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CONVERGENCE AND SUMMABILITY IN THE MEAN OF RANDOM FOURIER-STIELTJES SERIES

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Abstract

We show that the random Fourier-Stieltjes (RFS) series associated with a stochastic process of independent and symmetric increments whose laws belong to the domain of symmetric stable distribution converges in the mean to a stochastic integral. We also show that the conjugate RFS series converges in the mean to a stochastic integral. Both the series are also shown to be Abel summable.

1. Introduction

Samal [6] considered the question of the convergence of the "integrated" RFS series $\sum_{n=-\infty}^{\infty} \frac{A_n}{n} e^{2\pi int}$, where $A_n = \int_0^1 e^{-2\pi int}\,dX(t)$ and $X(t)$ is a continuous stochastic process with independent increments. Nayak, Pattanayak and Mishra [3] considered the RFS series $\sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi int}$ where $A_n = \int_0^1 e^{-2\pi int}\,dX(t)$ and $a_n = \int_0^1 f(t) e^{-2\pi int}\,dt$ for $f \in L^\alpha[0,1]$, $X(t)$ a stochastic process with independent increments which are symmetric stable random variables with index $\alpha \in (1,2]$. In this work they showed that it is possible to define stochastic integral $\int_0^1 g(t)\,dX(t)$ for $g \in L^\alpha[0,1]$, $X(t)$ a stable process of index α, in the sense of convergence in probability. They also showed that the RFS series $\sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi int}$ converges in probability to

$$\int_0^1 f(t-u)\,dX(u) \quad \text{for } f \in L^\alpha[0,1].$$

The proof depends, crucially, on the fact that $\lim_{n\to\infty} \int_0^1 |s_n(t) - f(t)|^\alpha\,dt = 0$ for

$$f \in L^\alpha[0,1], \quad a_n = \int_0^1 f(t) e^{-2\pi int}\,dt \quad \text{and} \quad s_n(t) = \sum_{k=-n}^{n} a_k e^{ikt}.$$

Pattanayak and Sahoo [4] were able to prove that this series in fact converges in the sense of mean. Pattanayak and Sharma [5] studied the question of convergence in the sense of probability of a RFS series associated with the

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stochastic process of independent and symmetric increments whose laws belong to the domain of symmetric stable distribution.

DEFINITION 1.1. $L_\phi([a,b])$ is the collection of measurable functions f, satisfying

$$\int_a^b \phi|f(t)|\,dt < \infty$$

for non-negative convex function $\phi(u)$ with $u \geq 0$.

By using similar techniques of Dash and Pattanayak [2] we can easily show that for $f \in L_\phi \cap L^\alpha$ the stochastic integral $\int_a^b f(t)\,dX(t)$ is well defined in the sense of **mean** for a stochastic process X of independent and symmetric increments whose laws belong to the domain of symmetric stable distribution.

We in this note are able to show the convergence of the RFS series and conjugate RFS series in the sense of **mean** and show that they are also Abel summable.

2. Our main results

THEOREM 2.1. *Let* $\phi(t) = |t|^\alpha h(t)$, *where* h *is a slowly varying function satisfying*

(i) $h(t) = O(t^\delta)$ *as* $t \to 0$ *for all* $\delta > 0$ *and*

(ii) $h(uv) \leq h(u) + h(v)$ *for* $u, v \geq 0$.

Let $X(t)$ *be a stochastic process of index* α, $1 < \alpha \leq 2$ *with independent and symmetric increments whose laws belong to the domain of symmetric stable distribution such that the increment* $X(t_1) - X(t_2)$ *has characteristic function* $e^{-|t_1-t_2|\phi(t)}$. *For* $f \in L_\phi \cap L^\alpha$, *let*

$$a_n = \int_0^1 e^{-2\pi int} f(t)\,dt \quad \text{and} \quad A_n = \int_0^1 e^{-2\pi int}\,dX(t),$$

then the RFS series

$$\sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi int} \tag{1}$$

converges in the sense of **mean** *to the stochastic integral* $\int_0^1 f(t-u)\,dX(u)$.

THEOREM 2.2. *If* $f \in L_\phi \cap L^\alpha$ *with*

$$a_n = \int_0^1 e^{-2\pi int} f(t)\,dt \quad \text{and} \quad A_n = \int_0^1 e^{-2\pi int}\,dX(t),$$

then the conjugate RFS series

$$\sum_{n=-\infty}^{\infty} \tilde{a}_n A_n e^{2\pi int} \tag{2}$$

converges in the sense of ***mean*** *to the stochastic integral* $\int_0^1 \tilde{f}(t-u)\,dX(u)$, *where* $\tilde{a}_n = -i\,\mathrm{sgn}(n)a_n$.

THEOREM 2.3. *Let* $X(t)$ *be a stochastic process of index* α, $1 < \alpha \leq 2$ *with independent and symmetric increments whose laws belong to the domain of symmetric stable distribution. If* $A_n = \int_0^1 e^{-2\pi int}\,dX(t)$ *and* $a_n = \int_0^1 e^{-2\pi int} f(t)\,dt$, *then the series* $\sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi int}$ *is Abel summable to* $\int_0^1 f(t-u)\,dX(u)$ *in the sense of* ***mean***.

To prove these results we need the following results:

LEMMA 2.4 (cf. Chow and Teicher [1], p. 285)
For a random variable X *with characteristic function* ψ *the absolute moment of the random variable* X *is given by*

$$E|X| = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1 - \mathrm{Re}\{\psi(t)\}}{t^2}\,dt$$

LEMMA 2.5 (Theorem of M. Riesz) (cf. Zygmund [7], Vol. I, p. 253)
If $f \in L^p$, $1 < p < \infty$ *and has the Fourier series* $\sum_{n=-\infty}^{\infty} a_n e^{2\pi int}$ *then the conjugate series* $\sum_{n=-\infty}^{\infty} \tilde{a}_n e^{2\pi int}$ *is also a Fourier series of a function* $\tilde{f} \in L^p$ *and for* $\tilde{s}_n(t) = \sum_{k=-n}^{n} \tilde{a}_k e^{2\pi ikt}$,

$$\lim_{n\to\infty} \int_0^1 |\tilde{s}_n(t) - \tilde{f}(t)|^p\,dt = 0.$$

Proof of Theorem 2.1
Let $S_n(t)$ be the n-th partial sum of the series $\sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi int}$ that is

$$S_n(t) = \sum_{k=-n}^{n} a_k A_k e^{2\pi ikt} \quad \text{and} \quad s_n(t) = \sum_{k=-n}^{n} a_k e^{2\pi ikt}.$$

We have

$$\begin{aligned} S_n(t) - S_m(t) &= \int_0^1 \{s_n(t-u) - s_m(t-u)\}\,dX(u) \\ &= \int_0^1 s_{n,m}(t-u)\,dX(u) \quad \text{(say)} \end{aligned}$$

By the Lemma 2.4, we have:

$$\begin{aligned} E|S_n(t) - S_m(t)| &= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1 - \exp(-\int_0^1 \phi(\xi s_{n,m}(t-u))\,du)}{\xi^2}\,d\xi \\ &= \frac{2}{\pi}\int_{|\xi|\le 1} \frac{1 - \exp(-\int_0^1 \phi(\xi s_{n,m}(t-u))\,du)}{\xi^2}\,d\xi \\ &\quad + \frac{2}{\pi}\int_{|\xi|>1} \frac{1 - \exp(-\int_0^1 \phi(\xi s_{n,m}(t-u))\,du)}{\xi^2}\,d\xi \\ &= I_1 + I_2 \quad \text{(say)} \end{aligned}$$

where

$$I_1 = \frac{2}{\pi}\int_{|\xi|\le 1} \frac{1 - \exp(-\int_0^1 \phi(\xi s_{n,m}(t-u))\,du)}{\xi^2}\,d\xi$$

and

$$I_2 = \frac{2}{\pi}\int_{|\xi|>1} \frac{1 - \exp(-\int_0^1 \phi(\xi s_{n,m}(t-u))\,du)}{\xi^2}\,d\xi.$$

It is easy to see that

$$\begin{aligned} |I_1| &\le \frac{4}{\pi}\int_0^1 \frac{\int_0^1 \phi(\xi s_{n,m}(t-u))\,du}{\xi^2}\,d\xi \\ &= \frac{4}{\pi}\int_0^1 \left(\xi^{\alpha-2}\int_0^1 |s_{n,m}(t-u)|^{\alpha} h(\xi s_{n,m}(t-u))\,du\right) d\xi. \end{aligned}$$

Now by the statement of the theorem, h is a slowly varying function satisfying (i) and (ii). Therefore for every $\delta > 0$, we have $h(u)u^{-\delta} \le 1$ for sufficiently small u. Hence for every $\delta > 0$, we can get a constant K_δ such that

$$h(u)u^{-\delta} \le K_\delta \quad \text{for } 0 \le u \le 1. \tag{3}$$

So

$$\begin{aligned} h(\xi s_{n,m}(t-u)) &\le h(\xi) + h(s_{n,m}(t-u)) \\ &\le K_\delta \xi^\delta + h(s_{n,m}(t-u)) \quad \text{(by (3) and for } 0 \le \xi \le 1). \end{aligned}$$

Therefore

$$\begin{aligned} |I_1| &\le \frac{4}{\pi}\int_0^1 \left(\xi^{\alpha-2}\int_0^1 |s_{n,m}(t-u)|^{\alpha} h(\xi s_{n,m}(t-u))\,du\right) d\xi \\ &\le \frac{4}{\pi}\int_0^1 \left(\xi^{\alpha-2}\int_0^1 |s_{n,m}(t-u)|^{\alpha}[K_\delta \xi^\delta + h(s_{n,m}(t-u))]\,du\right) d\xi \end{aligned}$$

$$= \frac{4}{\pi}\int_0^1 \left(K_\delta \xi^{\alpha+\delta-2} \int_0^1 |s_{n,m}(t-u)|^\alpha \, du \right) d\xi$$

$$+ \frac{4}{\pi}\int_0^1 \left(\xi^{\alpha-2} \int_0^1 |s_{n,m}(t-u)|^\alpha h(s_{n,m}(t-u)) \, du \right) d\xi$$

$$= \frac{4}{\pi}\left(\frac{K_\delta}{\alpha+\delta-1} \right) \int_0^1 |s_{n,m}(t-u)|^\alpha \, du + \frac{1}{\alpha-1}\int_0^1 |s_{n,m}(t-u)|^\alpha h(s_{n,m}(t-u)) \, du.$$

It is now seen that

$$\lim_{m,n\to\infty} |I_1| = 0.$$

Now for I_2, we observe that

$$\frac{1-\exp(-\int_0^1 \phi(\xi\{s_n(t-u) - s_m(t-u)\}) \, du)}{\xi^2} \le \frac{1}{\xi^2} \quad \text{and} \quad \int_{|\xi|>1} \frac{1}{\xi^2} \, d\xi < \infty.$$

So by Lebesgue dominated convergence theorem, we have:

$$\lim_{m,n\to\infty} |I_2| = 0.$$

Hence

$$\lim_{m,n\to\infty} E|S_n(t) - S_m(t)| = 0.$$

That is $S_n(t)$ converges in the sense of mean. It is not hard to show that $S_n(t)$ converges to the stochastic integral $\int_0^1 f(t-u) \, dX(u)$. □

Proof of Theorem 2.2

Let $\tilde{S}_n$ be the n-th partial sum of the conjugate RFS series $\sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi i n t}$. That is

$$\tilde{S}_n(t) = \sum_{k=-n}^{n} \tilde{a}_k A_k e^{2\pi i k t} \quad \text{and} \quad \tilde{s}_n(t) = \sum_{k=-n}^{n} \tilde{a}_k e^{2\pi i k t}.$$

We see that:

$$E|\tilde{S}_n(t) - \tilde{S}_m(t)| = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1-\exp(-\int_0^1 \phi(\xi\{\tilde{s}_n(t-u) - \tilde{s}_m(t-u)\}) \, du)}{\xi^2} \, d\xi$$

$$= \frac{2}{\pi}\int_{|\xi|\le 1} \frac{1-\exp(-\int_0^1 \phi(\xi\{\tilde{s}_n(t-u) - \tilde{s}_m(t-u)\}) \, du)}{\xi^2} \, d\xi$$

$$+ \frac{2}{\pi}\int_{|\xi|>1} \frac{1-\exp(-\int_0^1 \phi(\xi\{\tilde{s}_n(t-u) - \tilde{s}_m(t-u)\}) \, du)}{\xi^2} \, d\xi.$$

Using the techniques used in the Theorem 2.1 and by Lemma 2.5 it is easy to see that

$$\lim_{m,n\to\infty} E|\tilde{S}_n(t) - \tilde{S}_m(t)| = 0.$$

So $\tilde{S}_n(t)$ converges in the sense of mean. It is not hard to show that $\tilde{S}_n(t)$ converges to the stochastic integral $\int_0^1 \tilde{f}(t-u)\, dX(u)$. □

Proof of Theorem 2.3

We know that for $0 \le r < 1$ the series $\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{2\pi int}$ converges uniformly to a function $f \in L_\phi \cap L^p$, $p \ge \alpha$. Let us write

$$f_r(t) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{2\pi int}, \quad 0 \le r < 1.$$

Since $f \in L_\phi \cap L^p$ so $f_r \in L^p$, $p \ge \alpha$. By Theorem 2.1 the series $\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{2\pi int}$ converges to the stochastic integral $\int_0^1 f_r(t-u)\, dX(u)$ in the sense of mean. Now

$$\begin{aligned} &E\left|\int_0^1 f_r(t-u)\, dX(u) - \int_0^1 f(t-u)\, dX(u)\right| \\ &= E\left|\int_0^1 \{f_r(t-u) - f(t-u)\}\, dX(u)\right| \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1 - \exp(-\int_0^1 \phi(\xi\{f_r(t-u) - f(t-u)\})\, du)}{\xi^2}\, d\xi. \end{aligned}$$

Because $\lim_{r\to 1^-} \int_0^1 |f_r(t-u) - f(t-u)|^p\, du = 0$ for every $p \ge \alpha$, so we get

$$\lim_{r\to 1^-} E\left|\int_0^1 \{f_r(t-u) - f(t-u)\}\, dX(u)\right| = 0$$

by the same arguments used in Theorem 2.1. Therefore the RFS series $\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{2\pi int}$ is Abel summable to $\int_0^1 f(t-u)\, dX(u)$ in the sense of mean. □

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