

We have

$$2 \cdot \overline{4\sigma'} = 2\iota_7 \circ \overline{4\sigma'} \quad (\text{since } S^7 \text{ is an H-space})$$
$$\in \{2\iota_7, 4\sigma', 2\nu_{14}\} \circ q \quad (\text{by [9, Proposition 1.9]})$$

and

$$\mathrm{Indet}\{2\iota_7, 4\sigma', 2\nu_{14}\} = 2\pi_{18}(S^7) = \mathbf{Z}_4\{2\zeta_7\} \oplus \mathbf{Z}_{63}.$$

We shall show $\{2\iota_7, 4\sigma', 2\nu_{14}\} = \mathbf{Z}_4\{2\zeta_7\} \oplus \mathbf{Z}_{63}$ as follows. Since $\Sigma : \pi_{18}(S^7) \to \pi_{19}(S^8)$ is an isomorphism by [9] and $\mathrm{Indet}\{2\iota_8, \Sigma(4\sigma'), 2\nu_{15}\}_1 = 2\pi_{19}(S^8)$, we have

$$\Sigma\{2\iota_7, 4\sigma', 2\nu_{14}\} = (-1)\{2\iota_8, \Sigma(4\sigma'), 2\nu_{15}\}_1.$$

We have

$$\{2\iota_8, \Sigma(4\sigma'), 2\nu_{15}\}_1 \supset \{2\iota_8, \Sigma(4\sigma'), 2\iota_{15}\}_1 \circ \nu_{16} \quad (\text{by [9, Proposition 1.2]})$$

and

$$\{2\iota_8, \Sigma(4\sigma'), 2\iota_{15}\}_1 \ni \Sigma(4\sigma') \circ \eta_{15} = 0 \quad (\text{by [9, Corollary 3.7]}),$$
$$\mathrm{Indet}\{2\iota_8, \Sigma(4\sigma'), 2\iota_{15}\}_1 = 2\iota_8 \circ \Sigma\pi_{15}(S^7) + 2\pi_{16}(S^8) = 0.$$

Hence $\{2\iota_8, \Sigma(4\sigma'), 2\iota_{15}\}_1 = \{0\}$ so that $\{2\iota_8, \Sigma(4\sigma'), 2\nu_{15}\}_1 \ni 0$. Thus $\{2\iota_8, \Sigma(4\sigma'), 2\nu_{15}\}_1 = 2\pi_{19}(S^8)$ and

$$\{2\iota_7, 4\sigma', 2\nu_{14}\} = 2\pi_{18}(S^7) = \mathbf{Z}_4\{2\zeta_7\} \oplus \mathbf{Z}_{63}.$$

Therefore we can write

$$2 \cdot \overline{4\sigma'} = 2x\, q^*\zeta_7 \quad (0 \le x \le 3).$$

In this case, we have

$$\#(\overline{4\sigma'} - x\,q^*\zeta_7) = 2,$$

$$[C_{2\nu_{14}}, \mathrm{S}^7]_{(2)} = \mathbf{Z}_8\{q^*\zeta_7\} \oplus \mathbf{Z}_2^2\{q^*\bar{\nu}_7\nu_{16}, \overline{4\sigma'} - x\,q^*\zeta_7\}.$$

Hence, by rewriting $\overline{4\sigma'} - x\,q^*\zeta_7$ as $\overline{4\sigma'}$, we have

$$[C_{2\nu_{14}}, \mathrm{S}^7]_{(2)} = \mathbf{Z}_8\{q^*\zeta_7\} \oplus \mathbf{Z}_2^2\{q^*\bar{\nu}_7\nu_{16}, \overline{4\sigma'}\}.$$ □

Proof of Lemma 4.2 (3). Since $\Sigma : \pi_{12}(\mathrm{S}^5) \to \pi_{13}(\mathrm{S}^6)$ is injective by [9], we have $3\iota_5 \circ \pi_{12}(\mathrm{S}^5) = 3\pi_{12}(\mathrm{S}^5)$ as desired. □

Proof of Lemma 4.2 (4). It follows from [9, Theorem 13.9] that $\pi_{13}(\mathrm{S}^3)_{(3)} = \mathbf{Z}_3\{\alpha_1(3)\alpha_2(6)\}$ and from [9, Lemma 13.5] that

$$\alpha_2(3) \in \{\alpha_1(3), \Sigma(3\iota_5), \Sigma\alpha_1(5)\}_1 \subset \{\alpha_1(3), \Sigma(3\iota_5), \Sigma\alpha_1(5)\} \subset \pi_{10}(S^3).$$

Since $\mathrm{Indet}\{\alpha_1(3), \Sigma(3\iota_5), \Sigma\alpha_1(5)\} = \pi_7(S^3) \circ \alpha_1(7) + \alpha_1(3) \circ \pi_{10}(S^6) = 0$, we have

$$\alpha_2(3) = \{\alpha_1(3), 3\iota_6, \alpha_1(6)\}_1 = \{\alpha_1(3), 3\iota_6, \alpha_1(6)\}.$$

Then

$$\begin{aligned}
\alpha_2(3)\alpha_1(10) &= \{\alpha_1(3), 3\iota_6, \alpha_1(6)\}_1 \circ \alpha_1(10) \\
&= \alpha_1(3) \circ \Sigma\{3\iota_5, \alpha_1(5), \alpha_1(8)\} \quad \text{(by [9, Proposition 1.4])} \\
&\in \alpha_1(3) \circ (-\{3\iota_6, \alpha_1(6), \alpha_1(9)\}_1) \quad \text{(by [9, Proposition 1.3])} \\
&= -(\alpha_1(3) \circ \{3\iota_6, \alpha_1(6), \alpha_1(9)\}_1) \quad \text{(since } S^3 \text{ is an H-space).}
\end{aligned}$$

We have $\mathrm{Indet}\{3\iota_6, \alpha_1(6), \alpha_1(9)\}_1 = 3\iota_6 \circ \Sigma\pi_{12}(S^5) = 3 \cdot \Sigma\pi_{12}(\mathrm{S}^5)$ and so

$$\alpha_1(3) \circ \mathrm{Indet}\{3\iota_6, \alpha_1(6), \alpha_1(9)\}_1 = 0.$$

Thus $\alpha_1(3) \circ \{3\iota_3, \alpha_1(6), \alpha_1(9)\}_1$ consists of a single element. Hence

$$\begin{aligned}
\alpha_1(3) \circ \{3\iota_6, \alpha_1(6), \alpha_1(9)\}_1 &= \alpha_1(3) \circ (-\Sigma\{3\iota_5, \alpha_1(5), \alpha_1(8)\}) \\
&= \alpha_1(3) \circ (-2\alpha_2(6)) \quad \text{(by [3, Proposition 4.4 (1)])} \\
&= \alpha_1(3)\alpha_2(6).
\end{aligned}$$

Therefore $\alpha_2(3)\alpha_1(10) = -\alpha_1(3)\alpha_2(6)$ as desired. □

Proof of Lemma 4.2 (5). Since $\alpha_3(3) \in \{\alpha_2(3), 3\iota_{10}, \alpha_1(10)\}_1$ by [9, Lemma 13.5], we have

$$\begin{aligned}
(4.16) \qquad \alpha_3(3)\alpha_1(14) &\in \{\alpha_2(3), 3\iota_{10}, \alpha_1(10)\}_1 \circ \alpha_1(14) \\
&= \alpha_2(3) \circ \Sigma\{3\iota_9, \alpha_1(9), \alpha_1(12)\} \quad \text{(by [9, Proposition 1.4]).}
\end{aligned}$$

Since $\{3\iota_5, \alpha_1(5), \alpha_1(8)\} = 2\alpha_2(5) + 3\pi_{12}(S^5)$ by [3, Proposition 4.4(1)], we have

$$2\alpha_2(9) \in \Sigma^4\{3\iota_5, \alpha_1(5), \alpha_1(8)\} \subset \{3\iota_9, \alpha_1(9), \alpha_1(12)\}_4.$$

We have

$$\mathrm{Indet}\{3\iota_9, \alpha_1(9), \alpha_1(12)\} = 3\iota_9 \circ \pi_{16}(S^9) = 3\pi_{16}(S^9),$$

where the last equality follows from the fact $\Sigma\pi_{15}(S^8) = \pi_{16}(S^9)$. Hence $\{3\iota_9, \alpha_1(9), \alpha_1(12)\} = 2\alpha_2(9) + 3\pi_{16}(S^9)$. Since $\pi_{16}(S^9)$ is stable and is isomorphic to $\mathbf{Z}_3 \oplus \mathbf{Z}_{80}$ by [9], it follows that

$$\Sigma\{3\iota_9, \alpha_1(9), \alpha_1(12)\} = 2\alpha_2(10) + 3\pi_{17}(S^{10})$$

and so

$$\alpha_2(3) \circ \Sigma\{3\iota_9, \alpha_1(9), \alpha_1(12)\} = \alpha_2(3) \circ 2\alpha_2(10) = -\alpha_2(3)\alpha_2(10).$$

Hence (4.16) yields

$$\alpha_3(3)\alpha_1(14) = -\alpha_2(3)\alpha_2(10). \tag{4.17}$$

By [9, Proposition 13.3], we have

$$\pi_{17}(S^3)_{(3)} = \mathbf{Z}_3\{\alpha_1(3)\alpha_3'(6)\}.$$

Hence $\alpha_3(3)\alpha_1(14) = x \cdot \alpha_1(3)\alpha_3'(6)$ for some integer x. Then (4.17) yields

$$-\alpha_2(4)\alpha_2(11) = \alpha_3(4)\alpha_1(15) = x \cdot \alpha_1(4)\alpha_3'(7). \tag{4.18}$$

By the EHP-sequence, we see that $\alpha_1(4)\alpha_3'(7) \neq 0$. On the other hand, we have

$$\begin{aligned}\alpha_2(4) \circ \alpha_2(11) &\in \Sigma\{\alpha_1(3), 3\iota_6, \alpha_1(6)\} \circ \alpha_2(11) \subset -\{\alpha_1(4), 3\iota_7, \alpha_1(7)\} \circ \alpha_2(11)\\ &= \alpha_1(4) \circ \{3\iota_7, \alpha_1(7), \alpha_2(10)\}.\end{aligned}$$

Since $\mathrm{Indet}\{3\iota_7, \alpha_1(7), \alpha_2(10)\} = 3\iota_7 \circ \pi_{18}(S^7)$, it follows that $\alpha_1(4) \circ \{3\iota_7, \alpha_1(7), \alpha_2(10)\}$ is a single element. Thus

$$\alpha_2(4)\alpha_2(11) = \alpha_1(4) \circ \{3\iota_7, \alpha_1(7), \alpha_2(10)\}. \tag{4.19}$$

Since $\Sigma^\infty : \pi_{18}(S^7)_{(3)} = \mathbf{Z}_9\{\alpha_3'(7)\} \cong (\pi_{11}^s)_{(3)} = \mathbf{Z}_9\{\alpha_3'\}$, where $\pi_n^s = \lim_{k\to\infty} \pi_{n+k}(S^k)$, we have

$$\Sigma^\infty\{3\iota_7, \alpha_1(7), \alpha_2(10)\} = \langle 3\iota, \alpha_1, \alpha_2\rangle.$$

By [9, (3.9)], we have $\langle 3\iota, \alpha_1, \alpha_2\rangle = \langle \alpha_2, \alpha_1, 3\iota\rangle$. It follows from [8, Proposition 4.17 ii)] that $\langle \alpha_2, \alpha_1, 3\iota\rangle = 2\alpha_3' + 3\pi_{11}^s$ so that

$$\{3\iota_7, \alpha_1(7), \alpha_2(10)\} = 2\alpha_3'(7) + 3\pi_{18}(S^7).$$

Hence (4.18) and (4.19) yield

$$(-x) \cdot \alpha_1(4)\alpha_3'(7) = \alpha_2(4)\alpha_2(11) = \alpha_1(4) \circ 2\alpha_3'(7) = -\alpha_1(4)\alpha_3'(7).$$

Therefore $x \equiv 1 \pmod 3$ and $\alpha_3(3)\alpha_1(14) = \alpha_1(3)\alpha_3'(6)$ as desired. □

References

[1] W. Browder and E. Spanier, H-spaces and duality, Pacific J. Math. **12** (1970), 411–414.
[2] H. Kachi, J. Mukai, T. Nozaki, Y. Sumita and D. Tamaki, Some cohomotopy groups of suspended projective planes, Math. J. Okayama Univ. **43** (2001), 105–121.
[3] K. Maruyama and H. Ōshima, Homotopy groups of the spaces of self-maps of Lie groups, J. Math. Soc. Japan **60** (2008), 767–792.
[4] M. Mimura, The homotopy groups of Lie groups of low rank, J. Math. Kyoto Univ. **6** (1967), 131–176.
[5] M. Mimura and H. Ōshima, Self homotopy groups of Hopf spaces with at most three cells, J. Math. Soc. Japan **51** (1999), 71–92.
[6] M. Mimura and H. Toda, Homotopy groups of SU(3), SU(4) and Sp(2), J. Math. Kyoto Univ. **3** (1964), 217–250.
[7] K. Ôguchi, Generators of 2-primary components of homotopy groups of spheres, unitary groups and symplectic groups, J. Fac. Sci. Univ. Tokyo **11** (1964), 65–111.
[8] H. Toda, p-primary components of homotopy groups IV, Compositions and toric constructions, Memoirs Univ. of Kyoto **32** (1958), 143–160.
[9] H. Toda, Composition methods in homotopy groups of spheres, Ann. of Math. Studies **49**, Princeton, 1962.

Katsumi Ōshima
Ibaraki University
Mito, Ibaraki 310-8512
Japan
E-mail: 08nd402l@mcs.ibaraki.ac.jp

Hideaki Ōshima
Ibaraki University
Mito, Ibaraki 310-8512
Japan
E-mail: ooshima@mx.ibaraki.ac.jp