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# THE CENTRAL VALUE OF THE TRIPLE SINE FUNCTION

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**Abstract**

We study the central value of the triple sine function for a general period. We give an explicit integral expression and an inequality. As an application we obtain an expression for $\zeta(3)$.

## 1. Introduction

The triple sine function

$$S_3(x,(\omega_1,\omega_2,\omega_3)) = \prod_{n_1,n_2,n_3\geq 0}(n_1\omega_1+n_2\omega_2+n_3\omega_3+x) \times \prod_{m_1,m_2,m_3\geq 1}(m_1\omega_1+m_2\omega_2+m_3\omega_3-x)$$

constructed and studied in our previous papers [K] [KK] (cf. Manin [M]) is a generalization of the usual sine function

$$S_1(x,\omega) = \prod_{n\geq 0}(n\omega+x)\prod_{m\geq 1}(m\omega-x) = 2\sin\left(\frac{\pi x}{\omega}\right),$$

where we use the regularized product notation $\prod$ due to Deninger [D]:

$$\prod_{\lambda}\lambda = \exp\left(-\frac{\partial}{\partial s}\sum_{\lambda}\lambda^{-s}\Big|_{s=0}\right).$$

As is well-known, $S_1(x,\omega)$ is invariant under $x\leftrightarrow\omega-x$, and the central value of $S_1(x,\omega)$ is the simple value $S_1\left(\frac{\omega}{2},\omega\right)=2$. Similarly, the function $S_3(x,(\omega_1,\omega_2,\omega_3))$ has the symmetry $x\leftrightarrow\omega_1+\omega_2+\omega_3-x$, so the central value

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is $S_3\left(\frac{\omega_1+\omega_2+\omega_3}{2},(\omega_1,\omega_2,\omega_3)\right)$. This value is quite mysterious as seen from the simplest case

$$S_3\left(\frac{3}{2},(1,1,1)\right)=2^{-1/8}\exp\left(-\frac{3\zeta(3)}{16\pi^2}\right),$$

where the zeta value $\zeta(3)$ appears; see [KK].

In this paper we investigate the central value $S_3\left(\frac{\omega_1+\omega_2+\omega_3}{2},(\omega_1,\omega_2,\omega_3)\right)$ for general $\omega_1,\omega_2,\omega_3>0$. The first result is the explicit expression:

Theorem 1.

$$\begin{aligned}&S_3\left(\frac{\omega_1+\omega_2+\omega_3}{2},(\omega_1,\omega_2,\omega_3)\right)\\&\quad=\exp\left(-\int_0^\infty\left(\frac{1}{4}\prod_{k=1}^{3}\left(\sinh\left(\frac{\sqrt{2}\omega_k t}{\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}}\right)\right)^{-1}\right.\right.\\&\qquad\left.\left.-\frac{(\omega_1^2+\omega_2^2+\omega_3^2)^{3/2}}{8\sqrt{2}\omega_1\omega_2\omega_3 t^3}\left(1-\frac{t^2}{3}\right)\right)\frac{dt}{t}\right).\end{aligned}$$

The second result is the following estimate.

Theorem 2.

$$0<S_3\left(\frac{\omega_1+\omega_2+\omega_3}{2},(\omega_1,\omega_2,\omega_3)\right)<1.$$

We obtain an application of Theorem 2:

Theorem 3.

$$\prod_{i=1}^{3}S_3\left(\frac{\omega_i}{2},(\omega_1,\omega_2,\omega_3)\right)\times\prod_{i<j}S_3\left(\frac{\omega_i+\omega_j}{2},(\omega_1,\omega_2,\omega_3)\right)>2.$$

Using Theorems 2 and 3 we see the behavior of the triple sine function $S_3(x,(\omega_1,\omega_2,\omega_3))$ in the fundamental domain $0\le x\le\omega_1+\omega_2+\omega_3$:

Theorem 4. *The graph of* $S_3(x,(\omega_1,\omega_2,\omega_3))$ *is as in Fig.* 1. *It is symmetric with respect to the line* $x=\frac{\omega_1+\omega_2+\omega_3}{2}$, *and it has three extremal values*: *two*

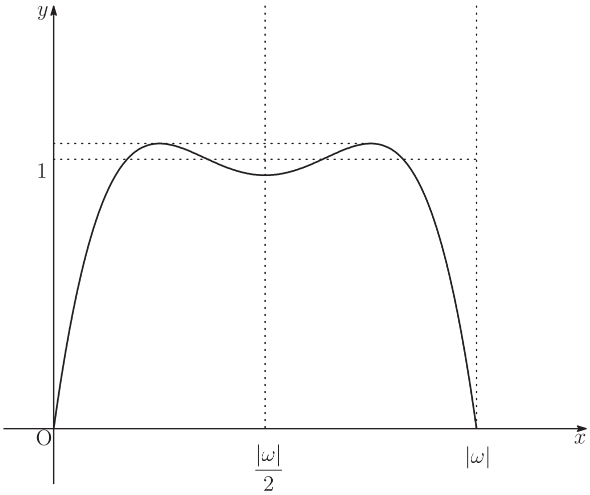


FIGURE 1. The graph of $S_3(x,(\omega_1,\omega_2,\omega_3))$.

*maximal values larger than* 1 *at two points and the local minimal less than* 1 *at* $x=\dfrac{\omega_1+\omega_2+\omega_3}{2}$.

We also obtain the following integral expression for $\zeta(3)$ from Theorem 1:

THEOREM 5.
$$\zeta(3)=\frac{16\pi^2}{3}\int_0^\infty\left(2(e^{\sqrt{2/3}t}-e^{-\sqrt{2/3}t})^{-3}+\frac{3}{16}\sqrt{\frac{2}{3}}\left(\frac{1}{t}-\frac{3}{t^3}\right)\right)\frac{dt}{t}-\frac{2}{3}\pi^2\log 2.$$

We remark that the formula
$$S_3\left(\frac{\omega_1+\omega_2+\omega_3}{2},(\omega_1,\omega_2,\omega_3)\right)=H(\omega_1,\omega_2,\omega_3)^2$$
with
$$H(\omega_1,\omega_2,\omega_3)=\prod_{n_1,n_2,n_3\geq 0}\left(\left(n_1+\frac{1}{2}\right)\omega_1+\left(n_2+\frac{1}{2}\right)\omega_2+\left(n_3+\frac{1}{2}\right)\omega_3\right)$$
reminds us the phenomenon that central values frequently become “squares” especially for zeta and $L$-functions. This is valid also for
$$S_1\left(\frac{\omega}{2},\omega\right)=H(\omega)^2$$
with

$$H(\omega) = \prod_{n=0}^{\infty}\left(\left(n+\frac{1}{2}\right)\omega\right) = \sqrt{2}.$$

Moreover, these $H(\omega_1, \omega_2, \omega_3)$ and $H(\omega)$ are considered as determinants of hamiltonians for harmonic oscillators in dimension 3 and 1 respectively. We refer to [KO] for studies from this viewpoint.

## 2. Integral expression: Proof of Theorem 1

We first recall needed facts on multiple Hurwitz zeta functions. The multiple Hurwitz zeta function $\zeta_r(s, x, (\omega_1, \dots, \omega_r))$ is defined (for $\omega_1, \dots, \omega_r > 0$ and $x > 0$) as

$$\zeta_r(s, x, (\omega_1, \dots, \omega_r)) = \sum_{n_1, \dots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}.$$

This converges absolutely in $\mathrm{Re}(s) > r$, and Barnes [B] shows that $\zeta_r(s, x, (\omega_1, \dots, \omega_r))$ has an analytic continuation to all $s \in \mathbf{C}$ as a meromorphic function. Moreover, it is holomorphic at $s = 0$. Hence we have the regularized product

$$\prod_{n_1, \dots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r + x) = \exp(-\zeta_r'(0, x, (\omega_1, \dots, \omega_r))),$$

where the differentiation concerns the first variable $s$. In particular, in our case, we have

$$\begin{aligned} S_3&\left(\frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) \\ &= \left(\prod_{n_1, \dots, n_r \geq 0}\left(\left(n_1+\frac{1}{2}\right)\omega_1 + \left(n_2+\frac{1}{2}\right)\omega_2 + \left(n_3+\frac{1}{2}\right)\omega_3\right)\right)^2 \\ &= \exp\left(-2\zeta_3'\left(0, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right)\right). \end{aligned}$$

Thus we must look at $\zeta_3(s, x, (\omega_1, \omega_2, \omega_3))$ around $s = 0$. We use the Riemann-Mellin integral expression for the zeta function. Here, we show the analytic continuation of $\zeta_3(s, x, (\omega_1, \omega_2, \omega_3))$ in $\mathrm{Re}(s) > -1$, which is sufficient for our purpose.

We start from the integral expression in $\mathrm{Re}(s) > 3$:

$$\begin{aligned} \zeta_3(s, x, (\omega_1, \omega_2, \omega_3)) &= \frac{1}{\Gamma(s)}\int_0^{\infty}\left(\sum_{n_1, n_2, n_3 \geq 0} e^{-(n_1\omega_1+n_2\omega_2+n_3\omega_3)t}\right)e^{-tx}t^{s-1}\,dt \\ &= \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{e^{-tx}t^{s-1}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})(1-e^{-\omega_3 t})}\,dt, \end{aligned}$$

which follows from the integral expression for the gamma function $\Gamma(s)$. Hence we have

$$\zeta_3\left(s, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) = \frac{1}{\Gamma(s)} \int_0^\infty \Theta(t, (\omega_1, \omega_2, \omega_3)) t^{s-1}\, dt$$

in $\mathrm{Re}(s) > 3$ with

$$\begin{aligned}\Theta(t, (\omega_1, \omega_2, \omega_3)) &= \frac{e^{-((\omega_1+\omega_2+\omega_3)/2)t}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})(1-e^{-\omega_3 t})} \\ &= \frac{1}{(e^{\omega_1 t/2}-e^{-\omega_1 t/2})(e^{\omega_2 t/2}-e^{-\omega_2 t/2})(e^{\omega_3 t/2}-e^{-\omega_3 t/2})} \\ &= \frac{1}{8}\prod_{k=1}^{3}\left(\sinh\left(\frac{\omega_k t}{2}\right)\right)^{-1}.\end{aligned}$$

We remark that $\Theta(t, (\omega_1, \omega_2, \omega_3))$ is an odd function of $t$ with the Laurent expansion

$$\Theta(t, (\omega_1, \omega_2, \omega_3)) = \frac{a_{-3}}{t^3} + \frac{a_{-1}}{t} + a_1 t + \cdots$$

around $t = 0$, where $a_j = a_j(\omega_1, \omega_2, \omega_3)$ is a rational function of $\omega_1$, $\omega_2$, $\omega_3$. In particular

$$\begin{aligned} a_{-3} &= \frac{1}{\omega_1\omega_2\omega_3}, \\ a_{-1} &= -\frac{\omega_1^2+\omega_2^2+\omega_3^2}{24\omega_1\omega_2\omega_3}. \end{aligned}$$

Now, the integral expression splits into three parts:

$$\begin{aligned}&\zeta_3\left(s, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) \\ &\quad= \frac{1}{\Gamma(s)} \int_1^\infty \Theta(t, (\omega_1, \omega_2, \omega_3)) t^{s-1}\, dt \\ &\qquad+ \frac{1}{\Gamma(s)} \int_0^1 \left(\Theta(t, (\omega_1, \omega_2, \omega_3)) - \frac{a_{-3}}{t^3} - \frac{a_{-1}}{t}\right) t^{s-1}\, dt \\ &\qquad+ \frac{1}{\Gamma(s)} \int_0^1 \left(\frac{a_{-3}}{t^3} + \frac{a_{-1}}{t}\right) t^{s-1}\, dt.\end{aligned}$$

Here, the first term is holomorphic for all $s \in \mathbf{C}$ since the integral converges absolutely. The second term is holomorphic in $\mathrm{Re}(s) > -1$ since

$$\Theta(t, (\omega_1, \omega_2, \omega_3)) - \frac{a_{-3}}{t^3} - \frac{a_{-1}}{t} = O(t)$$

as $t \to 0$. The third term is written as

$$\frac{1}{\Gamma(s)}\left(\frac{a_{-3}}{s-3}+\frac{a_{-1}}{s-1}\right)$$

and it is meromorphic in $s \in \mathbf{C}$ with possible (simple) poles at $s = 3, 1$ only. Thus we have shown the analytic continuation of $\zeta_3\left(s, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right)$ in $\mathrm{Re}(s) > -1$, and it is holomorphic at $s = 0$; in fact the above calculation implies that $\zeta_3\left(0, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) = 0$.

Hence, remarking that $\Gamma(s)^{-1}$ has a zero at $s = 0$ with $\frac{d}{ds}\Gamma(s)^{-1}\Big|_{s=0} = 1$, we see that

$$\begin{aligned}
&\zeta_3'\left(0, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) \\
&\quad = \int_1^\infty \Theta(t, (\omega_1, \omega_2, \omega_3))\frac{dt}{t} \\
&\qquad + \int_0^1 \left(\Theta(t, (\omega_1, \omega_2, \omega_3)) - \frac{a_{-3}}{t^3} - \frac{a_{-1}}{t}\right)\frac{dt}{t} - \frac{a_{-3}}{3} - a_{-1}.
\end{aligned}$$

Here we remark that

$$\zeta_3'\left(0, \frac{c\omega_1+c\omega_2+c\omega_3}{2}, (c\omega_1, c\omega_2, c\omega_3)\right) = \zeta_3'\left(0, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right)$$

for $c > 0$. This is seen as follows. The definition says that

$$\zeta_3\left(s, \frac{c\omega_1+c\omega_2+c\omega_3}{2}, (c\omega_1, c\omega_2, c\omega_3)\right) = c^{-s}\zeta_3\left(s, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right),$$

so we have

$$\begin{aligned}
&\zeta_3'\left(0, \frac{c\omega_1+c\omega_2+c\omega_3}{2}, (c\omega_1, c\omega_2, c\omega_3)\right) \\
&\quad = \zeta_3'\left(0, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) \\
&\qquad - (\log c)\zeta_3\left(0, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) \\
&\quad = \zeta_3'\left(0, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right).
\end{aligned}$$

from $\zeta_3\left(0, \frac{\omega_1+\omega_2+\omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) = 0$.

Take

$$c = \frac{2\sqrt{2}}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}},$$

and put

$$(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = (c\omega_1, c\omega_2, c\omega_3).$$

Then using $\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2 = 8$ we have

$$\begin{aligned}
&\zeta_3'\left(0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) \\
&\quad = \zeta_3'\left(0, \frac{\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3}{2}, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)\right) \\
&\quad = \int_1^\infty \Theta(t, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)) \frac{dt}{t} + \int_0^1 \left(\Theta(t, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)) - \frac{1}{\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3 t^3}\left(1 - \frac{t^2}{3}\right)\right)\frac{dt}{t}.
\end{aligned}$$

Thus we see that

$$\begin{aligned}
&\zeta_3'\left(0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) \\
&\quad = \int_0^\infty \left(\Theta(t, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)) - \frac{1}{\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3 t^3}\left(1 - \frac{t^2}{3}\right)\right)\frac{dt}{t}
\end{aligned}$$

since

$$\int_1^\infty \frac{1}{t^3}\left(1 - \frac{t^2}{3}\right)\frac{dt}{t} = 0.$$

This proves Theorem 1. ∎

### 3. Estimates: Proof of Theorem 2

Let

$$\tilde{\omega}_k = \frac{2\sqrt{2}\omega_k}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}$$

as in the proof of Theorem 1. To prove Theorem 2 it is sufficient to show that

$$\int_0^1 \left(\prod_{k=1}^3 (e^{\tilde{\omega}_k t/2} - e^{-\tilde{\omega}_k t/2})^{-1} - \frac{1}{\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3 t^3}\left(1 - \frac{t^2}{3}\right)\right)\frac{dt}{t} > 0$$

since

$$\int_1^\infty \left( \prod_{k=1}^{3} (e^{\tilde{\omega}_k t/2} - e^{-\tilde{\omega}_k t/2})^{-1} - \frac{1}{\tilde{\omega}_1 \tilde{\omega}_2 \tilde{\omega}_3 t^3} \left(1 - \frac{t^2}{3}\right) \right) \frac{dt}{t}$$

$$= \int_1^\infty \prod_{k=1}^{3} (e^{\tilde{\omega}_k t/2} - e^{-\tilde{\omega}_k t/2})^{-1} \frac{dt}{t} > 0.$$

Now, we prove the inequality

$$(*) \qquad \prod_{k=1}^{3} (e^{\tilde{\omega}_k t/2} - e^{-\tilde{\omega}_k t/2})^{-1} > \frac{1}{\tilde{\omega}_1 \tilde{\omega}_2 \tilde{\omega}_3 t^3} \left(1 - \frac{t^2}{3}\right)$$

for $0 < t \leq 1$. First we show the following two inequalities:

$$(1) \qquad (e^{\omega t/2} - e^{-\omega t/2})^{-1} \geq \frac{1}{\omega t} \left(1 - \frac{\omega^2 t^2}{24}\right)$$

for $0 < \omega < 2\sqrt{2}$ and $0 < t \leq 1$.

$$(2) \qquad (1 - au)(1 - bu)(1 - cu) > 1 - u$$

for $a, b, c > 0$ with $a + b + c = 1$ and $0 < u < 1$.

*Proof of* (1). Taylor expansion shows that

$$\begin{aligned} e^{\omega t/2} - e^{-\omega t/2} &= \omega t \sum_{n=0}^{\infty} \frac{\omega^{2n}}{(2n+1)! 2^{2n}} t^{2n} \\ &\leq \omega t \sum_{n=0}^{\infty} \left(\frac{\omega^2}{24}\right)^n t^{2n} \\ &= \frac{\omega t}{1 - \dfrac{\omega^2 t^2}{24}}, \end{aligned}$$

where we used the easy fact

$$(2n+1)! \geq 6^n$$

for $n = 0, 1, 2, \ldots$.

*Proof of* (2). Since

$$(1 - au)(1 - bu)(1 - cu) = 1 - u + abcu^2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - u\right),$$

it is sufficient to check that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 1.$$

Actually, the stronger inequality

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$$

follows from the famous inequality

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

with $a+b+c=1$.

*Proof of* $(*)$. By using (1) we have

$$\prod_{k=1}^{3}(e^{\tilde{\omega}_k t/2}-e^{-\tilde{\omega}_k t/2})^{-1} > \frac{1}{\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3 t^3}\left(1-\frac{\tilde{\omega}_1^2 t^2}{24}\right)\left(1-\frac{\tilde{\omega}_2^2 t^2}{24}\right)\left(1-\frac{\tilde{\omega}_3^2 t^2}{24}\right)$$

since $0<\tilde{\omega}_k<2\sqrt{2}$ from $\tilde{\omega}_1^2+\tilde{\omega}_2^2+\tilde{\omega}_3^2=8$. On the other hand, (2) shows that

$$\begin{aligned}\left(1-\frac{\tilde{\omega}_1^2 t^2}{24}\right)\left(1-\frac{\tilde{\omega}_2^2 t^2}{24}\right)\left(1-\frac{\tilde{\omega}_3^2 t^2}{24}\right) &= \left(1-\frac{\tilde{\omega}_1^2}{8}\cdot\frac{t^2}{3}\right)\left(1-\frac{\tilde{\omega}_2^2}{8}\cdot\frac{t^2}{3}\right)\left(1-\frac{\tilde{\omega}_3^2}{8}\cdot\frac{t^2}{3}\right)\\ &> 1-\frac{t^2}{3}\end{aligned}$$

since

$$\frac{\tilde{\omega}_1^2}{8}+\frac{\tilde{\omega}_2^2}{8}+\frac{\tilde{\omega}_3^2}{8}=1.$$

This proves $(*)$. Thus we have shown Theorem 2. ∎

## 4. An application: Proof of Theorem 3

We recall the following result proved in [KK]:

$$\prod_{k_1,\ldots,k_r}{}' S_r\left(\frac{k_1\omega_1+\cdots+k_r\omega_r}{N},(\omega_1,\ldots,\omega_r)\right)=N$$

for each integer $N\geq 2$. Especially, letting $N=2$ and $r=3$, we have

$$\begin{aligned}\prod_{i=1}^{3} S_3\left(\frac{\omega_i}{2},(\omega_1,\omega_2,\omega_3)\right)\prod_{i<j} S_3\left(\frac{\omega_i+\omega_j}{2},(\omega_1,\omega_2,\omega_3)\right)&\\ \times S_3\left(\frac{\omega_1+\omega_2+\omega_3}{2},(\omega_1,\omega_2,\omega_3)\right)=2.&\end{aligned}$$

Hence we see that

$$\prod_{i=1}^{3} S_3\left(\frac{\omega_i}{2},(\omega_1,\omega_2,\omega_3)\right)\prod_{i<j} S_3\left(\frac{\omega_i+\omega_j}{2},(\omega_1,\omega_2,\omega_3)\right)$$
$$=\frac{2}{S_3\left(\frac{\omega_1+\omega_2+\omega_3}{2},(\omega_1,\omega_2,\omega_3)\right)}$$
$$>2$$

from Theorem 2.

## 5. Proof of Theorem 4

Put $f(x)=S_3(x,\boldsymbol{\omega})$ for simplicity, and we restrict $x$ to $0<x<|\boldsymbol{\omega}|$ hereafter. Since

$$f(x)=\exp\left(-\left(\frac{\partial}{\partial s}\zeta_3\right)(s,x,\boldsymbol{\omega})\Big|_{s=0}-\left(\frac{\partial}{\partial s}\zeta_3\right)(s,|\boldsymbol{\omega}|-x,\boldsymbol{\omega})\Big|_{s=0}\right)$$

we have the following formulas:

$$\log f(x)=-\left(\frac{\partial}{\partial s}\zeta_3\right)(s,x,\boldsymbol{\omega})\Big|_{s=0}-\left(\frac{\partial}{\partial s}\zeta_3\right)(s,|\boldsymbol{\omega}|-x,\boldsymbol{\omega})\Big|_{s=0},$$
$$\frac{f'}{f}(x)=-\left(\frac{\partial^2}{\partial s\partial x}\zeta_3\right)(s,x,\boldsymbol{\omega})\Big|_{s=0}+\left(\frac{\partial^2}{\partial s\partial x}\zeta_3\right)(s,|\boldsymbol{\omega}|-x,\boldsymbol{\omega})\Big|_{s=0},$$
$$\left(\frac{f'}{f}\right)'(x)=-\left(\frac{\partial^3}{\partial s\partial x^2}\zeta_3\right)(s,x,\boldsymbol{\omega})\Big|_{s=0}-\left(\frac{\partial^3}{\partial s\partial x^2}\zeta_3\right)(s,|\boldsymbol{\omega}|-x,\boldsymbol{\omega})\Big|_{s=0},$$
$$\left(\frac{f'}{f}\right)''(x)=-\left(\frac{\partial^4}{\partial s\partial x^3}\zeta_3\right)(s,x,\boldsymbol{\omega})\Big|_{s=0}+\left(\frac{\partial^4}{\partial s\partial x^3}\zeta_3\right)(s,|\boldsymbol{\omega}|-x,\boldsymbol{\omega})\Big|_{s=0},$$
$$\left(\frac{f'}{f}\right)'''(x)=-\left(\frac{\partial^5}{\partial s\partial x^4}\zeta_3\right)(s,x,\boldsymbol{\omega})\Big|_{s=0}-\left(\frac{\partial^5}{\partial s\partial x^4}\zeta_3\right)(s,|\boldsymbol{\omega}|-x,\boldsymbol{\omega})\Big|_{s=0}.$$

Using

$$\left(\frac{\partial}{\partial x}\zeta_3\right)(s,x,\boldsymbol{\omega})=-s\zeta_3(s+1,x,\boldsymbol{\omega})$$

we have

$$\left(\frac{\partial^4}{\partial x^4}\zeta_3\right)(s,x,\boldsymbol{\omega})=s(s+1)(s+2)(s+3)\zeta_3(s+4,x,\boldsymbol{\omega}),$$

so we get

$$\left(\frac{\partial^5}{\partial s \partial x^4}\zeta_3\right)(s,x,\boldsymbol{\omega})\bigg|_{s=0} = 6\zeta_3(4,x,\boldsymbol{\omega}) = 6\sum_{\mathbf{n}\geq\mathbf{0}}(\mathbf{n}\cdot\boldsymbol{\omega}+x)^{-4} > 0.$$

Thus, we know that

$$\left(\frac{f'}{f}\right)'''(x) = -6\left(\sum_{\mathbf{n}\geq\mathbf{0}}(\mathbf{n}\cdot\boldsymbol{\omega}+x)^{-4} + \sum_{\mathbf{m}\geq\mathbf{1}}(\mathbf{m}\cdot\boldsymbol{\omega}-x)^{-4}\right)$$
$$< 0.$$

Noting

$$\left(\frac{f'}{f}\right)''\left(\frac{|\boldsymbol{\omega}|}{2}\right) = 0$$

from the symmetry (see the above formula for $(f'/f)''$), we see the shape of the graph of $(f'/f)''$ as in Fig. 2:

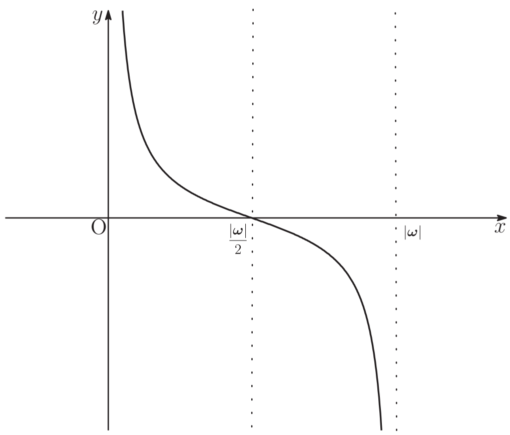


FIGURE 2. The graph of $\left(\frac{f'}{f}\right)''(x)$.

We remark a key observation

$$\left(\frac{f'}{f}\right)'\left(\frac{|\boldsymbol{\omega}|}{2}\right) > 0.$$

In fact, otherwise we see that

$$\left(\frac{f'}{f}\right)'(x) \leq 0$$

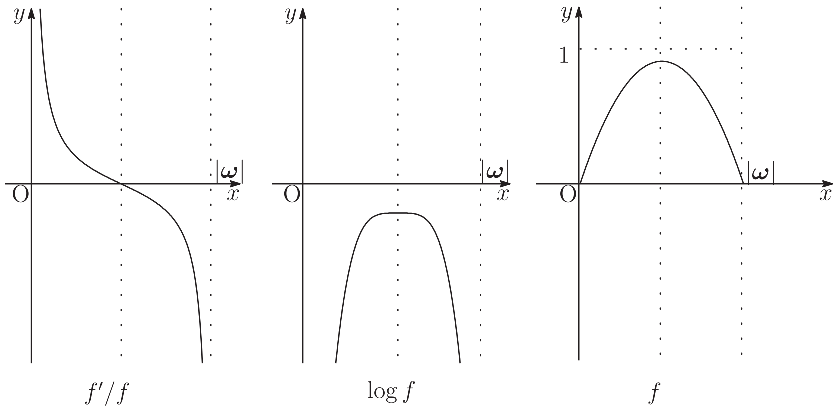


FIGURE 3.

for $0 < x < |\boldsymbol{\omega}|$ from the behavior of $(f'/f)''$. This implies that the shapes of the graphs of $f'/f$, $\log f$ and $f$ are as in Fig. 3, since we already know that $\log f\left(\frac{|\boldsymbol{\omega}|}{2}\right) < 0$ from Theorem 2.

Especially, this consideration shows that $0 < f(x) < 1$. This consequence contradicts to Theorem 3, since at least one of six values $S_3\left(\frac{\omega_i}{2}\right)$ and $S_3\left(\frac{\omega_i + \omega_j}{2}\right)$ are larger than 1 from Theorem 3.

Thus we know that $(f'/f)\left(\frac{|\boldsymbol{\omega}|}{2}\right) > 0$. Hence we see the true shapes of $(f'/f)'$, $f'/f$, $\log f$ and $f$ as in Fig. 4.

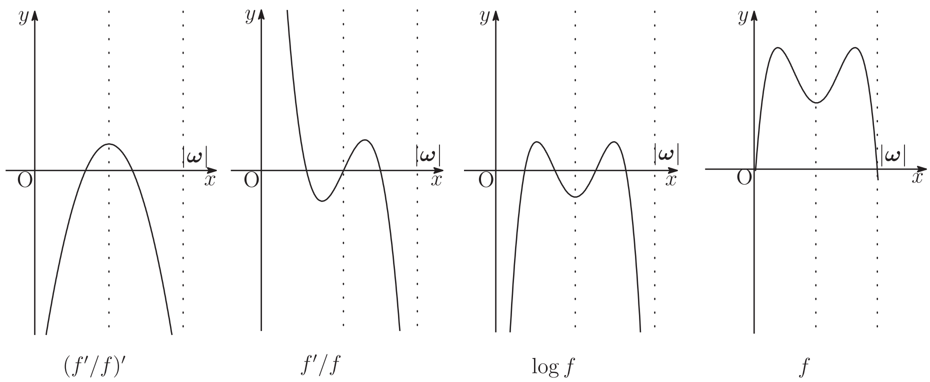


FIGURE 4.

This shows Theorem 4.

## 6. Special case: Proof of Theorem 5

Theorem 1 says in the special case $(\omega_1, \omega_2, \omega_3) = (1,1,1)$ that

$$S_3\left(\frac{3}{2},(1,1,1)\right) = \exp\left(-\int_0^{\infty}\left(\left(\frac{1}{4}\sinh\left(\sqrt{\frac{2}{3}}t\right)\right)^{-3} - \frac{3\sqrt{3}}{8\sqrt{2}t^3}\left(1-\frac{t^2}{3}\right)\right)\frac{dt}{t}\right)$$

$$= \exp\left(-\int_0^{\infty}\left(2(e^{\sqrt{2/3}t} - e^{-\sqrt{2/3}t})^{-3} + \frac{3}{16}\sqrt{\frac{2}{3}}\left(\frac{1}{t}-\frac{3}{t^3}\right)\right)\frac{dt}{t}\right).$$

Hence, using the result

$$S_3\left(\frac{3}{2},(1,1,1)\right) = 2^{-1/8}\exp\left(-\frac{3\zeta(3)}{16\pi^2}\right)$$

proved in [KK], we obtain Theorem 5. ∎

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