MEROMORPHIC FUNCTIONS THAT SHARE SOME PAIRS OF SMALL FUNCTIONS

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Abstract

We discuss possible relations between two meromorphic functions f and g when they share some pairs of small functions. By utilizing the generalized Nevanlinna's second main theorem for small functions obtained recently, we have been able to show that two meromorphic functions f and g must be linked by a quasi-Möbius transformation if they share three pairs of small functions CM* and share another pair of small function IM*. Moreover, we also improves a known result due to T. Czubiak and G. Gundersen on two meromorphic functions sharing five pairs of values and the results on the unicity of meromorphic functions that share five small functions obtained by Li Bao-Qin and Li Yu-Hua as well.

1. Introduction and results

Let f be a meromorphic functions defined on the complex plane $\mathbf{C}$. We assume the reader is familiar with the standard notations and basic results on Nevanlinna's value distribution theory such as the characteristic function $T(r, f)$, the counting functions above the poles of f: $N(r, f)$ and $\overline{N}(r, f)$ as well as the proximity function $m(r, f)$ (see, e.g., [3], [14]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure. A meromorphic function $a\ (\not\equiv \infty)$ is called a small function with respect to f provided that $T(r, a) = S(r, f)$. We denote by $\overline{N}_{k)}(r, f)$ the counting function of the poles of f of multiplicities $\leq k$, where every such a pole is counted only once, and denote by $\overline{N}_{(k}(r, f)$ the counting function of the poles of f of multiplicities $\geq k$, where every such a pole is counted only once. Let f and g be two nonconstant meromorphic functions, and a, b be two values in $\mathbf{C}$. We say that f and g share the value a IM (CM) provided that $f(z) - a$ and $g(z) - a$ have same zeros ignoring multiplicities (counting multiplicities). We say that f and g share the pair of values (a, b)

IM (CM) provided that $f(z) - a$ and $g(z) - b$ have same zeros ignoring multiplicities (counting multiplicities). The well-known Nevanlinna's five values theorem says that two meromorphic functions must be identical if they share five values IM. Nevanlinna's four values theorem says that two meromorphic functions must be linked by a Möbius transformation if they share four values CM. Since then, the subject on the unicity of meromorphic or entire functions that share some values has been studied by many complex analysts.

The studies of "sharing value or sharing pair of values IM or CM" can be extended to "sharing pairs of small functions IM* or CM* as follows (see, e.g., [1], [9]).

Let f and g be two nonconstant meromorphic functions, and a, b be two small meromorphic functions with respect to both f and g. Denote $\overline{N}(r, f = a, g = b)$ the counting function which counts the common zeros of $f - a$ and $g - b$, each such zero is counted only once. Denote $\overline{N}_E(r, f = a, g = b)$ the counting function which counts the common zeros of $f - a$ and $g - b$ with the same multiplicities, each such zero is counted only once. We say that f and g share the pair (a, b) in the sense of IM* provided that

$$\overline{N}\left(r, \frac{1}{f-a}\right) - \overline{N}(r, f = a, g = b) = S(r, f),$$

and

$$\overline{N}\left(r, \frac{1}{g-b}\right) - \overline{N}(r, f = a, g = b) = S(r, g).$$

We say that f and g share the pair (a, b) in the sense of CM* provided that

$$\overline{N}\left(r, \frac{1}{f-a}\right) - \overline{N}_E(r, f = a, g = b) = S(r, f),$$

and

$$\overline{N}\left(r, \frac{1}{g-b}\right) - \overline{N}_E(r, f = a, g = b) = S(r, g).$$

For convenience, we recall the notation $S^*(r, f)$ which is defined to be any quantity such that for any positive number ε there exists a $S(r, f)$ satisfying the following inequality:

$$|S^*(r, f)| \le \varepsilon T(r, f) + S(r, f).$$

Suppose that $\mathscr{M}(\mathbf{C})$ is the set of all meromorphic functions on $\mathbf{C}$. For $f \in \mathscr{M}(\mathbf{C})$, Let

$$\mathscr{S}(f) = \{g \in \mathscr{M}(\mathbf{C}) : T(r, g) = S(r, f)\},$$
$$\mathscr{S}^*(f) = \{g \in \mathscr{M}(\mathbf{C}) : T(r, g) = S^*(r, f)\}.$$

It is obvious that both $\mathscr{S}(f)$ and $\mathscr{S}^*(f)$ are fields of functions, which are closed under products and differentiating, and $\mathscr{S}(f) \subset \mathscr{S}^*(f)$. It is easily seen that we can not find any set I of infinite linear measure such that $T(r, f) \leq S^*(r, f)$, $r \in I$.

Nevanlinna's five values theorem has been generalized to small functions case (see, [7], [12], [5]), i.e., two nonconstant meromorphic functions must be identical if they share five small functions IM. The number 5 may be reduced to 4, if f and g have few poles. In fact, Ishizaki and Toda proved the following result.

THEOREM A ([6]). *Let f and g be two transcendental meromorphic functions, and let $a_1, \ldots, a_4$ be distinct small functions of f and g. If f and g share $a_1, \ldots, a_4$ IM, and if*

$$\overline{N}(r, f) \leq uT(r, f) + S(r, f) \quad and \quad \overline{N}(r, g) \leq vT(r, g) + S(r, g),$$

hold for some constants $u, v \in [0, 1/19)$, then $f = g$.

It follows that Nevanlinna's four values theorem can be generalized as follows.

THEOREM B ([8]). *Let f and g be nonconstant meromorphic functions and a_1, a_2, a_3, a_4 be four distinct small functions of f and g. If f and g share a_1, a_2 CM*, and share a_3, a_4 IM*, then f is a quasi-Möbius transformation of g, i.e., there exist four small functions α_i $(i = 1, 2, 3, 4)$ of g such that $f = \dfrac{\alpha_1 g + \alpha_2}{\alpha_3 g + \alpha_4}$.*

THEOREM C ([4], [9]). *Let f and g be nonconstant meromorphic functions and a_i, b_i $(i = 1, 2, 3, 4)$ $(a_i \neq a_j, b_i \neq b_j, i \neq j)$ be small functions of f and g. If f and g share the four pairs (a_i, b_i) CM*, then f is a quasi-Möbius transformation of g.*

The following two functions

$$(1) \qquad \hat{f}(z) = \frac{e^z + 1}{(e^z - 1)^2} \quad and \quad \hat{g}(z) = \frac{(e^z + 1)^2}{8(e^z - 1)},$$

which was found by G. G. Gundersen in 1979 (see [2]), shows that two meromorphic functions f and g sharing four values IM may not be linked by a Möbius transformation. In fact, it is easily seen that $\hat{f}$ and $\hat{g}$ share $0, 1, -1/8$, ∞ IM, but $\hat{f}$ is not a Möbius transformation of $\hat{g}$. Note that $\hat{f}$ and $\hat{g}$ also share the pair $(-1/2, 1/4)$ CM. So, two meromorphic functions that share five pairs of values may not be linked by a Möbius transformation. However, the following theorem shows that two meromorphic functions must be linked by a Möbius transformation when they share six pairs of values.

THEOREM D ([1]). *Let f and g be two nonconstant meromorphic functions that share six pairs of values (a_k, b_k), $1 \le k \le 6$, IM, where $a_i \ne a_j$ and $b_i \ne b_j$ whenever $i \ne j$. Then f is a Möbius transformation of g.*

In this paper, we shall prove the following results.

THEOREM 1. *Let f and g be nonconstant meromorphic functions and a_i, b_i $(i = 1, 2, 3, 4)$ $(a_i \ne a_j, b_i \ne b_j, i \ne j)$ be small functions of f and g. If f and g share three pairs (a_i, b_i) $(i = 1, 2, 3)$ CM^*, and share the fourth pair (a_4, b_4) IM^*, then f is a quasi-Möbius transformation of g.*

If the condition "f and g share the pair (a, b) IM^*" in Theorem 1 is replaced by "$f(z) - a(z) = 0$ implies $g(z) - b(z) = 0$", then the conclusion may not be true. In fact, the function

$$f(z) = \frac{1}{4}(e^{2z} - 2e^z + 4) \quad \text{and} \quad g(z) = e^{-2z}(e^{2z} - 2e^z + 4)$$

share 0, 1, ∞ CM, and $f(z) - 3/4 = 0$ implies $g(z) - 3 = 0$, but $f(z)$ can not be a Möbius transformation of $g(z)$.

The condition "f and g share three pairs CM^* and another pair IM^*" in Theorem 1 can not be replaced by "f and g share two pairs CM^* and another two pairs IM^*" either. For example, the functions

$$f(z) = \frac{-(e^z - 1)}{e^z - 2} \quad \text{and} \quad g(z) = \frac{-2(e^z - 1)^2}{e^z - 2}$$

share the pairs $(1, 1)$, (∞, ∞) CM, and share the pairs $(0, 0)$, $(-2, -8)$ IM, but $f(z)$ is not a Möbius transformation of $g(z)$.

THEOREM 2. *Let f and g be two nonconstant meromorphic functions, and a_j, b_j $(j = 1, \ldots, 5)$ be small functions with respect to f and g, and $a_i \ne a_j$, $b_i \ne b_j$ whenever $i \ne j$. If f and g share the four pairs (a_k, b_k) IM^*, $1 \le k \le 4$, and if the inequalities*

$$(2) \quad \overline{N}\left(r, \frac{1}{f - a_5}\right) \le \lambda T(r, f) + S(r, f) \quad \text{and} \quad \overline{N}\left(r, \frac{1}{g - b_5}\right) \le \lambda T(r, g) + S(r, g),$$

hold for $\lambda \in [0, 1/3)$, then f is a quasi-Möbius transformation of g.

From Theorem 2, we can get the following result for entire functions that share four pairs of finite values.

COROLLARY 1. *Let f and g be two nonconstant entire functions, and a_j, b_j $(j = 1, \ldots, 4)$ be finite values, and $a_i \ne a_j$, $b_i \ne b_j$ whenever $i \ne j$. If f and g share the four pairs (a_k, b_k) IM^*, $1 \le k \le 4$, then f is a Möbius transformation of g.*

THEOREM 3. *Let f and g be two nonconstant meromorphic functions, and a_j, b_j $(j = 1, \ldots, 6)$ be small functions with respect to f and g, and $a_i \neq a_j$, $b_i \neq b_j$ whenever $i \neq j$. If f and g share the five pairs (a_k, b_k) IM^*, $1 \leq k \leq 5$, and if f is not a quasi-Möbius transformation of g, then the following identities or inequalities hold*:

(a) $T(r, f) = T(r, g) + S^*(r, f)$;

(b) $\sum_{i=1}^{5} \overline{N}\left(r, \dfrac{1}{f - a_i}\right) = 3T(r, f) + S^*(r, f)$;

(c) $T(r, f) \leq \overline{N}\left(r, \dfrac{1}{f - a_i}\right) + \overline{N}\left(r, \dfrac{1}{f - a_j}\right) + S^*(r, f)$, $i \neq j$, $i, j = 1, \ldots, 5$;

(d) $T(r, f) \leq 3\overline{N}\left(r, \dfrac{1}{f - a_i}\right) + S^*(r, f)$, $i = 1, \ldots, 5$;

(e) $T(r, f) = \overline{N}\left(r, \dfrac{1}{f - a_6}\right) + S^*(r, f)$;

(f) $\overline{N}(r, f = a_6, g = b_6) \leq \frac{3}{5} T(r, f) + S^*(r, f)$;

(g) $T(r, f) = \overline{N}\left(r, \dfrac{1}{f - a_5}\right) + S^*(r, f)$ and $T(r, f) = 2\overline{N}\left(r, \dfrac{1}{f - a_i}\right) + S^*(r, f)$

   for $i = 1, \ldots, 4$ if $a_i = b_i$, $i = 1, \ldots, 4$.

Remark. From (g) of Theorem 3, we see that if f and g share five distinct small functions IM^*, then f is a quasi-Möbius transformation of g, and thus $f \equiv g$. This result was proved in [7] (entire case) and [12] (meromorphic case).

COROLLARY 2. *Let f and g be two nonconstant meromorphic functions, and a_j, b_j $(j = 1, \ldots, 6)$ be small functions with respect to f and g, and $a_i \neq a_j$, $b_i \neq b_j$ whenever $i \neq j$. If f and g share the five pairs (a_k, b_k) IM^*, $1 \leq k \leq 5$, and if there exists a number $\lambda \in [0, 2/5]$ such that*

$$\overline{N}\left(r, \dfrac{1}{f - a_6}\right) - \overline{N}(r, f = a_6, g = b_6) \leq \lambda T(r, f) + S(r, f),$$

then f must be a quasi-Möbius transformation of g.

Remark. The conclusion of Corollary 2 for the special case: $\lambda = 0$ and all a_i, b_i are values, can be found in [4].

Obviously, Theorem 2 is a generalization of Theorem A, Theorem 1 is a generalization of Theorem C, and Corollary 2 is a generalization of Theorem D.

2. Lemmas

LEMMA 1 ([9], [11]). *Let f and g be two nonconstant meromorphic functions, a_1, a_2 and a_3 be three distinct small functions with respect to f and g. If f and g*

share a_1, a_2, a_3 CM^*, and if f is not a quasi-Möbius transformation of g, then for any small function c $(\not\equiv a_1, a_2, a_3)$ with respect to f and g, we have

$$T(r,f) = N\left(r, \frac{1}{f-c}\right) + S(r,f) \quad and \quad N_{(3}\left(r, \frac{1}{f-c}\right) = S(r,f).$$

LEMMA 2 ([10]).   *Let h_1 and h_2 be two nonconstant meromorphic functions satisfying*

$$\overline{N}(r, h_i) + \overline{N}(r, 1/h_i) = S(r), \quad i = 1, 2.$$

If $h_1^s h_2^t - 1$ is not identically zero for all integers s and t $(|s| + |t| > 0)$, then for any positive number ε, we have

$$\overline{N}(r, h_1 = 1, h_2 = 1) \leq \varepsilon T(r) + S(r),$$

where $T(r) = T(r, h_1) + T(r, h_2)$ and $S(r) = o(T(r))$ as $r \to \infty$, except for a set of r of finite linear measure.

LEMMA 3 ([13]).   *Suppose that $f(z)$ is a nonconstant meromorphic function, and $a_1(z), a_2(z), \ldots, a_q(z)$ are distinct small functions of $f(z)$. Then for any positive number ε, we have*

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + \varepsilon T(r,f) + S(r,f).$$

LEMMA 4.   *Let h_1 h_2 and h be nonconstant meromorphic functions such that $T(r, h_i) \leq cT(r, h) + S(r, h)$ $(i = 1, 2)$, where c is a positive constant, and*

$$\overline{N}(r, h_i) + \overline{N}\left(r, \frac{1}{h_i}\right) = S(r, h), \quad i = 1, 2.$$

Let a and b be two small meromorphic functions of h. If the function $f = ah_1 + bh_2 + 1$ is not constant, then

$$\overline{N}_{(3}\left(r, \frac{1}{f}\right) = S(r, h).$$

Proof.   Let $\alpha_i = h_i'/h_i$ $(i = 1, 2)$. By the lemma of logarithmic derivative and the conditions of Lemma 4, we have $T(r, \alpha_i) = S(r, h)$ $(i = 1, 2)$. Let $a_1 = a' + a\alpha_1$, $b_1 = b' + b\alpha_2$, $a_2 = a_1' + a_1\alpha_1$ and $b_2 = b_1' + b_1\alpha_2$. It is obvious that $T(r, a_i) = S(r, h)$ and $T(r, b_i) = S(r, h)$, $i = 1, 2$. If both a_1 and b_1 are identically zero, then both ah_1 and bh_2 are constant, which implies that f is a constant. This contradicts the assumption. Without loss of generality, we may assume that b_1 is not identically zero.

Suppose that z_0 is zero of f of multiplicity ≥ 3, but not a zero or pole of α_i $(i = 1, 2)$. Then we have

(3) $f(z_0) = a(z_0)h_1(z_0) + b(z_0)h_2(z_0) + 1 = 0,$

(4) $f'(z_0) = a_1(z_0)h_1(z_0) + b_1(z_0)h_2(z_0) = 0,$

(5) $f''(z_0) = a_2(z_0)h_1(z_0) + b_2(z_0)h_2(z_0) = 0.$

If z_0 is not zero or pole of b_i, then from (4) and (5) we have

$$\frac{a_2(z_0)}{a_1(z_0)} = \frac{b_2(z_0)}{b_1(z_0)}.$$

If $a_2/a_1 \not\equiv b_2/b_1$, then we get

$$\overline{N}_{(3}\left(r, \frac{1}{f}\right) \leq \overline{N}\left(r, \frac{1}{a_2/a_1 - b_2/b_1}\right) \leq S(r, h).$$

Suppose that $a_2/a_1 \equiv b_2/b_1$. We get

(6) $$\frac{a_1'}{a_1} + \frac{h_1'}{h_1} = \frac{b_1'}{b_1} + \frac{h_2'}{h_2}.$$

By integration, we get

(7) $$a_1 h_1 = c b_1 h_2,$$

where c is a nonzero constant. From (4) and (7), we get $(c+1)b_1(z_0)h_2(z_0) = 0$. Note that $h_i(z_0) \neq 0$ $(i = 1, 2)$. We have $c = -1$. Thus $f' = a_1' h_1 + b_1' h_2 = 0$, it follows that f is a constant, a contradiction. Hence z_0 must be a zero or pole of b_1. Therefore, we have

$$\overline{N}_{(3}\left(r, \frac{1}{f}\right) \leq T(r, b_1) + S(r, h) \leq S(r, h),$$

which completes the proof of Lemma 4.   □

LEMMA 5. *Suppose that f and g are nonconstant meromorphic functions, $F = F(f, g)$ is a polynomial in f and g with coefficients being small functions with respect to f and g. The degree of F about f is p, and the degree about g is q. Then we have*

(8) $$T(r, F) \leq pT(r, f) + qT(r, g) + S(r, f).$$

Proof. The function F can be written as $F = \sum_{k=0}^{p} c_k f^k g^{n_k}$, where c_k are small functions with respect to f and g, and $0 \leq n_k \leq q$. It is obvious that

(9) $$N(r, F) \leq pN(r, f) + qN(r, g) + S(r, f).$$

To estimate $m(r, F)$, for a fixed positive number r, we set $A_1 = \{\theta \in [0, 2\pi] : |f(re^{i\theta})| \leq 1\}$, $A_2 = [0, 2\pi] \backslash A_1$, $B_1 = \{\theta \in [0, 2\pi] : |g(re^{i\theta})| \leq 1\}$, and $B_2 = [0, 2\pi] \backslash B_1$. Then

$$m(r, F) = \frac{1}{2\pi} \int_{A_1 \cap B_1} \log^+ |F(re^{i\theta})| \, d\theta + \frac{1}{2\pi} \int_{A_1 \cap B_2} \log^+ |F(re^{i\theta})| \, d\theta$$

$$+ \frac{1}{2\pi} \int_{A_2 \cap B_1} \log^+ |F(re^{i\theta})| \, d\theta \frac{1}{2\pi} \int_{A_2 \cap B_2} \log^+ |F(re^{i\theta})| \, d\theta.$$

Simple computation shows that

$$\int_{A_1 \cap B_1} \Psi \, d\theta \leq \int_{A_1 \cap B_1} \Phi \, d\theta,$$

$$\int_{A_1 \cap B_2} \Psi \, d\theta \leq q \int_{A_1 \cap B_2} \log^+ |g(re^{i\theta})| \, d\theta + \int_{A_1 \cap B_2} \Phi \, d\theta,$$

$$\int_{A_2 \cap B_1} \Psi \, d\theta \leq p \int_{A_2 \cap B_1} \log^+ |f(re^{i\theta})| \, d\theta + \int_{A_2 \cap B_1} \Phi \, d\theta,$$

$$\int_{A_2 \cap B_2} \Psi \, d\theta \leq p \int_{A_2 \cap B_2} \log^+ |f(re^{i\theta})| \, d\theta + q \int_{A_2 \cap B_2} \log^+ |g(re^{i\theta})| \, d\theta + \int_{A_2 \cap B_2} \Phi \, d\theta,$$

where $\Psi = \log^+ |F(re^{i\theta})|$ and $\Phi = \log^+ (\sum_{k=0}^{p} |c_k(re^{i\theta})|)$. By adding these inequalities together, we get

(10) $$m(r, F) \leq pm(r, f) + qm(r, g) + S(r, f).$$

The desired inequality follows from (9) and (10). $\qquad\qquad\square$

3. Proofs of the results

Proof of Theorem 1. Without loss of generality, we assume that f and g share the pairs 0, 1, ∞ CM^*, and share the pair (a, b) IM^*, where a $(\not\equiv 0, 1, \infty)$ and b $(\not\equiv 0, 1, \infty)$ are small functions of f and g, otherwise, we can consider the following transformation

$$F = \frac{f - a_1}{f - a_3} \cdot \frac{a_2 - a_3}{a_2 - a_1} \quad \text{and} \quad G = \frac{g - b_1}{g - b_3} \cdot \frac{b_2 - b_3}{b_2 - b_1}.$$

By Nevanlinna's second main theorem, we have

$$T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f)$$

$$\leq \bar{N}_E(r, f = \infty, g = \infty) + \bar{N}_E(r, f = 1, g = 1)$$

$$+ \bar{N}_E(r, f = 0, g = 0) + S(r, f)$$

$$\leq 3T(r, g) + S(r, f).$$

Similarly, we have $T(r, g) \leq 3T(r, f) + S(r, g)$. Therefore, an $S(r, f)$ is an $S(r, g)$, and vice versa. We write $S(r) = S(r, f) = S(r, g)$. Let

(11) $$h_1 = \frac{b}{a} \cdot \frac{f}{g}, \quad h_2 = \frac{b-1}{a-1} \cdot \frac{f-1}{g-1}.$$

Since f and g share 0, 1, ∞ CM^*, we have

(12) $$\overline{N}(r, h_i) + \overline{N}\left(r, \frac{1}{h_i}\right) = S(r), \quad i = 1, 2.$$

It is obvious that $T(r, h_i) \leq T(r, f) + T(r, g) + S(r) \leq 4T(r, f) + S(r)$.

Suppose that f is not a quasi-Möbius transformation of g. Then h_1 and h_2 can not be small functions of f and g. Since f and g share the pair (a, b) IM^*, by Theorem A, we have $a \not\equiv b$. By Lemma 1, we get

$$T(r, f) \leq 2\overline{N}\left(r, \frac{1}{f-a}\right) + S(r) \leq 2\overline{N}(r, f = a, g = b) + S(r)$$

$$\leq 2\overline{N}(r, h_1 = 1, h_2 = 1) + S(r).$$

By Lemma 2, there exist two nonzero integers s_1 and t_1 such that $h_1^{s_1} = h_2^{t_1}$. Let d be the greatest common factor of s_1 and t_1. Then there exist a nonzero constant c such that $h_1^s = ch_2^t$, where $s = s_1/d$ and $t = t_1/d$. Note that there many common 1-points of h_1 and h_2. Therefore, $c = 1$. Thus we have $h_1^s = h_2^t$. Since s and t are relatively prime to each other, there exist two nonzero integers u and v such that $us + vt = 1$. Let

(13) $$h = h_1^v h_2^u.$$

Then we have

(14) $$h_1 = h^t, \quad h_2 = h^s.$$

Without loss of generality, we can assume that $s \geq 1$. From (11) and (14), we get

(15) $$f - a = \frac{a(a-1)}{b(b-1)} \cdot \frac{h^{s+t} - bh^s + (b-1)h^t}{\frac{a-1}{b-1}h^s - \frac{a}{b}h^t}, \quad g - b = \frac{(1-a)h^s + ah^t - 1}{\frac{a-1}{b-1}h^s - \frac{a}{b}h^t}.$$

If $\overline{N}_{(2}(r, 1/f - a) = S(r)$ and $\overline{N}_{(2}(r, 1/g - b) = S(r)$, then f and g share the pair (a, b) CM^*. Thus, by Theorem B, f and g must be linked by a quasi-Möbius transformation. Thus we may assume, without loss of generality, that

$$\overline{N}_{(2}\left(r, \frac{1}{f-a}\right) \neq S(r).$$

Suppose that z_0 is a multiple zero of $f - a$, but not the zero or pole of a, b, not the 1-point of a, b either. It follows from (15) that z_0 must be a multiple zero of $h^{s+t} - bh^s + (b-1)h^t$, i.e.,

$$\{h^{s+t} - bh^s + (b-1)h^t\}(z_0) = 0,$$

$$\{((s+t)h^{s+t} - sbh^s + t(b-1)h^t)\alpha - b'h^s + b'h^t\}(z_0) = 0,$$

where $\alpha = h'/h \not\equiv 0$, which is a small function of f. Note that f and g share the pair (a, b) IM*. From (11) and (13), we get $h(z_0) = 1$. It follows from the above equation that $\alpha(z_0) = 0$ or $s - (s - t)b(z_0) = 0$. Since $\overline{N}_{(2}(r, 1/(f - a)) \neq S(r)$ and $\alpha \not\equiv 0$, we get

$$s - (s - t)b \equiv 0.$$

Hence $b = s/(s - t)$ is a constant. Since f and g share ∞ CM* and share the pair (a, b) IM*, we can see from (15) that the two functions

(16) $F(h) := h^{s+t} - bh^s + (b - 1)h^t$ and $G(h) := (1 - a)h^s + ah^t - 1$

share 0 IM*. Suppose that z_1 is a common zero of F and G, but not the zero or pole of a, not the 1-point of a either. Then we have

$$h^s(z_1) = h^t(z_1) = 1 \quad \text{or} \quad h^s(z_1) = \frac{b(z_1) - 1}{a(z_1) - 1}, \quad h^t(z_1) = \frac{b(z_1)}{a(z_1)}.$$

It follows that $h(z_1) = 1$ or $h(z_1) = r_0(z_1)$, where $r_0 := \{(b - 1)/(a - 1)\}^u (b/a)^v$ is a small function of f, and $r_0 \not\equiv 0$. Therefore, F and G can be expressed as

(17) $F = A_1 h^{k_1}(h - 1)^{p_1}(h - r_0)^{q_1}$ and $G = A_2 h^{k_2}(h - 1)^{p_2}(h - r_0)^{q_2}$,

where A_i is a small function of f, and k_i, p_i, q_i are non-negative integers. By Lemma 4, we see that $p_i \leq 2$ and $q_i \leq 2$. Since $h = 1$ is a root of $G(h) = 0$, and a multiple root of $F(h) = 0$. We have $p_1 = 2$ and $p_2 \geq 1$. Note that there are three terms in $F(h)$. We get $q_1 \leq 1$.

   If $p_2 = 2$, then $h = 1$ is a multiple root of $G(h) = 0$. By the arguments similar to that in the above, we can prove that $a \equiv s/(s - t) = b$. Therefore, f and g share a IM*. By Theorem B, f is a quasi-Möbius transformation of g, which contradicts the assumption. Hence $p_2 = 1$.

   From (16), we see that there are at most three terms in $G(h)$. Thus $q_2 \geq 1$. And then $q_1 \geq 1$, otherwise F and G can not share 0 IM*. Hence $q_1 = 1$. Then we have

(18) $$F = \frac{A_1}{A_2} h^{k_1 - k_2}(h - 1)G.$$

By computation,

(19) $(h - 1)G = (1 - a)h^{s+1} - (1 - a)h^s + 1 + ah^{t+1} - ah^t - h.$

However, there are at most three terms in $F(h)$. This may be occur only when $t = 1$, $s = 2$, $b = 2$ or $t = -1$, $s = 1$, $b = \frac{1}{2}$. In both cases, F can be expressed as $A_1 h^{k_1}(h - 1)^2$, which shows that $F(h)$ can not be the form $A_1 h^{k_1}(h - 1)^2(h - r_0)$, $r_0 \not\equiv 0$. So, if f is not a quasi-Möbius transformation of g, then we will arrive at a contradiction. This also completes the proof of Theorem 1.   □

   Proof of Theorem 2.   By utilizing quasi-Möbius transformation, we assume, without loss of generality, that none of a_j and b_j $(j = 1, \ldots, 5)$ is infinity. Let $\mathcal{L}$

be the quasi-Möbius transformation such that $a_j \equiv \mathcal{L}(b_j)$, $j = 1, 2, 3$. Note that f and g share (a_j, b_j) IM* $(1 \leq j \leq 4)$. A quantity $S(r, f)$ is also a $S(r, g)$, and vise versa. For convenience, in the sequel we write $S(r) := S(r, f) = S(r, g)$ and $S^*(r) := S^*(r, f) = S^*(r, g)$.

Assume f is not a quasi-Möbius transformation of g, then we have

$$(20) \quad \sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f - a_j}\right) \leq \bar{N}\left(r, \frac{1}{f - \mathcal{L}(g)}\right) + S(r) \leq T(r, f) + T(r, g) + S(r).$$

By Lemma 3, we have

$$(21) \quad 3T(r, f) \leq \sum_{j=1}^{5} \bar{N}\left(r, \frac{1}{f - a_j}\right) + S^*(r).$$

From (2), (20) and (21), we get

$$3T(r, f) \leq T(r, f) + T(r, g) + \bar{N}\left(r, \frac{1}{f - a_4}\right) + \lambda T(r, f) + S^*(r).$$

That is

$$(22) \quad 2T(r, f) \leq T(r, g) + \bar{N}\left(r, \frac{1}{f - a_4}\right) + \lambda T(r, f) + S^*(r).$$

Similarly, we have

$$(23) \quad 2T(r, f) \leq T(r, g) + \bar{N}\left(r, \frac{1}{f - a_j}\right) + \lambda T(r, f) + S^*(r), \quad j = 1, 2, 3.$$

Adding the three inequalities in (23) together and using (20) again yield

$$6T(r, f) \leq 3T(r, g) + T(r, f) + T(r, g) + 3\lambda T(r, f) + S^*(r).$$

Hence

$$(24) \quad 5T(r, f) \leq 4T(r, g) + 3\lambda T(r, f) + S^*(r).$$

Symmetrically, we have

$$(25) \quad 5T(r, g) \leq 4T(r, f) + 3\lambda T(r, g) + S^*(r).$$

Add the above two inequalities yield

$$T(r, f) + T(r, g) \leq 3\lambda(T(r, f) + T(r, g)) + S^*(r).$$

This is impossible for the number $\lambda < 1/3$. Hence f must be a quasi-Möbius transformation of g. □

Proof of Theorem 3. Without loss of generality, we assume that $a_j \not\equiv \infty$, $b_j \not\equiv \infty$ $(j = 1, \ldots, 6)$, and $a \not\equiv \infty$. Furthermore, we may assume $(a_1, b_1) = (0, 0)$, $(a_2, b_2) = (1, 1)$, and $(a_3, b_3) = (-1, -1)$. It is not difficult to find five

small functions c_j $(j = 1, \ldots, 5)$ (at least one of them is not identically zero) such that the following function

$$(26) \qquad F := F(f, g) = c_1 f^2 g + c_2 fg + c_3 f^2 + c_4 f + c_5 g$$

satisfy $F(a_j, b_j) \equiv 0$ for $j = 1, \ldots, 5$. By Lemma 5, we have

$$(27) \qquad T(r, F) \leq 2T(r, f) + T(r, g) + S(r).$$

If $F \equiv 0$, then $(c_1 f^2 + c_2 f + c_5)g \equiv -(c_3 f^2 + c_4 f)$. Note that at least one of c_j is not zero. Therefore, $c_1 f^2 + c_2 f + c_5 \not\equiv 0$. Hence

$$(28) \qquad g = -\frac{c_3 f^2 + c_4 f}{c_1 f^2 + c_2 f + c_5}.$$

Since g is not a quasi-Möbius transformation of f, the right-hand side of the above equation is irreducible. Therefore, $T(r, g) = 2T(r, f) + S(r)$. By Lemma 3, we have

$$3T(r, g) \leq \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g - b_j}\right) + S^*(r)$$

$$\leq \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S^*(r)$$

$$\leq 5T(r, f) + S^*(r).$$

Therefore, $6T(r, f) \leq 5T(r, f) + S^*(r)$, which is impossible. Hence $F \not\equiv 0$.

Since f and g share the five pairs (a_j, b_j) IM*, and $F(a_j, b_j) \equiv 0$ for $j = 1, \ldots, 5$, by Lemma 3 and Lemma 5, we have

$$4T(r, f) \leq \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_j}\right) + \overline{N}\left(r, \frac{1}{f - a_6}\right) + S^*(r)$$

$$\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{f - a_6}\right) + S^*(r)$$

$$\leq 2T(r, f) + T(r, g) + \overline{N}\left(r, \frac{1}{f - a_6}\right) + S^*(r).$$

Therefore,

$$(29) \qquad 2T(r, f) \leq T(r, g) + \overline{N}\left(r, \frac{1}{f - a_6}\right) + S^*(r),$$

which implies that

$$(30) \qquad T(r, f) \leq T(r, g) + S^*(r).$$

Symmetrically, we have

$$(31) \qquad T(r, g) \leq T(r, f) + S^*(r).$$

This proves (a). It follows from (31) and Lemma 5 that

$$\sum_{j=1}^{5} \bar{N}\left(r, \frac{1}{f - a_j}\right) \le \bar{N}\left(r, \frac{1}{F}\right) + S(r) \le 3T(r, f) + S^*(r),$$

By Lemma 3, the opposite inequality also holds. Therefore, (b) holds. From (21), (20) and (31), we get

$$3T(r, f) \le 2T(r, f) + \bar{N}\left(r, \frac{1}{f - a_4}\right) + \bar{N}\left(r, \frac{1}{f - a_5}\right) + S^*(r).$$

That is

$$T(r, f) \le \bar{N}\left(r, \frac{1}{f - a_4}\right) + \bar{N}\left(r, \frac{1}{f - a_5}\right) + S^*(r).$$

Similarly, we can deduce that

$$(32) \qquad T(r, f) \le \bar{N}\left(r, \frac{1}{f - a_i}\right) + \bar{N}\left(r, \frac{1}{f - a_j}\right) + S^*(r)$$

holds for $i, j = 1, \ldots, 5$ and $i \ne j$. Therefore, (c) holds. From (32), (20) and (31), we have

$$3T(r, f) \le \sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f - a_j}\right) + 3\bar{N}\left(r, \frac{1}{f - a_4}\right) + S^*(r)$$

$$\le 2T(r, f) + 3\bar{N}\left(r, \frac{1}{f - a_4}\right) + S^*(r).$$

This gives

$$T(r, f) \le 3\bar{N}\left(r, \frac{1}{f - a_4}\right) + S^*(r)$$

Similarly, we can obtain

$$T(r, f) \le 3\bar{N}\left(r, \frac{1}{f - a_i}\right) + S^*(r), \quad i = 1, \ldots, 5,$$

which shows that (d) holds. From (29), (30) and (31), we can deduce

$$(33) \qquad T(r, f) \le \bar{N}\left(r, \frac{1}{f - a_6}\right) + S^*(r).$$

Therefore, (e) holds. By arguing similarly to that in the proof of (b), we have

$$\sum_{i=1}^{4} \bar{N}\left(r, \frac{1}{f - a_i}\right) + \bar{N}(r, f = a_6, g = b_6) \le 3T(r, f) + S^*(r).$$

Therefore,

$$\sum_{i=1}^{5} \overline{N}\left(r, \frac{1}{f - a_i}\right) \leq 3T(r, f) + \overline{N}\left(r, \frac{1}{f - a_5}\right) - \overline{N}(r, f = a_6, g = b_6) + S^*(r).$$

From this and Lemma 3, we get

$$\overline{N}(r, f = a_6, g = b_6) \leq \overline{N}\left(r, \frac{1}{f - a_5}\right) + S^*(r).$$

Similarly, we have

$$\overline{N}(r, f = a_6, g = b_6) \leq \overline{N}\left(r, \frac{1}{f - a_i}\right) + S^*(r), \quad i = 1, \ldots, 4.$$

Add the five inequalities together and then use (b), we get

$$5\overline{N}(r, f = a_6, g = b_6) \leq 3T(r, f) + S^*(r).$$

So, (f) holds.

Suppose furthermore that $a_i = b_i$ for $i = 1, \ldots, 4$. We have

$$(34) \qquad \sum_{i=1}^{4} \overline{N}\left(r, \frac{1}{f - a_i}\right) \leq \overline{N}\left(r, \frac{1}{f - g}\right) \leq 2T(r, f) + S^*(r).$$

From this and by (c), we get

$$(35) \qquad \overline{N}\left(r, \frac{1}{f - a_i}\right) + \overline{N}\left(r, \frac{1}{f - a_j}\right) \leq T(r, f) + S^*(r).$$

And thus

$$(36) \quad \overline{N}\left(r, \frac{1}{f - a_i}\right) + \overline{N}\left(r, \frac{1}{f - a_j}\right) = T(r, f) + S^*(r), \quad i, j = 1, \ldots, 4, \, i \neq j,$$

which yields

$$(37) \qquad \overline{N}\left(r, \frac{1}{f - a_i}\right) = \frac{1}{2}T(r, f) + S^*(r), \quad i = 1, \ldots, 4.$$

From this and (b), we get $\overline{N}\left(r, \frac{1}{f - a_5}\right) = T(r, f) + S^*(r)$. This also completes the proof of Theorem 3. □

4. Concluding remark and questions

We are unable to show whether the number $1/3$ in Theorem 2 is best or not. So we propose the following question for further study.

QUESTION 1. Suppose that f and g are two nonconstant meromorphic functions, a_j, b_j $(j = 1, \ldots, 5)$ are small functions with respect to f and g, and $a_i \neq a_j$, $b_i \neq b_j$ whenever $i \neq j$. Can one find a number $d > 1/3$ such that f must be a quasi-Möbius transformation of g as long as f and g share the four pairs (a_k, b_k) IM*, $1 \leq k \leq 4$, and the inequalities

$$\overline{N}\left(r, \frac{1}{f - a_5}\right) \leq \lambda T(r, f) + S(r, f) \quad and \quad \overline{N}\left(r, \frac{1}{g - b_5}\right) \leq \lambda T(r, g) + S(r, g)$$

hold for all $\lambda \in [0, d)$?

From (d) in Theorem 3, we can see that $\Theta(a_i, f) \leq \frac{2}{3}$, $i = 1, \ldots, 5$, provided that f and g share five pairs (a_i, b_i) IM*, and f is not a quasi-Möbius transformation of g. Thus we have the following question.

QUESTION 2. Suppose that f and g are two nonconstant meromorphic functions sharing five pairs of small functions (a_i, b_i) $(i = 1, \ldots, 5)$, and f is not a quasi-Möbius transformation of g. What is the minimal number μ such that $\Theta(a_i, f) \leq \mu$, $i = 1, \ldots, 5$?

By consideration of the functions $\hat{f}$ and $\hat{g}$ in (1), we see that the number d in Question 1 must be less than or equal to $1/2$, and the minimal number μ in Question 2 can not be less than $1/2$.

From Theorem 1, we see that two meromorphic functions must be linked by a quasi-Möbius transformation if they share three pairs of small functions CM*, and share another pair of small functions IM*. The following conjecture is reasonable.

CONJECTRUE. *Suppose that f and g are two nonconstant meromorphic functions sharing five pairs of small functions (a_i, b_i) IM* $(i = 1, \ldots, 5)$. If two of the pairs are shared in the sense CM*, then f must be a quasi-Möbius transformation of g.*

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