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KODAI MATH. J.
32 (2009), 471–500

ITERATION METHOD FOR MULTIPLE ROGERS-RAMANUJAN IDENTITIES

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Abstract

Inspired by the recursive lemma due to Bressoud (1983), we present an iteration process for constructing transformations from unilateral multiple basic hypergeometric series to bilateral univariate one. Applications are illustrated to multiple series transformation formulae and multiple Rogers-Ramanujan identities.

1. Introduction and motivation

Let $\mathbf{N}$, $\mathbf{N}_0$ and $\mathbf{Z}$ stand respectively for the sets of natural numbers, nonnegative integers and integers. For two indeterminate x and q, the shifted factorial of x with base q is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad \text{for } n \in \mathbf{N}.$$

When $|q| < 1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = (x;q)_\infty/(xq^n;q)_\infty.$$

With the multiparameter forms of shifted factorials being abbreviated to

$$[\alpha,\beta,\ldots,\gamma;q]_n = (\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n,$$
$$\left[\begin{matrix}\alpha,\beta,\ldots,\gamma\\ A,B,\ldots,C\end{matrix}\middle|q\right]_n = \frac{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n(B;q)_n\cdots(C;q)_n};$$

we define the unilateral and bilateral basic hypergeometric series, respectively, by

$${}_{1+r}\phi_s\left[\begin{matrix}a_0, & a_1,\ldots,a_r\\ & b_1,\ldots,b_s\end{matrix}\middle|q;z\right] = \sum_{n=0}^{\infty}\{(-1)^n q^{\binom{n}{2}}\}^{s-r}\left[\begin{matrix}a_0, & a_1,\ldots,a_r\\ q, & b_1,\ldots,b_s\end{matrix}\middle|q\right]_n z^n,$$

2000 *Mathematics Subject Classification*. Primary 33D15, Secondary 05A15.

Key words and phrases. Basic hypergeometric series (q-series); Jacobi's triple product identity; Iteration procedure; Multiple series transformations; Multiple Rogers-Ramanujan identities.

Received October 7, 2008; revised February 18, 2009.

$$ {}_r\psi_s\left[\begin{matrix} a_1,\ a_2,\dots,a_r\\ b_1,\ b_2,\dots,b_s\end{matrix}\middle| q;z\right] = \sum_{n=-\infty}^{+\infty}\{(-1)^n q^{\binom{n}{2}}\}^{s-r}\left[\begin{matrix} a_1,\ a_2,\dots,a_r\\ b_1,\ b_2,\dots,b_s\end{matrix}\middle| q\right]_n z^n; $$

where the base q will be restricted to $|q|<1$ for nonterminating q-series. More comprehensive coverage of q-series theory can be found in three monographs by Bailey [6, Chapter 8], Slater [30, Chapters 3, 5, 7] and Gasper-Rahman [20].

In 1983, utilizing the following well-known q-analogue of the binomial theorem

$$ (1) \qquad \frac{1}{(qa;q)_n} = \sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}\frac{q^{m^2}a^m}{(qa;q)_m} \quad \text{where} \quad \begin{bmatrix} n\\ m\end{bmatrix} = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}} $$

Bressoud [13, Lemma 2] devised ingeniously the recursive lemma on finite sums

$$ (2) \qquad \sum_{k=-n}^{n}\frac{q^{\lambda k^2}x^k}{(q;q)_{n+k}(q;q)_{n-k}} = \sum_{m=0}^{n}\frac{q^{m^2}}{(q;q)_{n-m}}\sum_{k=-m}^{m}\frac{q^{(\lambda-1)k^2}x^k}{(q;q)_{m+k}(q;q)_{m-k}}. $$

Iterating the last equation ℓ-times and then putting $\lambda=\ell+1/2$, he discovered the following multiple series transformation theorem [13, Theorem]

$$ (3a) \qquad \sum_{n\geq m_1\geq m_2\geq\cdots\geq m_\ell\geq 0}\frac{q^{m_1^2+m_2^2+\cdots+m_\ell^2}(x;q)_{m_\ell}(q/x;q)_{m_\ell}}{(q;q)_{n-m_1}(q;q)_{m_1-m_2}\cdots(q;q)_{m_{\ell-1}-m_\ell}(q;q)_{2m_\ell}} $$

$$ (3b) \qquad = \sum_{k=-n}^{n}(-1)^k\begin{bmatrix} 2n\\ n+k\end{bmatrix}\frac{q^{k^2\ell+\binom{k}{2}}}{(q;q)_{2n}}x^k. $$

The limiting case $n\to\infty$ of the last formula yields easy proofs and generalizations of the celebrated Rogers-Ramanujan identities (cf. Watson [34] and Slater [29, Eqs 14 and 18])

$$ \frac{[q^5,q^2,q^3;q^5]_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad \frac{[q^5,q,q^4;q^5]_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}. $$

There exist numerous identities expressing infinite series in terms of infinite products in q-series, generally called Rogers-Ramanujan identities. Slater [28, 29] has made a collection of 130 such identities by means of Bailey's lemma [8, 9]. For the multiple q-series, there are less identities of Rogers-Ramanujan type, that scattered mainly in the following literatures [1, 2, 4, 12, 14, 15, 18, 22, 23, 26, 27, 31–33]. Inspired by the recursive approach of Bressoud [13], we shall establish another remarkably useful transformation theorem involving an arbitrary sequence $\{W_k\}_{k\geq 0}$. By specifying the W-sequence, several transformation formulae from unilateral multiple series to bilateral univariate one will be derived. Their limiting cases lead to numerous multiple Rogers-Ramanujan identities with most of them having not appeared previously.

The paper will be organized as follows. The next section will be devoted to the main theorem of this paper, which transforms a unilateral multiple series to a bilateral univariate series. Then we shall present, in the third section, its applications to terminating and nonterminating multiple series transformation formulae as well as multiple Rogers-Ramanujan identities. The identities examined in this paper show that the iteration process is efficient and simple for dealing with multiple Rogers-Ramanujan identities, just like Bressoud's approach to the classical Rogers-Ramanujan identities.

2. The main theorem and proof

Recall the q-Chu-Vandermonde-Gauss theorem (cf. Gasper-Rahman [20, II-7])

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix}\middle| q; q^n c/a\right] = \frac{(c/a;q)_n}{(c;q)_n} $$

which can explicitly be expressed in terms of the following q-binomial convolution

$$ \frac{(c/a;q)_n}{(c;q)_n} = \sum_{m=0}^{n} q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(a;q)_m}{(c;q)_m} \left(-\frac{c}{a}\right)^m. \tag{4} $$

This formula plays the similar role as (1) for deriving multiple series transformations. In fact for $m \to m-k$, $n \to n-k$, $a = -q^{k+(1+\delta)/2}$ and $c = q^{1+\delta+2k}$ with $\delta = 0, 1$, the last equation may be reformulated as

$$ \frac{(-q^{(1+\delta)/2};q)_n}{(q;q)_{n+k+\delta}} = \sum_{m=k}^{n} \begin{bmatrix} n-k \\ m-k \end{bmatrix} \frac{(-q^{(1+\delta)/2};q)_m}{(q;q)_{m+k+\delta}} q^{(m-k)(m+k+\delta)/2}. $$

Let $\{W_k\}_{k\in\mathbf{Z}}$ be an arbitrary sequence. Multiplying by $W_k/(q;q)_{n-k}$ across the last equation, we may manipulate the following bilateral finite sum with respect to k over $-n-\delta \le k \le n$

$$ \begin{aligned} &\sum_{k=-n-\delta}^{n} \frac{(-q^{(1+\delta)/2};q)_n W_k}{(q;q)_{n-k}(q;q)_{n+k+\delta}} \\ &= \sum_{k=-n-\delta}^{n} \frac{W_k}{(q;q)_{n-k}} \sum_{m=k}^{n} \begin{bmatrix} n-k \\ m-k \end{bmatrix} \frac{(-q^{(1+\delta)/2};q)_m}{(q;q)_{m+k+\delta}} q^{(m-k)(m+k+\delta)/2} \\ &= \sum_{m=-n-\delta}^{n} \frac{q^{(m)(m+\delta)/2}}{(q;q)_{n-m}} \sum_{k=-n-\delta}^{m} \frac{(-q^{(1+\delta)/2};q)_m q^{-(k)(k+\delta)/2}}{(q;q)_{m-k}(q;q)_{m+k+\delta}} W_k \\ &= \sum_{m=0}^{n} \frac{q^{(m)(m+\delta)/2}}{(q;q)_{n-m}} \sum_{k=-m-\delta}^{m} \frac{(-q^{(1+\delta)/2};q)_m q^{-(k)(k+\delta)/2}}{(q;q)_{m-k}(q;q)_{m+k+\delta}} W_k \end{aligned} $$

where the last line has been justified by the fact that the innermost summand vanishes for $m<0$ and $k<-m-\delta$. Replacing k by $-k$ in the two extreme sums with respect to k, we find the following generalized recursive lemma.

Lemma 1 (Recursive sums).

$$\sum_{k=-n}^{n+\delta}\frac{(-q^{(1+\delta)/2};q)_n W_k}{(q;q)_{n+k}(q;q)_{n-k+\delta}}=\sum_{m=0}^{n}\frac{q^{(m)(m+\delta)/2}}{(q;q)_{n-m}}\sum_{k=-m}^{m+\delta}\frac{(-q^{(1+\delta)/2};q)_m q^{(k)(\delta-k)/2}}{(q;q)_{m+k}(q;q)_{m-k+\delta}}W_k.$$

Iterating ℓ-times the recursion in Lemma 1 leads to the following equation

$$\sum_{k=-n}^{n+\delta}\frac{(-q^{(1+\delta)/2};q)_n W_k}{(q;q)_{n+k}(q;q)_{n-k+\delta}}=\sum_{n\geq r_1\geq r_2\geq\cdots\geq r_\ell\geq 0}\frac{q^{(r_1)(r_1+\delta)/2+(r_2)(r_2+\delta)/2+\cdots+(r_\ell)(r_\ell+\delta)/2}}{(q;q)_{n-r_1}(q;q)_{r_1-r_2}\cdots(q;q)_{r_{\ell-1}-r_\ell}}$$
$$\times\sum_{k=-r_\ell}^{r_\ell+\delta}\frac{(-q^{(1+\delta)/2};q)_{r_\ell}q^{(k\ell)(\delta-k)/2}W_k}{(q;q)_{r_\ell+k}(q;q)_{r_\ell-k+\delta}}.$$

In order to shorten the long expressions, we make the replacements on summation indices and fix the compact notations as follows

$$\left.\begin{array}{c} n-r_1\to m_0\\ r_1-r_2\to m_1\\ \cdots\\ r_{\ell-1}-r_\ell\to m_{\ell-1}\\ r_\ell\to m_\ell\end{array}\right\}\quad\text{and}\quad\begin{cases}\tilde{m}=(m_1,m_2,\ldots,m_\ell);\\ M_k=\sum_{\imath=k}^{\ell}m_\imath,\quad(0\leq k\leq\ell).\end{cases}$$

Further replacing W_k by $(-1)^k q^{\binom{k}{2}+(k\ell)(k-\delta)/2}W_k$ in the last finite series transformation, we may reformulate the result as the following main theorem of this paper.

Theorem 2 (Multiple series transformation). *For an arbitrary bilateral sequence* $\{W_k\}_{k\in\mathbf{Z}}$, *there holds the multiple series transformation*

$$\sum_{M_0=n}\frac{(q;q)_{2n+\delta}(-q^{(1+\delta)/2};q)_{m_\ell}}{(q;q)_{m_0}(q;q)_{m_\ell+\delta}}\prod_{\imath=1}^{\ell}\frac{q^{(M_\imath)(M_\imath+\delta)/2}}{(q;q)_{m_\imath}}\sum_{k=-m_\ell}^{m_\ell+\delta}q^{k(m_\ell+\delta)}\frac{(q^{-m_\ell-\delta};q)_k}{(q^{1+m_\ell};q)_k}W_k \tag{5a}$$

$$=(-q^{(1+\delta)/2};q)_n\sum_{k=-n}^{n+\delta}(-1)^k\begin{bmatrix}2n+\delta\\ n+k\end{bmatrix}q^{\binom{k}{2}+(k\ell)(k-\delta)/2}W_k \tag{5b}$$

where the multiple sum on the left runs over $(m_0,m_1,\ldots,m_\ell)\in\mathbf{N}_0^{1+\ell}$ *subject to the condition* $M_0=m_0+m_1+\cdots+m_\ell=n$.

This theorem is remarkably useful for deriving concrete multiple transformation formulae and multiple Rogers-Ramanujan identities. Here we illustrate two examples by utilizing Ramanujan's identity of bilateral ${}_1\psi_1$-series [20, II-29]

$$(6) \qquad {}_1\psi_1\left[\begin{matrix}a\\c\end{matrix}\middle|q;z\right]=\left[\begin{matrix}q, & c/a, & az, & q/az\\c, & q/a, & z, & c/az\end{matrix}\middle|q\right]_\infty \quad \text{where } |c/a|<|z|<1.$$

More applications will systematically be exhibited in the next section.

First, letting $\delta=1$ and $W_k=x^k$ in Theorem 2 and then evaluating the sum with respect to k displayed in (5a) by means of the terminating form of (6) as

$$\sum_{k=-m}^{1+m} q^{k(1+m)}\frac{(q^{-1-m};q)_k}{(q^{1+m};q)_k}x^k=\frac{(q;q)_m(q;q)_{1+m}(x;q)_{1+m}(q/x;q)_m}{(q;q)_{1+2m}}$$

we derive the following variant of Bressoud's theorem stated in (3).

Theorem 3 (Terminating series transformation).

$$\sum_{M_0=n}\frac{(q;q)_{2n+1}[qx,q/x;q]_{m_\ell}}{(q;q)_{m_0}(q;q^2)_{1+m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}=\frac{(-q;q)_n}{1-x}\sum_{k=-n}^{n+1}(-1)^k\begin{bmatrix}2n+1\\n+k\end{bmatrix}q^{\binom{k}{2}(\ell+1)}x^k.$$

Letting $n\to\infty$ and then factorizing the last sum through Jacobi's triple product identity [21] (see [5, P497] for historical notes)

$$(7) \qquad [q,x,q/x;q]_\infty=\sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}}x^k$$

we establish the nonterminating multiple series identity.

Proposition 4 (Multiple series identity).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{[qx,q/x;q]_{m_\ell}}{(q;q^2)_{1+m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{1+\ell},q^{1+\ell}x,q^{1+\ell}/x;q^{1+\ell}]_\infty.$$

For the rest of this paper, $\begin{bmatrix}n\\k\end{bmatrix}_{q^m}$ will denote the Gaussian binomial coefficient under the base change $q\to q^m$ for $m\in\mathbf{N}$. For the sake of brevity, we shall also fix $\varepsilon=\pm1$ and $\delta_{m,n}$, the usual Kronecker symbol.

Similarly, letting $\delta=0$ and $W_k=x^k$ in Theorem 2, we can evaluate the corresponding sum with respect to k displayed in (5a) by means of (6) as

$$\sum_{k=-m}^{m} q^{km}\frac{(q^{-m};q)_k}{(q^{1+m};q)_k}x^k=\frac{(q;q)_m^2(x;q)_m(q/x;q)_m}{(q;q)_{2m}}.$$

Making further the replacements $q \to q^2$ and $x \to qx$, we derive another variant of Bressoud's theorem stated in (3).

Theorem 5 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q^2;q^2)_{2n}[q^2,qx,q/x;q^2]_{m_\ell}}{(q^2;q^2)_{m_0}(q;q^2)_{m_\ell}(q^4;q^4)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$
$$= (-q;q^2)_n \sum_{k=-n}^{n} (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} q^{k^2(1+\ell)} x^k.$$

Its limiting case $n \to \infty$ leads analogously to the multiple series identity.

Proposition 6 (Multiple series identity).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{[q^2,qx,q/x;q^2]_{m_\ell}}{(q;q^2)_{m_\ell}(q^4;q^4)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [q^{2+2\ell}, q^{1+\ell}x, q^{1+\ell}/x; q^{2+2\ell}]_\infty.$$

It is interesting to observe that when $\ell = 1$, Propositions 4 and 6 reduce respectively to the instances of the q-analogue of Gauss' ${}_2F_1\left(\frac{1}{2}\right)$-sum

$${}_2\phi_2\left[\begin{matrix} a, & b \\ \sqrt{qab}, & -\sqrt{qab} \end{matrix}\middle| q; -q\right] = \left[\begin{matrix} qa, & qb \\ q, & qab \end{matrix}\middle| q^2\right]_\infty$$

and the q-analogue of Bailey's ${}_2F_1\left(\frac{1}{2}\right)$-sum

$${}_2\phi_2\left[\begin{matrix} a, & q/a \\ -q, & c \end{matrix}\middle| q; -c\right] = \left[\begin{matrix} ac, & qc/a \\ c, & qc \end{matrix}\middle| q^2\right]_\infty$$

due to Andrews [3, Eqs 1.8 and 1.9] (cf. Gasper-Rahman [20, II-10 and II-11]).

3. Transformations and multiple series identities

Following the two examples shown in the last section, this section will systematically present applications of Theorem 2 to multiple series transformation formulae and multiple Rogers-Ramanujan identities.

This will be realized through the following procedure. First by specifying concretely W-sequence, we shall evaluate the corresponding finite sum displayed in (5a) by Bailey's summation formula of very well-poised ${}_6\psi_6$-series [7] (see also [20, II-33])

$$(8a)\qquad {}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\middle| q; \frac{qa^2}{bcde}\right]$$

$$(8b)\qquad = \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix}\middle| q\right]_\infty$$

provided that $|qa^2/bcde| < 1$ for convergence, together with two particular cases essentially due to Bailey [10] (cf. Chu [17, Eqs 3.16a-b-c])

$$(9)\qquad {}_4\psi_4\left[\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d \end{matrix}\middle| q; \frac{q}{bcd}\right] = \left[\begin{matrix} q, & q/bc, & q/bd, & q/cd \\ q/b, & q/c, & q/d, & q/bcd \end{matrix}\middle| q\right]_\infty,$$

$$(10)\qquad {}_5\psi_5\left[\begin{matrix} qu, & qv, & b, & c, & d \\ u, & v, & 1/b, & 1/c, & 1/d \end{matrix}\middle| q; \frac{q^{-1}}{bcd}\right]$$
$$= \frac{uv - 1/q}{(1-u)(1-v)}\left[\begin{matrix} q, & 1/bc, & 1/bd, & 1/cd \\ q/b, & q/c, & q/d, & q^{-1}/bcd \end{matrix}\middle| q\right]_\infty.$$

Then we shall factorize the limiting case $n \to \infty$ of bilateral sum (5b) by invoking Jacobi's triple product identity (7), its variant, in view of the parity of summation index, due to Bailey [11, Eq 4.1]

$$(11)\qquad [q^2, qy, q/y; q^2]_\infty = \sum_{n=-\infty}^{+\infty} \{1 - yq^{1+4n}\} q^{4n^2} y^{2n}$$

as well as the quintuple product identity [16, 19, 35]

$$(12)\qquad [q, z, q/z; q]_\infty \times [qz^2, q/z^2; q^2]_\infty = \sum_{n=-\infty}^{+\infty} \{1 - zq^n\} q^{3\binom{n}{2}} (qz^3)^n.$$

The contents will be divided into sixteen subsections. Each subsection will consist of two main transformation formulae of terminating and nonterminating multiple series as well as a few exemplified multiple Rogers-Ramanujan identities.

§3.1. For $\delta = 0$, taking W_k in Theorem 2 as

$$W_k = \frac{1 - wq^k}{1 - w}\left[\begin{matrix} b, & d \\ q/b, & q/d \end{matrix}\middle| q\right]_k \left(\frac{q}{bd}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_4\psi_4$-series identity (9) as

$$\sum_{k=-m}^{m} q^{km} \frac{(q^{-m}; q)_k}{(q^{1+m}; q)_k} \frac{1 - wq^k}{1 - w}\left[\begin{matrix} b, & d \\ q/b, & q/d \end{matrix}\middle| q\right]_k \left(\frac{q}{bd}\right)^k = \left[\begin{matrix} q, & q/bd \\ q/b, & q/d \end{matrix}\middle| q\right]_m.$$

Making further the replacement $q \to q^2$, we derive the following transformation.

THEOREM 7 (Terminating series transformation).

$$\sum_{M_0=n}\frac{(q^2;q^2)_{2n}}{(q^2;q^2)_{m_0}}\left[\begin{matrix}-q, & q^2/bd\\ q^2/b, & q^2/d\end{matrix}\middle|q^2\right]_{m_\ell}\prod_{\imath=1}^{\ell}\frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$

$$=(-q;q^2)_n\sum_{k=-n}^{n}(-1)^k\frac{1-wq^{2k}}{1-w}\begin{bmatrix}2n\\ n+k\end{bmatrix}_{q^2}\left[\begin{matrix}b, & d\\ q^2/b, & q^2/d\end{matrix}\middle|q^2\right]_k\frac{q^{k^2(1+\ell)+k}}{(bd)^k}.$$

Letting $n\to\infty$, we get the nonterminating multiple series transformation.

PROPOSITION 8 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\left[\begin{matrix}-q, & q^2/bd\\ q^2/b, & q^2/d\end{matrix}\middle|q^2\right]_{m_\ell}\prod_{\imath=1}^{\ell}\frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$

$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k\frac{1-wq^{2k}}{1-w}\left[\begin{matrix}b, & d\\ q^2/b, & q^2/d\end{matrix}\middle|q^2\right]_k\frac{q^{k^2(1+\ell)+k}}{(bd)^k}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

COROLLARY 9 ($b=w=-1,\ d\to 0$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}(-1)^{m_\ell}\frac{(-q;q^2)_{m_\ell}}{(-q^2;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2\ell},q^\ell,q^\ell;q^{2\ell}]_\infty.$$

COROLLARY 10 ($b=-q\varepsilon,\ d\to 0$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\left(-\frac{\varepsilon}{q}\right)^{m_\ell}\frac{(-q;q^2)_{m_\ell}}{(-q\varepsilon;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2\ell},q^{\ell-1}\varepsilon,q^{1+\ell}\varepsilon;q^{2\ell}]_\infty.$$

COROLLARY 11 ($b=-q\varepsilon,\ d=w=-1$: Chu [18, Example 17]).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(q\varepsilon;q^2)_{m_\ell}^2}{(q;-q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2+2\ell},q^{1+\ell}\varepsilon,q^{1+\ell}\varepsilon;q^{2+2\ell}]_\infty.$$

COROLLARY 12 ($b=-q\varepsilon,\ d\to\infty$: Stembridge [32, Eq b] Warnaar [33, Thm 4.4] for $\varepsilon=1$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q;q^2)_{m_\ell}}{(-q\varepsilon;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{4+2\ell},q^{1+\ell}\varepsilon,q^{3+\ell}\varepsilon;q^{4+2\ell}]_\infty.$$

COROLLARY 13 ($b = w = -1,\ d \to \infty$: Chu [18, Example 18]).

$$\sum_{\tilde{m}\in\mathbf{N}_0^{\ell}} \frac{(-q;q^2)_{m_\ell}}{(-q^2;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2;q^2)_{m_k}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [q^{4+2\ell}, q^{2+\ell}, q^{2+\ell}; q^{4+2\ell}]_\infty.$$

COROLLARY 14 ($b, d \to \infty$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^{\ell}} (-q;q^2)_{m_\ell} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2;q^2)_{m_k}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [q^{6+2\ell}, q^{2+\ell}, q^{4+\ell}; q^{6+2\ell}]_\infty.$$

When $\ell = 1$, the last corollary reduces to the analytic version of the first Göllnitz-Gordon partition identity (cf. Slater [29, Eq 36] and Stanton [31]).

§3.2. For $\delta = 0$, taking W_k in Theorem 2 as

$$W_k = \frac{1 - q^{-k}w}{1 - w} \left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix} \middle| q \right]_k \left(\frac{q}{bd} \right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_5\psi_5$-series identity (10) as

$$\sum_{k=-m}^{m} q^{km} \frac{(q^{-m};q)_k}{(q^{1+m};q)_k} \frac{1 - q^{-k}w}{1 - w} \left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix} \middle| q \right]_k \left(\frac{q}{bd} \right)^k = \frac{1 - wq^m}{1 - w} \left[\begin{matrix} q, & 1/bd \\ q/b, & q/d \end{matrix} \middle| q \right]_m.$$

Making further the replacement $q \to q^2$, we derive the following transformation.

THEOREM 15 (Terminating series transformation).

$$\sum_{M_0=n} \frac{1 - wq^{2m_\ell}}{1 - w} \frac{(q^2;q^2)_{2n}}{(q^2;q^2)_{m_0}} \left[\begin{matrix} -q, & 1/bd \\ q^2/b, & q^2/d \end{matrix} \middle| q^2 \right]_{m_\ell} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$
$$= (-q;q^2)_n \sum_{k=-n}^{n} (-1)^k \frac{1 - q^{-2k}w}{1 - w} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} \left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix} \middle| q^2 \right]_k \frac{q^{k^2(1+\ell)+k}}{(bd)^k}.$$

Letting $n \to \infty$, we get the nonterminating multiple series transformation.

PROPOSITION 16 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^{\ell}} \frac{1 - wq^{2m_\ell}}{1 - w} \left[\begin{matrix} -q, & 1/bd \\ q^2/b, & q^2/d \end{matrix} \middle| q^2 \right]_{m_\ell} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k \frac{1 - q^{-2k}w}{1 - w} \left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix} \middle| q^2 \right]_k \frac{q^{k^2(1+\ell)+k}}{(bd)^k}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

Corollary 17 $(b=-1,\ d\to 0,\ w=0)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}(-1)^{m_\ell}\frac{(-q;q^2)_{m_\ell}}{(-q^2;q^2)_{m_\ell}}q^{-2m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2\ell},q^{\ell-2},q^{2+\ell};q^{2\ell}]_\infty.$$

Corollary 18 $(b=-1,\ w=-d=q\varepsilon)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\left[\begin{matrix}q^3\varepsilon, & q^{-1}\varepsilon\\ q, & -q^2\end{matrix}\middle|q^2\right]_{m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2+2\ell},q^{\ell-1}\varepsilon,q^{3+\ell}\varepsilon;q^{2+2\ell}]_\infty.$$

Corollary 19 $(b,d\to 0$ and $w\to\infty)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}(-1)^{m_\ell}(-q;q^2)_{m_\ell}q^{-m_\ell^2-m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2\ell-2},q^{\ell-2},q^{\ell};q^{2\ell-2}]_\infty.$$

Corollary 20 $(b,d\to 0,\ w=0)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}(-1)^{m_\ell}(-q;q^2)_{m_\ell}q^{-m_\ell^2-3m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}$$
$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2\ell-2},q^{\ell-4},q^{2+\ell};q^{2\ell-2}]_\infty.$$

Corollary 21 $(b=-1,\ d\to\infty,\ w\to\infty)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q;q^2)_{m_\ell}}{(-q^2;q^2)_{m_\ell}}q^{2m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{4+2\ell},q^{\ell},q^{4+\ell};q^{4+2\ell}]_\infty.$$

Corollary 22 $(b,d\to\infty,\ w\to\infty)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}(-q;q^2)_{m_\ell}q^{2m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{6+2\ell},q^{\ell},q^{6+\ell};q^{6+2\ell}]_\infty.$$

When $\ell=1$, the last corollary reduces to the analytic version of the second Göllnitz-Gordon partition identity (cf. Slater [29, Eq 34] and Stembridge [32, Eq 3.10]).

§3.3. For $\delta = 1$, taking W_k in Theorem 2 as

$$W_k = \frac{(1-uq^k)(1-vq^k)}{uv-q^{-1}}\left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix}\middle| q\right]_k \left(\frac{1}{qbd}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_5\psi_5$-series identity (10) as

$$\sum_{k=-m}^{1+m} \frac{(q^{-1-m};q)_k}{(q^{1+m};q)_k} \frac{(1-uq^k)(1-vq^k)}{uv-q^{-1}}\left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix}\middle| q\right]_k \left(\frac{q^m}{bd}\right)^k = \frac{(q;q)_{1+m}(1/bd;q)_m}{[q/b,q/d;q]_m}.$$

According to Theorem 2, we have the following transformation.

Theorem 23 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q;q)_{2n+1}}{(q;q)_{m_0}}\left[\begin{matrix} -q, & 1/bd \\ q/b, & q/d \end{matrix}\middle| q\right]_{m_\ell} \prod_{\imath=1}^{\ell} \frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}$$
$$= (-q;q)_n \sum_{k=-n}^{n+1} (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix} \frac{(1-uq^k)(1-vq^k)}{(uv-q^{-1})(qbd)^k}\left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix}\middle| q\right]_k q^{\binom{k}{2}(1+\ell)}.$$

Letting $n \to \infty$, we get the nonterminating multiple series transformation.

Proposition 24 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \left[\begin{matrix} -q, & 1/bd \\ q/b, & q/d \end{matrix}\middle| q\right]_{m_\ell} \prod_{\imath=1}^{\ell} \frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k \frac{(1-uq^k)(1-vq^k)}{(uv-q^{-1})(qbd)^k}\left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix}\middle| q\right]_k q^{\binom{k}{2}(1+\ell)}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

Corollary 25 ($b = -1,\ d \to 0,\ u = v = 0$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \left(-\frac{1}{q}\right)^{m_\ell} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^\ell, q, q^{\ell-1}; q^\ell]_\infty.$$

Corollary 26 ($b, d \to 0,\ u = v = 0$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} (-1)^{m_\ell}(-q;q)_{m_\ell} q^{-\binom{m_\ell}{2}-2m_\ell} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^{\ell-1}, q, q^{\ell-2}; q^{\ell-1}]_\infty.$$

COROLLARY 27 ($b=-1,\ d\to\infty,\ u=v=0$: Stembridge [32, Eq a]).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{2+\ell},q,q^{1+\ell};q^{2+\ell}]_\infty.$$

COROLLARY 28 ($b,d\to\infty,\ u=v=0$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}(-q;q)_{m_\ell}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{3+\ell},q,q^{2+\ell};q^{3+\ell}]_\infty.$$

COROLLARY 29 ($b=u=q^{-1/2},\ d\to 0,\ v=0\,|\,q\to q^2$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}q^{-m_\ell}\frac{(-q^2;q^2)_{m_\ell}}{(q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q^2;q^2)_{m_k}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2\ell},-q,-q^{2\ell-1};q^{2\ell}]_\infty.$$

§3.4. For $\delta=0$, replacing q by q^2 and then taking W_k in Theorem 2 as

$$W_k=\frac{1+q^{2k}}{2}\begin{bmatrix}b, & d\\ -q/b, & -q/d\end{bmatrix}q\bigg]_k\left(-\frac{q}{bd}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (8a–8b) as

$$\sum_{k=-m}^{m}q^{2km}\frac{(q^{-2m};q^2)_k}{(q^{2+2m};q^2)_k}\frac{1+q^{2k}}{2}\begin{bmatrix}b, & d\\ -q/b, & -q/d\end{bmatrix}q\bigg]_k\left(-\frac{q}{bd}\right)^k$$
$$=\frac{(-q/bd;q)_{2m}}{(q;q)_{2m}}\begin{bmatrix}q^2, & q^2\\ q^2/b^2, & q^2/d^2\end{bmatrix}q^2\bigg]_m.$$

According to Theorem 2, we derive the following transformation.

THEOREM 30 (Terminating series transformation).

$$\sum_{M_0=n}\frac{(q^2;q^2)_{2n}}{(q^2;q^2)_{m_0}}\frac{(-q/bd;q)_{2m_\ell}}{(q;q)_{2m_\ell}}\begin{bmatrix}-q, & q^2\\ q^2/b^2, & q^2/d^2\end{bmatrix}q^2\bigg]_{m_\ell}\prod_{\imath=1}^{\ell}\frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$
$$=(-q;q^2)_n\sum_{k=-n}^{n}\frac{1+q^{2k}}{2}\begin{bmatrix}2n\\ n+k\end{bmatrix}_{q^2}\begin{bmatrix}b, & d\\ -q/b, & -q/d\end{bmatrix}q\bigg]_k\frac{q^{k^2(1+\ell)}}{(bd)^k}.$$

Letting $n\to\infty$, we get the nonterminating multiple series transformation.

Proposition 31 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q/bd;q)_{2m_\ell}}{(q;q)_{2m_\ell}}\left[\begin{matrix}-q, & q^2\\ q^2/b^2, & q^2/d^2\end{matrix}\middle|q^2\right]_{m_\ell}\prod_{\imath=1}^{\ell}\frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$
$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{k=-\infty}^{+\infty}\frac{1+q^{2k}}{2}\left[\begin{matrix}b, & d\\ -q/b, & -q/d\end{matrix}\middle|q\right]_k\frac{q^{k^2(1+\ell)}}{(bd)^k}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

Corollary 32 ($b=\sqrt{-1},\ d\to 0$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}q^{m_\ell^2}\frac{(-q;q^2)_{m_\ell}}{(q;-q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^{1+2\ell},q^\ell,-q^{1+\ell};-q^{1+2\ell}]_\infty.$$

Corollary 33 ($b=-d=\sqrt{-q}$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-1;q^2)_{m_\ell}}{(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2+2\ell},-q^\ell,-q^{2+\ell};q^{2+2\ell}]_\infty.$$

Corollary 34 ($b=\sqrt{-1},\ d\to\infty$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q;q^2)_{m_\ell}}{(q;-q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^{3+2\ell},-q^{1+\ell},q^{2+\ell};-q^{3+2\ell}]_\infty.$$

§3.5. For $\delta=1$, replacing q by q^2 and then taking W_k in Theorem 2 as

$$W_k=\frac{q+q^{2k}}{2}\left[\begin{matrix}b, & d\\ -1/b, & -1/d\end{matrix}\middle|q\right]_k\left(-\frac{1}{qbd}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (8a–8b) as

$$\sum_{k=-m}^{1+m}q^{2km}\frac{(q^{-2-2m};q^2)_k}{(q^{2+2m};q^2)_k}\frac{q+q^{2k}}{2}\left[\begin{matrix}b, & d\\ -1/b, & -1/d\end{matrix}\middle|q\right]_k\left(-\frac{q}{bd}\right)^k$$
$$=\frac{1+bd}{(1+b)(1+d)}\frac{(-q/bd;q)_{2m}}{(q;q)_{1+2m}}\frac{(q^2;q^2)_m(q^2;q^2)_{1+m}}{[q^2/b^2,q^2/d^2;q^2]_m}.$$

According to Theorem 2, we derive the following transformation.

Theorem 35 (Terminating series transformation).

$$\sum_{M_0=n}\frac{(q^2;q^2)_{2n+1}}{(q^2;q^2)_{m_0}}\frac{(-q/bd;q)_{2m_\ell}}{(q;q)_{1+2m_\ell}}\frac{(q^4;q^4)_{m_\ell}}{[q^2/b^2,q^2/d^2;q^2]_{m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{M_\imath^2+M_\imath}}{(q^2;q^2)_{m_\imath}}$$
$$=\frac{(1+b)(1+d)}{2(1+bd)}(-q^2;q^2)_n\sum_{k=-n}^{n+1}\frac{q+q^{2k}}{(qbd)^k}\begin{bmatrix}2n+1\\ n+k\end{bmatrix}_{q^2}\left[\begin{matrix}b, & d\\ -1/b, & -1/d\end{matrix}\middle| q\right]_k q^{(k^2-k)(1+\ell)}.$$

Letting $n\to\infty$, we get the nonterminating multiple series transformation.

Proposition 36 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q/bd;q)_{2m_\ell}}{(q;q)_{1+2m_\ell}}\frac{(q^4;q^4)_{m_\ell}}{[q^2/b^2,q^2/d^2;q^2]_{m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{M_\imath^2+M_\imath}}{(q^2;q^2)_{m_\imath}}$$
$$=\frac{(1+b)(1+d)}{2(1+bd)}\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{k=-\infty}^{+\infty}\frac{q+q^{2k}}{(qbd)^k}\left[\begin{matrix}b, & d\\ -1/b, & -1/d\end{matrix}\middle| q\right]_k q^{(k^2-k)(1+\ell)}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

Corollary 37 ($b=-d=\sqrt{-q^{-1}}\,|\,q\to q^{1/2}$: Chu [18, Example 7]).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q;q)_{m_\ell}^2}{(q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{1+\ell},-q^{1+\ell},-q^{1+\ell};q^{1+\ell}]_\infty.$$

Corollary 38 ($b=\sqrt{-q^{-1}},\ d\to 0\,|\,q\to q^{1/2}$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}q^{\binom{1+m_\ell}{2}}\frac{(-q;q)_{m_\ell}}{(q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{2+4\ell},q^\ell,q^{2+3\ell};q^{2+4\ell}]_\infty.$$

Corollary 39 ($b=\sqrt{-q^{-1}},\ d\to\infty\,|\,q\to q^{1/2}$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q;q)_{m_\ell}}{(q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{6+4\ell},q^{2+\ell},q^{4+3\ell};q^{6+4\ell}]_\infty.$$

Corollary 40 ($b,d\to\infty$: Chu [18, Example 27]).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q^2;q^2)_{m_\ell}}{(q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q^2;q^2)_{m_k}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{4+2\ell},-q,-q^{3+2\ell};q^{4+2\ell}]_\infty.$$

The univariate case of Corollary 39 can be found in Slater [29, Eq 45].

§3.6. For $\delta = 0$, replacing q by q^2 and then taking W_k in Theorem 2 as

$$W_k = \frac{1 - wq^k}{1 - w} \frac{(c;q)_k}{(q/c;q)_k} \left(-\frac{q}{c}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_4\psi_4$-series identity (9) as

$$\sum_{k=-m}^{m} q^{2km} \frac{(q^{-2m};q^2)_k}{(q^{2+2m};q^2)_k} \frac{1 - wq^k}{1 - w} \frac{(c;q)_k}{(q/c;q)_k} \left(-\frac{q}{c}\right)^k = \frac{(-q/c;q)_{2m}(q^2;q^2)_m}{(-q;q)_{2m}(q^2/c^2;q^2)_m}.$$

According to Theorem 2, we have the following transformation.

Theorem 41 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q^2;q^2)_{2n}}{(q^2;q^2)_{m_0}} \frac{(-q/c;q)_{2m_\ell}}{(-q^2;q^2)_{m_\ell}(q^2/c^2;q^2)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$
$$= (-q;q^2)_n \sum_{k=-n}^{n} \frac{1 - wq^k}{1 - w} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} \frac{(c;q)_k}{(q/c;q)_k} \frac{q^{k^2(1+\ell)}}{c^k}.$$

Letting $n \to \infty$, we get the nonterminating multiple series transformation.

Proposition 42 (Nonterminating series transformation).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(-q/c;q)_{2m_\ell}}{(-q^2;q^2)_{m_\ell}(q^2/c^2;q^2)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2;q^2)_{m_\imath}}$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{+\infty} \frac{1 - wq^k}{1 - w} \frac{(c;q)_k}{(q/c;q)_k} \frac{q^{k^2(1+\ell)}}{c^k}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

Corollary 43 ($c \to 0$).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(-1)^{m_\ell} q^{m_\ell^2}}{(-q^2;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2;q^2)_{m_k}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [q^{1+2\ell}, q^\ell, q^{1+\ell}; q^{1+2\ell}]_\infty.$$

Corollary 44 ($c = w = -1$).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(q;q^2)_{m_\ell}}{(-q^2;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2;q^2)_{m_k}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [q^{2+2\ell}, q^{1+\ell}, q^{1+\ell}; q^{2+2\ell}]_\infty.$$

Corollary 45 ($c \to \infty$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{1}{(-q^2;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2;q^2)_{m_k}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{3+2\ell}, q^{1+\ell}, q^{2+\ell}; q^{3+2\ell}]_\infty.$$

Corollary 46 ($c = -q^{1/2} \mid q \to q^2$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{(q;q^2)_{2m_\ell}}{(q^2;-q^2)_{2m_\ell}} \prod_{k=1}^{\ell} \frac{q^{2M_k^2}}{(q^4;q^4)_{m_k}} = \frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}[q^{4+4\ell}, q^{1+2\ell}, q^{3+2\ell}; q^{4+4\ell}]_\infty.$$

The univariate case of Corollary 45 has been discovered by Rogers [24, 1894].

§3.7. For $\delta = 1$, replacing q by q^2 and then taking W_k in Theorem 2 as

$$W_k = \frac{(1-uq^k)(1-vq^k)}{uv-q^{-1}} \frac{(c;q)_k}{(1/c;q)_k}\left(-\frac{1}{qc}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_5\psi_5$-series identity (10) as

$$\sum_{k=-m}^{1+m} q^{2km} \frac{(q^{-2-2m};q^2)_k}{(q^{2+2m};q^2)_k} \frac{(1-uq^k)(1-vq^k)}{uv-q^{-1}} \frac{(c;q)_k}{(1/c;q)_k}\left(\frac{-q}{c}\right)^k$$
$$= \frac{(-q/c;q)_{2m}(q^2;q^2)_{1+m}}{(-q;q)_{1+2m}(q^2/c^2;q^2)_m}.$$

According to Theorem 2, we establish the following transformation.

Theorem 47 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q^2;q^2)_{2n+1}}{(q^2;q^2)_{m_0}} \frac{(-q/c;q)_{2m_\ell}}{(-q;q^2)_{1+m_\ell}(q^2/c^2;q^2)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2+M_\imath}}{(q^2;q^2)_{m_\imath}}$$
$$= (-q^2;q^2)_n \sum_{k=-n}^{n+1} \frac{(1-uq^k)(1-vq^k)}{(uv-q^{-1})(qc)^k} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_{q^2} \frac{(c;q)_k}{(1/c;q)_k} q^{(k^2-k)(1+\ell)}.$$

Letting $n \to \infty$, we get the nonterminating multiple series transformation.

PROPOSITION 48 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-q/c;q)_{2m_\ell}}{(-q;q^2)_{1+m_\ell}(q^2/c^2;q^2)_{m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{M_\imath^2+M_\imath}}{(q^2;q^2)_{m_\imath}}$$

$$=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{k=-\infty}^{+\infty}\frac{(1-uq^k)(1-vq^k)}{(uv-q^{-1})(qc)^k}\frac{(c;q)_k}{(1/c;q)_k}q^{(k^2-k)(1+\ell)}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

COROLLARY 49 $(c\to 0)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-1)^{m_\ell}q^{m_\ell^2}}{(-q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q^2;q^2)_{m_k}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{1+2\ell},q,q^{2\ell};q^{1+2\ell}]_\infty.$$

COROLLARY 50 $(c\to\infty)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{1}{(-q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q^2;q^2)_{m_k}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{3+2\ell},q,q^{2+2\ell};q^{3+2\ell}]_\infty.$$

COROLLARY 51 $(c=u=-q^{-1/2},\ v=0\,|\,q\to q^2)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(q;q^2)_{1+2m_\ell}}{(q^4;q^8)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2+2M_k}}{(q^4;q^4)_{m_k}}=\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}[q^{4+4\ell},q,q^{3+4\ell};q^{4+4\ell}]_\infty.$$

§3.8. For $\delta=0$, replacing q by q^3 and then taking W_k in Theorem 2 as

$$W_k=\frac{1-\omega^2q^{2k}}{1-\omega^2}\frac{(\omega c;q)_k}{(q\omega/c;q)_k}\left(\frac{q}{c}\right)^k$$

where $\omega:=e^{2\pi i/3}$ is the cubic root of unity, we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (8a–8b) as

$$\sum_{k=-m}^{m}q^{3km}\frac{(q^{-3m};q^3)_k}{(q^{3+3m};q^3)_k}\frac{1-\omega^2q^{2k}}{1-\omega^2}\frac{(\omega c;q)_k}{(q\omega/c;q)_k}\left(\frac{q}{c}\right)^k=\frac{(q/c;q)_{3m}(q^3;q^3)_m^2}{(q^3;q^3)_{2m}(q^3/c^3;q^3)_m}.$$

Making further the replacement $q\to q^2$, we derive the following transformation.

THEOREM 52 (Terminating series transformation).

$$\sum_{M_0=n}\frac{(q^6;q^6)_{2n}}{(q^6;q^6)_{m_0}}\frac{(q^2/c;q^2)_{3m_\ell}}{(q^3;q^6)_{m_\ell}(-q^6;q^6)_{m_\ell}(q^6/c^3;q^6)_{m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{3M_\imath^2}}{(q^6;q^6)_{m_\imath}}$$

$$=(-q^3;q^6)_n\sum_{k=-n}^{n}(-1)^k\frac{1-\omega^2q^{4k}}{1-\omega^2}\begin{bmatrix}2n\\n+k\end{bmatrix}_{q^6}\frac{(\omega c;q^2)_k}{(q^2\omega/c;q^2)_k}\frac{q^{3k^2(1+\ell)}}{(qc)^k}.$$

Letting $n\to\infty$, we get the nonterminating multiple series transformation.

PROPOSITION 53 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(q^2/c;q^2)_{3m_\ell}}{(q^3;q^6)_{m_\ell}(-q^6;q^6)_{m_\ell}(q^6/c^3;q^6)_{m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{3M_\imath^2}}{(q^6;q^6)_{m_\imath}}$$

$$=\frac{(-q^3;q^6)_\infty}{(q^6;q^6)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k\frac{1-\omega^2q^{4k}}{1-\omega^2}\frac{(\omega c;q^2)_k}{(q^2\omega/c;q^2)_k}\frac{q^{3k^2(1+\ell)}}{(qc)^k}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

COROLLARY 54 $(c\to 0\,|\,q\to q^{1/3})$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{q^{2m_\ell^2}}{(q;q^2)_{m_\ell}(-q^2;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}$$

$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{4+6\ell},q^{\ell},q^{4+5\ell};q^{4+6\ell}]_\infty[q^{4+8\ell},q^{4+4\ell};q^{8+12\ell}]_\infty.$$

COROLLARY 55 $(c\to\infty\,|\,q\to q^{1/3})$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{1}{(q;q^2)_{m_\ell}(-q^2;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q^2;q^2)_{m_k}}$$

$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{8+6\ell},q^{2+\ell},q^{6+5\ell};q^{8+6\ell}]_\infty[q^{12+8\ell},q^{4+4\ell};q^{16+12\ell}]_\infty.$$

Corollary 56 ($c = q\varepsilon$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{(q\varepsilon;q^2)_{3m_\ell}}{(q^3;q^6)_{m_\ell}(-q^6;q^6)_{m_\ell}(q^3\varepsilon;q^6)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{3M_k^2}}{(q^6;q^6)_{m_k}}$$
$$= \frac{(-q^3;q^6)_\infty}{(q^6;q^6)_\infty} [q^{6+6\ell}, q^{1+3\ell}\varepsilon, q^{5+3\ell}\varepsilon; q^{6+6\ell}]_\infty.$$

Corollary 57 ($c = \varepsilon$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{(q^2\varepsilon;q^2)_{3m_\ell}}{(q^3;q^6)_{m_\ell}(-q^6;q^6)_{m_\ell}(q^6\varepsilon;q^6)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{3M_k^2}}{(q^6;q^6)_{m_k}}$$
$$= \frac{(-q^3;q^6)_\infty}{(q^6;q^6)_\infty} [q^{6+6\ell}, q^{2+3\ell}\varepsilon, q^{4+3\ell}\varepsilon; q^{6+6\ell}]_\infty.$$

When $\ell = 1$, the two identities in quintuple products become respectively Bailey [11, Eq 4.3] and Slater [29, Eq 117], while Corollary 57 reduces to Bailey [9, Eq 7.5]. However, the identity displayed in Corollary 56 seems new even for $\ell = 1$.

§3.9. For $\delta = 1$, replacing q by q^3 and then taking W_k in Theorem 2 as

$$W_k = (q^{2k} - q\omega)\frac{(\omega c;q)_k}{(\omega/c;q)_k}\left(\frac{q^{-1}}{c}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (8a–8b) as

$$\sum_{k=-m}^{1+m} q^{3km}\frac{(q^{-3-3m};q^3)_k}{(q^{3+3m};q^3)_k}(q^{2k} - q\omega)\frac{(\omega c;q)_k}{(\omega/c;q)_k}\left(\frac{q^2}{c}\right)^k$$
$$= \frac{1-\omega^2}{1-\omega/c}\,\frac{(q/c;q)_{1+3m}(q^3;q^3)_{1+m}}{(-q^3;q^3)_m(q^3;q^6)_{1+m}(q^3/c^3;q^3)_m}.$$

According to Theorem 2, we derive the following transformation.

Theorem 58 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q^3;q^3)_{2n+1}}{(q^3;q^3)_{m_0}}\,\frac{(q/c;q)_{1+3m_\ell}}{(q^3;q^6)_{1+m_\ell}(q^3/c^3;q^3)_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{3\binom{1+M_\iota}{2}}}{(q^3;q^3)_{m_\iota}}$$
$$= (1-\omega/c)(-q^3;q^3)_n \sum_{k=-n}^{n+1} \frac{1-q^{1-2k}\omega}{1-\omega^2} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_{q^3} \left(-\frac{q}{c}\right)^k \frac{(\omega c;q)_k}{(\omega/c;q)_k} q^{3\binom{k}{2}(1+\ell)}.$$

Letting $n \to \infty$, we get the nonterminating multiple series transformation.

Proposition 59 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^{\ell}} \frac{(q/c;q)_{1+3m_\ell}}{(q^3;q^6)_{1+m_\ell}(q^3/c^3;q^3)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{3\binom{1+M_\imath}{2}}}{(q^3;q^3)_{m_\imath}}$$
$$= (1-\omega/c)\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty} \sum_{k=-\infty}^{+\infty} \frac{1-q^{1-2k}\omega}{1-\omega^2}\, \frac{(\omega c;q)_k}{(\omega/c;q)_k} \left(-\frac{q}{c}\right)^k q^{3\binom{k}{2}(1+\ell)}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

Corollary 60 $(c=\varepsilon)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^{\ell}} \frac{(q\varepsilon;q)_{1+3m_\ell}}{(q^3;q^6)_{1+m_\ell}(q^3\varepsilon;q^3)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{3\binom{1+M_k}{2}}}{(q^3;q^3)_{m_k}} = \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}[q^{3+3\ell}, q\varepsilon, q^{2+3\ell}\varepsilon; q^{3+3\ell}]_\infty.$$

Corollary 61 $(c=q^{-1/2}\,|\,q\to q^2)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^{\ell}} \frac{(q;q^2)_{2+3m_\ell}}{(q^3;q^6)_{1+m_\ell}(q^6;q^{12})_{1+m_\ell}} \prod_{k=1}^{\ell} \frac{q^{3M_k^2+3M_k}}{(q^6;q^6)_{m_k}} = \frac{(-q^6;q^6)_\infty}{(q^6;q^6)_\infty}[q^{6+6\ell}, q, q^{5+6\ell}; q^{6+6\ell}]_\infty.$$

Corollary 62 $(c\to 0\,|\,q\to q^{1/3})$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^{\ell}} \frac{q^{m_\ell^2+m_\ell}}{(q;q^2)_{1+m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty}[q^{2+3\ell}, q^{1+\ell}, q^{1+2\ell}; q^{2+3\ell}]_\infty [q^{4+5\ell}, q^{\ell}; q^{4+6\ell}]_\infty.$$

Corollary 63 $(c\to \infty\,|\,q\to q^{1/3})$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^{\ell}} \frac{1}{(q;q^2)_{1+m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty}[q^{4+3\ell}, q^{1+\ell}, q^{3+2\ell}; q^{4+3\ell}]_\infty [q^{6+5\ell}, q^{2+\ell}; q^{8+6\ell}]_\infty.$$

The last two corollaries generalize the identities in quintuple products respectively due to Slater [29, Eq 63] and Rogers [25, 1917].

§3.10. For $\delta = 0$, taking W_k in Theorem 2 as

$$W_{2k+1} = 0 \quad \text{and} \quad W_{2k} = \frac{1 - wq^{2k}}{1 - w} \frac{(c; q^2)_k}{(q^2/c; q^2)_k} \left(\frac{q}{c}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_4\psi_4$-series identity (9) as

$$\sum_k q^{2km} \frac{(q^{-m}; q)_{2k}}{(q^{1+m}; q)_{2k}} \frac{1 - wq^{2k}}{1 - w} \frac{(c; q^2)_k}{(q^2/c; q^2)_k} \left(\frac{q}{c}\right)^k = \frac{(q; q)_m (q/c; q^2)_m}{(q; q^2)_m (q/c; q)_m}.$$

Making further the replacement $q \to q^2$, we derive the following transformation.

THEOREM 64 (Terminating series transformation).

$$\sum_{M_0 = n} \frac{(q^2; q^2)_{2n}}{(q^2; q^2)_{m_0}} \frac{(q^2/c; q^4)_{m_\ell}}{(q; q^2)_{m_\ell} (q^2/c; q^2)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2; q^2)_{m_\imath}}$$
$$= (-q; q^2)_n \sum_k \frac{1 - wq^{4k}}{1 - w} \begin{bmatrix} 2n \\ n + 2k \end{bmatrix}_{q^2} \frac{(c; q^4)_k}{(q^4/c; q^4)_k} \frac{q^{4k^2(1+\ell)}}{c^k}.$$

Letting $n \to \infty$, we get the nonterminating multiple series transformation.

PROPOSITION 65 (Nonterminating series transformation).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(q^2/c; q^4)_{m_\ell}}{(q; q^2)_{m_\ell} (q^2/c; q^2)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2; q^2)_{m_\imath}}$$
$$= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{k=-\infty}^{+\infty} \frac{1 - wq^{4k}}{1 - w} \frac{(c; q^4)_k}{(q^4/c; q^4)_k} \frac{q^{4k^2(1+\ell)}}{c^k}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

COROLLARY 66 $(c \to 0)$.

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{q^{m_\ell^2 - m_\ell}}{(q; q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2; q^2)_{m_k}} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} [q^{4+8\ell}, q^{4\ell}, q^{4+4\ell}; q^{4+8\ell}]_\infty.$$

COROLLARY 67 $(c = -q^2\varepsilon)$.

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(-\varepsilon; q^4)_{m_\ell}}{(-\varepsilon; -q\varepsilon)_{2m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2; q^2)_{m_k}} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} [q^{8+8\ell}, q^{2+4\ell}\varepsilon, q^{6+4\ell}\varepsilon; q^{8+8\ell}]_\infty.$$

Corollary 68 ($c = w = -1$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{(-q^2;q^4)_{m_\ell}}{(q;-q)_{2m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2;q^2)_{m_k}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [q^{8+8\ell}, q^{4+4\ell}, q^{4+4\ell}; q^{8+8\ell}]_\infty.$$

Corollary 69 ($c \to \infty$: Warnaar [33, Thm 4.4]).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{1}{(q;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2;q^2)_{m_k}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [q^{12+8\ell}, q^{4+4\ell}, q^{8+4\ell}; q^{12+8\ell}]_\infty.$$

The case $\ell = 1$ of the last corollary was first discovered by Rogers [24, 1894].

§3.11. For $\delta = 1$, taking W_k in Theorem 2 as

$$W_{2k+1} = 0 \quad \text{and} \quad W_{2k} = \frac{1-q^{-2k}w}{1-w} \frac{(c;q^2)_k}{(1/c;q^2)_k} \left(\frac{q}{c}\right)^k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_5\psi_5$-series identity (10) as

$$\sum_k q^{2km} \frac{(q^{-1-m};q)_{2k}}{(q^{1+m};q)_{2k}} \frac{1-q^{-2k}w}{1-w} \frac{(c;q^2)_k}{(1/c;q^2)_k} \left(\frac{q^3}{c}\right)^k = \frac{1-wq^m}{1-w} \frac{(q;q)_{1+m}(q/c;q^2)_m}{(q;q^2)_{1+m}(q/c;q)_m}.$$

According to Theorem 2, we have the following transformation.

Theorem 70 (Terminating series transformation).

$$\sum_{M_0=n} \frac{1-wq^{m_\ell}}{1-w} \frac{(q;q)_{2n+1}}{(q;q)_{m_0}} \frac{(-q;q)_{m_\ell}(q/c;q^2)_{m_\ell}}{(q;q^2)_{1+m_\ell}(q/c;q)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}$$
$$= (-q;q)_n \sum_k \frac{1-q^{-2k}w}{1-w} \begin{bmatrix} 2n+1 \\ n+2k \end{bmatrix} \left(\frac{q}{c}\right)^k \frac{(c;q^2)_k}{(1/c;q^2)_k} q^{\binom{2k}{2}(1+\ell)}.$$

Letting $n \to \infty$, we get the nonterminating multiple series transformation.

Proposition 71 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{1-wq^{m_\ell}}{1-w} \frac{(-q;q)_{m_\ell}(q/c;q^2)_{m_\ell}}{(q;q^2)_{1+m_\ell}(q/c;q)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{k=-\infty}^{+\infty} \frac{1-q^{-2k}w}{1-w} \left(\frac{q}{c}\right)^k \frac{(c;q^2)_k}{(1/c;q^2)_k} q^{\binom{2k}{2}(1+\ell)}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

Corollary 72 $(c \to 0,\ w = 0)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} q^{\binom{m_\ell}{2}} \frac{(-q;q)_{m_\ell}}{(q;q^2)_{1+m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}} = \frac{(-q;q)_\infty}{(q;q)_\infty} [q^{2+4\ell}, q^{2+\ell}, q^{3\ell}; q^{2+4\ell}]_\infty.$$

Corollary 73 $(c = -1,\ w \to \infty)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} q^{m_\ell} \frac{(-q;q^2)_{m_\ell}}{(q;q^2)_{1+m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}} = \frac{(-q;q)_\infty}{(q;q)_\infty} [q^{4+4\ell}, q^{\ell}, q^{4+3\ell}; q^{4+4\ell}]_\infty.$$

Corollary 74 $(c = \varepsilon/q,\ w = q\varepsilon)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{(-q;q)_{m_\ell}(q^2\varepsilon;q^2)_{m_\ell}}{(q;q^2)_{1+m_\ell}(q\varepsilon;q)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}} = \frac{(-q;q)_\infty}{(q;q)_\infty} [q^{4+4\ell}, -q^{1+\ell}\varepsilon, -q^{3+3\ell}\varepsilon; q^{4+4\ell}]_\infty.$$

Corollary 75 $(c = -1,\ w = 0)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} \frac{(-q;q^2)_{m_\ell}}{(q;q^2)_{1+m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}} = \frac{(-q;q)_\infty}{(q;q)_\infty} [q^{4+4\ell}, q^{2+\ell}, q^{2+3\ell}; q^{4+4\ell}]_\infty.$$

Corollary 76 $(c \to \infty,\ w \to \infty)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell} q^{m_\ell} \frac{(-q;q)_{m_\ell}}{(q;q^2)_{1+m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}} = \frac{(-q;q)_\infty}{(q;q)_\infty} [q^{6+4\ell}, q^{\ell}, q^{6+3\ell}; q^{6+4\ell}]_\infty.$$

§3.12. For $\delta = 1$, taking W_k in Theorem 2 as

$$W_{2k+1} = 0 \quad \text{and} \quad W_{2k} = \frac{q^{4k} - q}{(qbd)^k} \left[\begin{matrix} b, & d \\ q/b, & q/d \end{matrix} \middle| q^2 \right]_k$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (8a–8b) as

$$\sum_k q^{2k(1+m)} \frac{(q^{-1-m};q)_{2k}}{(q^{1+m};q)_{2k}} \left[\begin{matrix} b, & d \\ q/b, & q/d \end{matrix} \middle| q^2 \right]_k \frac{q^{4k} - q}{(qbd)^k} = \frac{(q;q)_{1+m}(q/bd;q^2)_m}{(-q;q)_m [q/b, q/d; q]_m}.$$

According to Theorem 2, we have the following transformation.

THEOREM 77 (Terminating series transformation).

$$\sum_{M_0=n}\frac{(q;q)_{2n+1}}{(q;q)_{m_0}}\frac{(q/bd;q^2)_{m_\ell}}{[q/b,q/d;q]_{m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}$$
$$=(-q;q)_n\sum_k\frac{q^{4k}-q}{(qbd)^k}\begin{bmatrix}2n+1\\n+2k\end{bmatrix}\left[\begin{matrix}b, & d\\q/b, & q/d\end{matrix}\middle|q^2\right]_k q^{\binom{2k}{2}(1+\ell)}.$$

Letting $n\to\infty$, we get the nonterminating multiple series transformation.

PROPOSITION 78 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(q/bd;q^2)_{m_\ell}}{[q/b,q/d;q]_{m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}\frac{q^{4k}-q}{(qbd)^k}\left[\begin{matrix}b, & d\\q/b, & q/d\end{matrix}\middle|q^2\right]_k q^{\binom{2k}{2}(1+\ell)}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

COROLLARY 79 ($b=-d=q^{1/2}$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-1;q^2)_{m_\ell}}{(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[-q^{1+\ell},q,-q^{\ell};-q^{1+\ell}]_\infty.$$

COROLLARY 80 ($b=-q^{1/2},\ d\to 0\,|\,q\to q^2$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}(-1)^{m_\ell}\frac{q^{m_\ell^2-2m_\ell}}{(-q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q^2;q^2)_{m_k}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{1+2\ell},q^2,q^{2\ell-1};q^{1+2\ell}]_\infty.$$

COROLLARY 81 ($b=q^{1/2}\varepsilon,\ d=q^{-1/2}\,|\,q\to q^2$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(q^2\varepsilon;q^4)_{m_\ell}}{(q\varepsilon;q^2)_{m_\ell}(q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q^2;q^2)_{m_k}}$$
$$=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{2+2\ell}\varepsilon,-q,-q^{1+2\ell}\varepsilon;q^{2+2\ell}\varepsilon]_\infty.$$

COROLLARY 82 ($b=-q^{1/2},\ d\to\infty\,|\,q\to q^2$).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{1}{(-q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q^2;q^2)_{m_k}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{3+2\ell},q^2,q^{1+2\ell};q^{3+2\ell}]_\infty.$$

§3.13. For $\delta = 1$, taking W_k in Theorem 2 as

$$W_{3k+1} = W_{3k+2} = 0 \quad \text{and} \quad W_{3k} = \frac{q^{6k} - q}{(q^2c)^k} \frac{(c; q^3)_k}{(q^2/c; q^3)_k}$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (8a–8b) as

$$\sum_k q^{k(1+3m)} \frac{(q^{-1-m}; q)_{3k}}{(q^{1+m}; q)_{3k}} \frac{(c; q^3)_k}{(q^2/c; q^3)_k} \frac{q^{6k} - q}{c^k} = \frac{(q; q)_{1+m}(q/c; q^3)_m}{(-q; q)_m (q; q^2)_m (q/c; q)_m}.$$

According to Theorem 2, we have the following transformation.

THEOREM 83 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q; q)_{2n+1}}{(q; q)_{m_0}} \frac{(q/c; q^3)_{m_\ell}}{(q; q^2)_{m_\ell} (q/c; q)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{\binom{1+M_\imath}{2}}}{(q; q)_{m_\imath}}$$
$$= (-q; q)_n \sum_k \frac{q^{6k} - q}{(-q^2c)^k} \begin{bmatrix} 2n+1 \\ n+3k \end{bmatrix} \frac{(c; q^3)_k}{(q^2/c; q^3)_k} q^{\binom{3k}{2}(1+\ell)}.$$

Letting $n \to \infty$, we get the nonterminating multiple series transformation.

PROPOSITION 84 (Nonterminating series transformation).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(q/c; q^3)_{m_\ell}}{(q; q^2)_{m_\ell} (q/c; q)_{m_\ell}} \prod_{\imath=1}^{\ell} \frac{q^{\binom{1+M_\imath}{2}}}{(q; q)_{m_\imath}} = \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{k=-\infty}^{+\infty} \frac{q^{6k} - q}{(-q^2c)^k} \frac{(c; q^3)_k}{(q^2/c; q^3)_k} q^{\binom{3k}{2}(1+\ell)}.$$

Special cases of this transformation result in multiple Rogers-Ramanujan identities.

COROLLARY 85 ($c = -q^{-1/2} \mid q \to q^2$).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(-q^3; q^6)_{m_\ell}}{(-q; q^2)_{1+m_\ell} (q^2; q^4)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2 + M_k}}{(q^2; q^2)_{m_k}}$$
$$= \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} [q^{6+6\ell}, q, q^{5+6\ell}; q^{6+6\ell}]_\infty [q^{8+6\ell}, q^{4+6\ell}; q^{12+12\ell}]_\infty.$$

COROLLARY 86 ($c \to 0$).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{q^{m_\ell^2 - m_\ell}}{(q; q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q; q)_{m_k}} = \frac{(-q; q)_\infty}{(q; q)_\infty} [q^{2+3\ell}, q, q^{1+3\ell}; q^{2+3\ell}]_\infty [q^{4+3\ell}, q^{3\ell}; q^{4+6\ell}]_\infty.$$

Corollary 87 $(c=-q)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(-1;q^3)_{m_\ell}}{(-1;q)_{m_\ell}(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{3+3\ell},q,q^{2+3\ell};q^{3+3\ell}]_\infty[q^{5+3\ell},q^{1+3\ell};q^{6+6\ell}]_\infty.$$

Corollary 88 $(c\to\infty)$.

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{1}{(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{4+3\ell},q,q^{3+3\ell};q^{4+3\ell}]_\infty[q^{6+3\ell},q^{2+3\ell};q^{8+6\ell}]_\infty.$$

The case $\ell=1$ of the last corollary was originally found by Rogers [25, 1917].

§3.14. For $\delta=1$, taking W_k in Theorem 2 as

$$W_{3k+1}=W_{3k+2}=0 \quad\text{and}\quad W_{3k}=1$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_4\psi_4$-series identity (9) as

$$\sum_k q^{3k(1+m)}\frac{(q^{-1-m};q)_{3k}}{(q^{1+m};q)_{3k}}=\frac{(q;q)_{1+m}(q^3;q^3)_m}{(q;q)_{1+2m}}.$$

According to Theorem 2, we have the following transformation.

Theorem 89 (Terminating series transformation).

$$\sum_{M_0=n}\frac{(q;q)_{2n+1}}{(q;q)_{m_0}}\frac{(q^3;q^3)_{m_\ell}}{(q;q)_{m_\ell}(q;q^2)_{1+m_\ell}}\prod_{\imath=1}^{\ell}\frac{q^{\binom{1+M_\imath}{2}}}{(q;q)_{m_\imath}}$$
$$=(-q;q)_n\sum_k(-1)^k\begin{bmatrix}2n+1\\ n+3k\end{bmatrix}q^{\binom{3k}{2}(1+\ell)}.$$

Letting $n\to\infty$, we get the following multiple Rogers-Ramanujan identity.

Corollary 90 (Multiple series identity).

$$\sum_{\tilde{m}\in\mathbf{N}_0^\ell}\frac{(q^3;q^3)_{m_\ell}}{(q;q)_{m_\ell}(q;q^2)_{1+m_\ell}}\prod_{k=1}^{\ell}\frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{9+9\ell},q^{3+3\ell},q^{6+6\ell};q^{9+9\ell}]_\infty.$$

§3.15. For $\delta = 0$, taking W_k in Theorem 2 as

$$W_{3k+1} = W_{3k+2} = 0 \quad \text{and} \quad W_{3k} = \frac{1 - wq^{3k}}{1 - w}$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_4\psi_4$-series identity (9) as

$$\sum_k q^{3km} \frac{(q^{-m}; q)_{3k}}{(q^{1+m}; q)_{3k}} \frac{1 - wq^{3k}}{1 - w} = \frac{(q; q)_m (q^3; q^3)_{m-1+\delta_{0,m}}}{(q; q)_{2m-1+\delta_{0,m}}}.$$

Making further the replacement $q \to q^2$, we derive the following transformation.

THEOREM 91 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q^2; q^2)_{2n}}{(q^2; q^2)_{m_0}} \frac{(q^6; q^6)_{m_\ell - 1 + \delta_{0,m_\ell}}}{(q; q^2)_{m_\ell} (q^4; q^4)_{m_\ell - 1 + \delta_{0,m_\ell}}} \prod_{\imath=1}^{\ell} \frac{q^{M_\imath^2}}{(q^2; q^2)_{m_\imath}}$$
$$= (-q; q^2)_n \sum_k (-1)^k \frac{1 - wq^{6k}}{1 - w} \begin{bmatrix} 2n \\ n + 3k \end{bmatrix}_{q^2} q^{2\binom{3k}{2} + 9k^2\ell}.$$

Letting $n \to \infty$, we obtain the following multiple Rogers-Ramanujan identity.

COROLLARY 92 (Multiple series identity).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(q^6; q^6)_{m_\ell - 1 + \delta_{0,m_\ell}}}{(q; q^2)_{m_\ell} (q^4; q^4)_{m_\ell - 1 + \delta_{0,m_\ell}}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q^2; q^2)_{m_k}}$$
$$= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} [q^{18+18\ell}, q^{6+9\ell}, q^{12+9\ell}; q^{18+18\ell}]_\infty.$$

The last corollary may be considered as an extension of an identity in quintuple product due to Bailey [8, Eq C3] (see also Slater [29, Eq 114]).

§3.16. For $\delta = 1$, taking W_k in Theorem 2 as

$$W_{4k+1} = W_{4k+2} = W_{4k+3} = 0 \quad \text{and} \quad W_{4k} = q^{4k} - q^{1-4k}$$

we can evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (8a–8b) as

$$\sum_k q^{4k(1+m)} \frac{(q^{-1-m}; q)_{4k}}{(q^{1+m}; q)_{4k}} (q^{4k} - q^{1-4k}) = \frac{(q; q)_{1+m} (-q^2; q^2)_{m-1+\delta_{0,m}}}{(-q; q)_m (q; q^2)_m}.$$

According to Theorem 2, we have the following transformation.

THEOREM 93 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q;q)_{2n+1}}{(q;q)_{m_0}} \frac{(-q^2;q^2)_{m_\ell-1+\delta_{0,m_\ell}}}{(q;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}$$
$$= (-q;q)_n \sum_k \begin{bmatrix} 2n+1 \\ n+4k \end{bmatrix} q^{\binom{4k}{2}(1+\ell)} (q^{4k} - q^{1-4k}).$$

Letting $n \to \infty$, we obtain the nonterminating multiple series identity.

COROLLARY 94 (Difference of infinite products).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{(-q^2;q^2)_{m_\ell-1+\delta_{0,m_\ell}}}{(q;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \left\{ \begin{matrix} [q^{16+16\ell}, -q^{6+10\ell}, -q^{10+6\ell}; q^{16+16\ell}]_\infty \\ -q[q^{16+16\ell}, -q^{2+6\ell}, -q^{14+10\ell}; q^{16+16\ell}]_\infty \end{matrix} \right\}.$$

There exist more identities expressing nonterminating multiple sums in terms of differences of infinite products. For example, taking $c \to q$ in Proposition 84 leads to another identity of this type.

COROLLARY 95 (Difference of infinite products).

$$\sum_{\tilde{m} \in \mathbf{N}_0^\ell} \frac{1+\delta_{0,m_\ell} - q^{m_\ell}}{1+\delta_{0,m_\ell} - q^{3m_\ell}} \frac{(q^3;q^3)_{m_\ell}}{(q;q)_{m_\ell}(q;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{\binom{1+M_k}{2}}}{(q;q)_{m_k}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \left\{ \begin{matrix} [q^{9+9\ell}, q^{6+3\ell}, q^{3+6\ell}; q^{9+9\ell}]_\infty \\ -q[q^{9+9\ell}, q^{3\ell}, q^{9+6\ell}; q^{9+9\ell}]_\infty \end{matrix} \right\}.$$

However, we are not going to produce further identities involving differences of infinite products due to the space limitation.

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