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AUTOMORPHY OF THE PRINCIPAL EISENSTEIN SERIES OF WEIGHT 1: AN APPLICATION OF THE DOUBLE SINE FUNCTION

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Abstract

We calculate cuspidal values of the modular transform of the principal Eisenstein series of weight 1. We obtain this by looking at derivatives of the double sine function. Our result is considered to be an explicit calculation of values of a Stirling modular function.

1. Introduction

For each non-zero complex number k we put

$$E_k(\tau) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

with

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1},$$

where τ is belonging to the upper half plane and $q = e^{2\pi i \tau}$. As is well-known, for an even integer $k \geq 4$, $E_k(\tau)$ is the Eisenstein series of weight k with respect to the modular group $SL_2(\mathbf{Z})$. Especially, it satisfies the automorphy

$$E_k\left(-\frac{1}{\tau}\right) = \tau^k E_k(\tau)$$

and the corresponding zeta function is given by

$$L(s, E_k) = \zeta(s)\zeta(s-k+1).$$

We consider $E_k(\tau)$ as the principal Eisenstein series of weight k for $k \in \mathbf{C} \backslash \{0\}$ in general, where "principal" is indicating the "principal character." It seems to be an interesting problem to see the automorphy of $E_k(\tau)$. This means to calculate

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$$R_k(\tau) = E_k\left(-\frac{1}{\tau}\right) - \tau^k E_k(\tau)$$

concretely. We know the result in the following cases:

(1) $R_k(\tau) = 0$ for each even integer $k \geq 4$,

(2) $R_2(\tau) = -\dfrac{\tau}{4\pi i}$.

From these results we obtain

$$E_6(i) = 0$$

and

$$E_2(i) = -\frac{1}{8\pi}.$$

In other words,

$$\sum_{n=1}^{\infty} \frac{n^5}{e^{2\pi n} - 1} = \frac{1}{504}$$

and

$$\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n} - 1} = \frac{1}{24} - \frac{1}{8\pi}$$

as noted by Ramanujan. We notice that the case (2) is shown from the transformation formula

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau)$$

for the Dedekind η-function by taking the logarithmic derivative, for example. It would be a non-trivial problem to investigate $R_k(\tau)$ for $k \in \mathbf{C}\backslash\{0\}$ in general.

In papers [K7] [KK8] we proved

$$\lim_{\tau \to 1} R_k(\tau) = \frac{(-1)^k B_{k-1}}{2\pi i}$$

and its higher dimensional analogue for each positive integer k, where B_k is the Bernoulli number.

In this paper we look at the case $k = 1$, where

$$E_1(\tau) = -\frac{1}{4} + \sum_{n=1}^{\infty} d(n) q^n = -\frac{1}{4} + \sum_{m,n \geq 1} q^{mn}$$

and

$$R_1(\tau) = E_1\left(-\frac{1}{\tau}\right) - \tau E_1(\tau).$$

We notice that this k is the only one case when $L(s, E_k)$ has the double pole; in fact $L(s, E_1) = \zeta(s)^2$ is having the double pole at $s = 1$. We calculate boundary values of $R_1(\tau)$ at cusps N and $\frac{1}{N}$ for each integer $N \geq 1$.

THEOREM 1. *Let N be a positive integer. Then $R_1(\tau)$ has the following transcendental numbers as boundary values at cusps N and $\frac{1}{N}$.*

(1) $\lim_{\tau \to N} R_1(\tau) = \frac{1}{2i}\left(\frac{1}{N}\sum_{k=1}^{[N/2]}(N-2k)\cot\left(\frac{\pi k}{N}\right) - \frac{1}{\pi}\right).$

(2) $\lim_{\tau \to 1/N} R_1(\tau) = \frac{1}{2Ni}\left(\frac{1}{N}\sum_{k=1}^{[N/2]}(N-2k)\cot\left(\frac{\pi k}{N}\right) - \frac{1}{\pi}\right).$

Examples (We omit "lim".)

$$R_1(1) = -\frac{1}{2\pi i}.$$

$$R_1(2) = -\frac{1}{2\pi i}.$$

$$R_1\left(\frac{1}{2}\right) = -\frac{1}{4\pi i}.$$

$$R_1(3) = -\frac{1}{2\pi i} + \frac{1}{6\sqrt{3}i}.$$

$$R_1\left(\frac{1}{3}\right) = -\frac{1}{6\pi i} + \frac{1}{18\sqrt{3}i}.$$

$$R_1(4) = -\frac{1}{2\pi i} + \frac{1}{4i}.$$

$$R_1\left(\frac{1}{4}\right) = -\frac{1}{8\pi i} + \frac{1}{16i}.$$

$$R_1(5) = -\frac{1}{2\pi i} + \frac{1}{10i}\left(\sqrt{1-\frac{2}{\sqrt{5}}} + 3\sqrt{1+\frac{2}{\sqrt{5}}}\right).$$

$$R_1\left(\frac{1}{5}\right) = -\frac{1}{10\pi i} + \frac{1}{50i}\left(\sqrt{1-\frac{2}{\sqrt{5}}} + 3\sqrt{1+\frac{2}{\sqrt{5}}}\right).$$

$$R_1(6) = -\frac{1}{2\pi i} + \frac{7}{6\sqrt{3}i}.$$

$$R_1\left(\frac{1}{6}\right) = -\frac{1}{12\pi i} + \frac{7}{36\sqrt{3}i}.$$

We notice that it would be difficult to say something about the nature of the interior value such as

$$R_1(i) = (1-i)\left(-\frac{1}{4} + \sum_{n=1}^{\infty} d(n)e^{-2\pi n}\right)$$
$$= (1-i)\left(-\frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{e^{2\pi n}-1}\right)$$

for example.

Our result is obtained by using the theory of the double sine function. This theory was originated by Shintani [S1] to investigate Kronecker's Jugendtraum for a real quadratic field. We refer to [K1]–[K7] and [KK1]–[KK8] for detailed studies of multiple sine functions including the double sine function. We recommend to read the excellent survey [M1] of Manin. Here we need the double sine function

$$S_2(x,(\omega_1,\omega_2)) = \frac{\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))}{\Gamma_2(x,(\omega_1,\omega_2))}$$

where

$$\Gamma_2(x,(\omega_1,\omega_2)) = \exp\left(\frac{\partial}{\partial s}\zeta_2(s,x,(\omega_1,\omega_2))\bigg|_{s=0}\right)$$

is the normalized double gamma function of Barnes [B] and

$$\zeta_2(s,x,(\omega_1,\omega_2)) = \sum_{n_1,n_2\geq 0}(n_1\omega_1+n_2\omega_2+x)^{-s}$$

is the double Hurwitz zeta function. By using the regularized product notation $\coprod$ of Deninger [D] $S_2(x,(\omega_1,\omega_2))$ is neatly written as

$$S_2(x,(\omega_1,\omega_2)) = \frac{\displaystyle\coprod_{n_1,n_2\geq 0}(n_1\omega_1+n_2\omega_2+x)}{\displaystyle\coprod_{m_1,m_2\geq 1}(m_1\omega_1+m_2\omega_2-x)},$$

where

$$\coprod_{n=1}^{\infty} a_n = \exp\left(-\frac{d}{ds}\left(\sum_{n=1}^{\infty} a_n^{-s}\right)\bigg|_{s=0}\right).$$

We prove Theorem 1 from the following three theorems.

Theorem 2. *For* $\mathrm{Im}(\tau) > 0$

$$R_1(\tau) = \frac{\tau^{3/2}}{8\pi^2 i}S_2''(0,(1,\tau)).$$

Theorem 3. *For each positive real number* α

$$\lim_{\tau\to\alpha} R_1(\tau) = \frac{\alpha^{3/2}}{8\pi^2 i} S_2''(0,(1,\alpha)).$$

Theorem 4. *For each integer* $N \geq 1$

$$S_2''(0,(1,N)) = \frac{4\pi^2}{N^{3/2}}\left(\frac{1}{N}\sum_{k=1}^{[N/2]}(N-2k)\cot\left(\frac{\pi k}{N}\right) - \frac{1}{\pi}\right).$$

In the proof it is important to proceed from the view point of the Stirling modular function suggested by Barnes [B]. We notice that "boundary values" are interesting from the study of Shintani [S1] and also from the view point of "real multiplication" of Manin [M2].

We notice that our method is applicable also to calculate $\lim_{\tau\to M/N} R_1(\tau)$ for positive integers M and N in exactly the same way. Thus we will have boundary values at all the positive rational numbers. Since the full treatment would make the calculation a bit complicated, we will treat it in another paper.

2. Multiple sine functions

We recall the theory of the multiple sine function for our use in this paper. Let

$$\begin{aligned} S_r(x,(\omega_1,\dots,\omega_r)) &= \prod_{\mathbf{n}\geq\mathbf{0}}(\mathbf{n}\cdot\boldsymbol{\omega}+x)\left(\prod_{\mathbf{m}\geq\mathbf{1}}(\mathbf{m}\cdot\boldsymbol{\omega}-x)\right)^{(-1)^{r-1}} \\ &= \Gamma_r(x,\boldsymbol{\omega})^{-1}\Gamma_r(|\boldsymbol{\omega}|-x,\boldsymbol{\omega})^{(-1)^r} \end{aligned}$$

be the multiple sine function, where

$$\Gamma_r(x,\boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s,x,\boldsymbol{\omega})\bigg|_{s=0}\right)$$

is the normalized multiple gamma function obtained from the multiple Hurwitz zeta function

$$\zeta_r(s,x,\boldsymbol{\omega}) = \sum_{\mathbf{n}\geq\mathbf{0}}(\mathbf{n}\cdot\boldsymbol{\omega}+x)^{-s}.$$

Here we use notation $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r)$ and $|\boldsymbol{\omega}| = \omega_1+\cdots+\omega_r$. It is obvious that $S_r(x,(\omega_1,\dots,\omega_r))$ is symmetric with respect to $\omega_1,\dots,\omega_r$.

We need the following properties proved in [KK1].

Proposition 1.

(1) (*periodicity*)

$$S_r(x+\omega_i,\boldsymbol{\omega}) = S_r(x,\boldsymbol{\omega})S_{r-1}(x,\boldsymbol{\omega}(i))^{-1},$$

where $\boldsymbol{\omega}(i) = (\omega_1,\dots,\omega_{i-1},\omega_{i+1},\dots,\omega_r)$.

(2) (*homogeneity*)

$$S_r(cx,c\boldsymbol{\omega}) = S_r(x,\boldsymbol{\omega})$$

for $c>0$.

(3) (*differential equation*)

$$\frac{S_r'}{S_r}(x,(1,\dots,1)) = (-1)^{r-1}\pi\binom{x-1}{r-1}\cot(\pi x).$$

(4)

$$S_1(x,\omega) = 2\sin\left(\frac{\pi x}{\omega}\right).$$

(5)

$$S_2(\omega_1,(\omega_1,\omega_2)) = \sqrt{\frac{\omega_2}{\omega_1}},$$

$$S_2(\omega_2,(\omega_1,\omega_2)) = \sqrt{\frac{\omega_1}{\omega_2}}.$$

We notice that these properties are at first proved for $\omega_1,\dots,\omega_r>0$ (or $\mathrm{Re}(\omega_1)>0,\dots,\mathrm{Re}(\omega_r)>0$), but the same proofs are applicable to the general case where $\omega_1,\dots\omega_r\in\mathbf{C}$ are belonging to one side with respect to a line crossing the origin 0.

3. Proof of Theorem 4

We use the following result.

Proposition 2.

(1) $S_2''(0,(\omega_1,\omega_2)) = \dfrac{4\pi}{\omega_2}S_2'(\omega_1,(\omega_1,\omega_2))$.

(2) *For an integer* $N\geq 1$

$$S_2(x,(1,N)) = \prod_{k=0}^{N-1} S_2\left(\frac{x+k}{N},(1,1)\right).$$

Proof. We refer (1) to [K4] and (2) to [KK3]. For convenience we sketch the proof here. To see (1) we start from the periodicity (Proposition 1(1))

$$S_2(x,(\omega_1,\omega_2)) = S_2(x+\omega_1,(\omega_1,\omega_2))S_1(x,\omega_2).$$

Then the differentiation gives

$$S_2''(0,(\omega_1,\omega_2)) = S_2''(\omega_1,(\omega_1,\omega_2))S_1(0,\omega_2) + 2S_2'(\omega_1,(\omega_1,\omega_2))S_1'(0,\omega_2) + S_2(\omega_1,(\omega_1,\omega_2))S_1''(0,\omega_2).$$

Hence, using Proposition 1(4)

$$S_1(x,\omega_2) = 2\sin\left(\frac{\pi x}{\omega_2}\right)$$

we have

$$S_2''(0,(\omega_1,\omega_2)) = \frac{4\pi}{\omega_2}S_2'(\omega_1,(\omega_1,\omega_2)).$$

This proves (1). Concerning (2) we notice that

$$\begin{aligned}
\zeta_2(s,x,(1,N)) &= \sum_{n,m\geq 0}(n+mN+x)^{-s} \\
&= \sum_{k=0}^{N-1}\sum_{l,m\geq 0}((lN+k)+mN+x)^{-s} \\
&= N^{-s}\sum_{k=0}^{N-1}\sum_{l,m\geq 0}\left((l+m)+\frac{x+k}{N}\right)^{-s} \\
&= N^{-s}\sum_{k=0}^{N-1}\zeta_2\left(s,\frac{x+k}{N},(1,1)\right).
\end{aligned}$$

Hence, by differentiating with respect to s we have

$$\begin{aligned}
\zeta_2'(0,x,(1,N)) &= \sum_{k=0}^{N-1}\zeta_2'\left(0,\frac{x+k}{N},(1,1)\right) - (\log N)\sum_{k=0}^{N-1}\zeta_2\left(0,\frac{x+k}{N},(1,1)\right) \\
&= \sum_{k=0}^{N-1}\zeta_2'\left(0,\frac{x+k}{N},(1,1)\right) - (\log N)\zeta_2(0,x,(1,N)).
\end{aligned}$$

This gives

$$\Gamma_2(x,(1,N)) = \prod_{k=0}^{N-1}\Gamma_2\left(\frac{x+k}{N},(1,1)\right)N^{-\zeta_2(0,x,(1,N))}$$

and

$$\begin{aligned}
\Gamma_2(N+1-x,(1,N)) &= \prod_{k=0}^{N-1}\Gamma_2\left(\frac{N+1-x+k}{N},(1,1)\right)N^{-\zeta_2(0,N+1-x,(1,N))} \\
&= \prod_{k=0}^{N-1}\Gamma_2\left(2-\frac{x+k}{N},(1,1)\right)N^{-\zeta_2(0,N+1-x,(1,N))};
\end{aligned}$$

in the latter product the correspondence $k \leftrightarrow N-1-k$ is used. Thus

$$S_2(x,(1,N)) = \prod_{k=0}^{N-1} S_2\left(\frac{x+k}{N},(1,1)\right) N^{\zeta_2(0,x,(1,N))-\zeta_2(0,N+1-x,(1,N))}.$$

Hence the vanishing

$$\zeta_2(0,x,(1,N)) - \zeta_2(0,N+1-x,(1,N)) = 0$$

(see [KK1, Theorem 2.1 and the proof]) implies (2). This proves Proposition 2. ∎

Now we prove Theorem 4. From Proposition 2 we see that

$$S_2''(0,(1,N)) = \frac{4\pi}{N} S_2'(1,(1,N))$$

and

$$\frac{S_2'}{S_2}(x,(1,N)) = \frac{1}{N}\sum_{k=0}^{N-1}\frac{S_2'}{S_2}\left(\frac{x+k}{N},(1,1)\right).$$

Hence we have

$$\begin{aligned} S_2''(0,(1,N)) &= \frac{4\pi}{N} S_2(1,(1,N)) \cdot \frac{1}{N}\sum_{k=0}^{N-1}\frac{S_2'}{S_2}\left(\frac{k+1}{N},(1,1)\right) \\ &= \frac{4\pi}{N^{3/2}}\sum_{k=1}^{N}\frac{S_2'}{S_2}\left(\frac{k}{N},(1,1)\right) \end{aligned}$$

from

$$S_2(1,(1,N)) = \sqrt{N}$$

contained in Proposition 1(5). Then using the differential equation Proposition 1(3) for $S_2(x,(1,1))$ we obtain

$$\begin{aligned} S_2''(0,(1,N)) &= \frac{4\pi}{N^{3/2}} \cdot \pi \sum_{k=1}^{N}\left(1-\frac{k}{N}\right)\cot\left(\frac{\pi k}{N}\right) \\ &= \frac{4\pi^2}{N^{3/2}}\left(-\frac{1}{\pi} - \sum_{k=1}^{N-1}\frac{k}{N}\cot\left(\frac{\pi k}{N}\right)\right) \\ &= \frac{4\pi^2}{N^{3/2}}\left(-\frac{1}{\pi} + \sum_{k=1}^{[N/2]}\frac{N-2k}{N}\cot\left(\frac{\pi k}{N}\right)\right). \end{aligned}$$

Thus we have Theorem 4.

4. Proof of Theorem 2

This is contained in Theorem 3(1) of [K4]. We sketch the proof. Exactly as in the proof of Proposition 2(1) above (after interchanging under $\omega_1 \leftrightarrow \omega_2$) we have

$$\begin{aligned} S_2''(0,(1,\tau)) &= 4\pi S_2'(\tau,(1,\tau)) \\ &= \frac{4\pi}{\sqrt{\tau}} \cdot \frac{S_2'}{S_2}(\tau,(1,\tau)) \end{aligned}$$

since

$$S_2(\tau,(1,\tau)) = \frac{1}{\sqrt{\tau}}.$$

Then, using Shintani's product expression (Proposition 5 of [S1])

$$S_2(x,(1,\tau)) = \frac{\prod_{n=0}^{\infty}(1-e^{2\pi ix}e^{2\pi in\tau})}{\prod_{n=1}^{\infty}(1-e^{2\pi ix/\tau}e^{-2\pi in/\tau})}\exp\left(\frac{\pi i}{2}\left(\frac{x^2}{\tau}-\left(1+\frac{1}{\tau}\right)x+\frac{1}{6}(\tau+1)+\frac{1}{2}\right)\right)$$

we have

$$\begin{aligned} \frac{S_2'}{S_2}(\tau,(1,\tau)) &= \frac{\pi i}{2}\left(1-\frac{1}{\tau}\right) - 2\pi i\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} e^{2\pi inm\tau}e^{2\pi im\tau} + \frac{2\pi i}{\tau}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{-2\pi inm/\tau} \\ &= \frac{\pi i}{2}\left(1-\frac{1}{\tau}\right) - 2\pi i\left(E_1(\tau)+\frac{1}{4}\right) + \frac{2\pi i}{\tau}\left(E_1\left(-\frac{1}{\tau}\right)+\frac{1}{4}\right) \\ &= \frac{2\pi i}{\tau}R_1(\tau). \end{aligned}$$

Thus

$$S_2''(0,(1,\tau)) = \frac{8\pi^2 i}{\tau^{3/2}}R_1(\tau). \qquad \blacksquare$$

5. Proofs of Theorems 3 and 1

From the continuity of $S_2''(0,(1,\omega))$ in $\omega \in \mathbf{C}\backslash(-\infty,0]$, Theorem 3 follows from Theorem 2. Consequently, Theorems 3 and 4 give (1) of Theorem 1. To see (2) of Theorem 1 it is sufficient to use the relation

$$S_2''\left(0,\left(1,\frac{1}{N}\right)\right) = N^2 S_2''(0,(1,N)).$$

In fact, combining with Theorem 3, we have

$$R_1\left(\frac{1}{N}\right) = \frac{N^{-3/2}}{8\pi^2 i} S_2''\left(0, \left(1, \frac{1}{N}\right)\right)$$
$$= \frac{\sqrt{N}}{8\pi^2 i} S_2''(0, (1, N))$$
$$= \frac{1}{N} R_1(N),$$

where we simply write $R_1(\alpha)$ for $\lim_{\tau\to\alpha} R_1(\tau)$. Lastly, the needed relation for S_2'' is a special case of a bit general relation

$$S_r^{(k)}(0, c\boldsymbol{\omega}) = c^{-k} S_r^{(k)}(0, \boldsymbol{\omega}),$$

which in turn is obtained from differentiating k-times the homogeneity identity

$$S_r(cx, c\boldsymbol{\omega}) = S_r(x, \boldsymbol{\omega}).$$

For our case, put $r = 2$, $k = 2$, $(\omega_1, \omega_2) = (1, N)$, and $c = 1/N$. This completes the proof of Theorem 1. ■

6. Stirling modular functions

We add a brief explanation of our calculation from the view point of Stirling modular functions. Stirling modular functions were first suggested by Barnes [B]. His example is

$$\rho_r(\omega_1, \dots, \omega_r) = \prod_{n_1,\dots,n_r\geq 0}{}' (n_1\omega_1 + \cdots + n_r\omega_r).$$

For example

$$\rho_1(\omega) = \prod_{n=1}^{\infty}(n\omega) = \sqrt{\frac{2\pi}{\omega}}.$$

We notice the name "Stirling" comes from the famous Stirling's formula

$$\sqrt{2\pi} = \lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}}$$

since the regularized product

$$\sqrt{2\pi} = \prod_{n=1}^{\infty} n$$

explains the constant term via the Euler-Maclaurin summation formula; see Hardy [H] for details. The idea of the Stirling modular function is to consider a function of the semi-lattice $\mathbf{Z}_{\geq 0}\boldsymbol{\omega} = \mathbf{Z}_{\geq 0}\omega_1 + \cdots + \mathbf{Z}_{\geq 0}\omega_r$ instead of the lattice $\mathbf{Z}\boldsymbol{\omega} = \mathbf{Z}\omega_1 + \cdots + \mathbf{Z}\omega_r$ used in the usual theory of modular functions.

We put

$$\mathbf{C}_+^r = \left\{ (\omega_1, \dots, \omega_r) \in \mathbf{C}^r \,\middle|\, \begin{array}{l} \omega_1, \dots, \omega_r \text{ are belonging to one side} \\ \text{with respect to a line crossing } 0 \end{array} \right\}$$

We say that a function

$$f : \mathbf{R}_{>0}^r \to \mathbf{C}$$

or

$$f : \mathbf{C}_+^r \to \mathbf{C}$$

is a Stirling modular function if it satisfies the following two conditions:

(1) f is *symmetric*, i.e.

$$f(\omega_{\sigma(1)}, \dots, \omega_{\sigma(r)}) = f(\omega_1, \dots, \omega_r)$$

for all $\sigma \in S_r$,

(2) f is *homogeneous*, i.e.

$$f(c\omega_r, \dots, c\omega_r) = f(\omega_1, \dots, \omega_r)$$

for all $c > 0$ (or $c \in \mathbf{C} \setminus \{0\}$).

There exist many interesting examples of (real analytic) Stirling modular functions associated to multiple sine functions. We notice two of them:

(A) For an integer $N \geq 2$, the N-division value

$$H_{r,N}(\omega_1, \dots, \omega_r) = S_r\left(\frac{\omega_1 + \cdots + \omega_r}{N}, (\omega_1, \dots, \omega_r)\right)$$

is a Stirling modular function.

(B) Let

$$S_r(x+y, \boldsymbol{\omega}) = S_r(x, \boldsymbol{\omega}) + S_r(y, \boldsymbol{\omega}) + c_{11}(\boldsymbol{\omega}) S_r(x, \boldsymbol{\omega}) S_r(y, \boldsymbol{\omega}) + \sum_{\substack{i,j \geq 1 \\ i+j \geq 3}} c_{ij}(\boldsymbol{\omega}) S_r(x, \boldsymbol{\omega})^i S_r(y, \boldsymbol{\omega})^j$$

be a local addition relation around $x = y = 0$. For example

$$c_{11}(\boldsymbol{\omega}) = \frac{S_r''(0, \boldsymbol{\omega})}{S_r'(0, \boldsymbol{\omega})^2}$$

and

$$c_{12}(\boldsymbol{\omega}) = c_{21}(\boldsymbol{\omega}) = \frac{S_r'''(0, \boldsymbol{\omega}) S_r'(0, \boldsymbol{\omega}) - S_r''(0, \boldsymbol{\omega})^2}{2S_r'(0, \boldsymbol{\omega})^4}.$$

Then, the coefficients $c_{ij}(\omega_1, \dots, \omega_r)$ are Stirling modular functions.

Example (A) is important for Kronecker's Jugendtraum as indicated by Shintani [S1] in case of $r = 2$ leading to interesting algebraic values. We notice an example of special value in $r = 3$:

$$H_{3,2}(1,1,1) = S_3\left(\frac{3}{2},(1,1,1)\right) = 2^{-1/8}\exp\left(-\frac{3\zeta(3)}{16\pi^2}\right).$$

Example (B) is directly related to the theme of this paper. From this view point, our result is expressed as the calculation of the special value

$$c_{11}(1,N) = c_{11}\left(1,\frac{1}{N}\right) = \frac{1}{\sqrt{N}}\left(\frac{1}{N}\sum_{k=1}^{[N/2]}(N-2k)\cot\left(\frac{k\pi}{N}\right) - \frac{1}{\pi}\right)$$

for an integer $N \geq 1$. ∎

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