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SEIBERG-WITTEN-FLOER HOMOLOGY AND THE GEOMETRIC STRUCTURE $\mathbf{R} \times H^2$

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Abstract[1]

The Seiberg-Witten-Floer homology of an oriented closed 3-manifold M with the geometric structure $\mathbf{R} \times H^2$ is computed.

1. Introduction

In [5], A. Floer constructed a remarkable invariant for an oriented closed 3-manifold, so-called Floer homology, whose developments of this work are widely discussed in [4]. Variants of Floer homology are described in [8], [17]. Floer's work is based on Yang-Mills gauge theory. So it is natural to attempt to define a similar homology for Seiberg-Witten gauge theory.

By the efforts of several geometers, one can obtain a notion of Floer homology in the framework of Seiberg-Witten gauge theory, so-called Seiberg-Witten-Floer homology. In several geometric situations, Seiberg-Witten-Floer homology is computed. See [3], [11], [15], for example.

In Seiberg-Witten gauge theory, the monopole class $\alpha = c_1(L)$ plays an essential role in computing the Seiberg-Witten invariant. Also the scalar curvature of a 3-manifold crucially appears in Seiberg-Witten gauge theory as in [10].

In fact, we introduce in [6] a certain equality of the L^2-norm between the monopole class and the scalar curvature of an oriented closed 3-manifold M, an equality which is closely related to the dual Thurston norm. Moreover in [7], we show that this equality holds if and only if M admits the geometric structure $\mathbf{R} \times H^2$ which is one of the eight model geometries introduced by Thurston. Taking a suitable complex line bundle L associated with a $Spin(3)^c$ structure, we make clear the structure of the moduli space of the solutions to the 3-dimensional Seiberg-Witten equations. These results are stated as follows.

Theorem 1.1 ([7]). *Let M be an oriented closed 3-manifold with a monopole*

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class $\alpha = c_1(L)$ *associated with the principal* $Spin(3)^c$ *bundle induced by TM of M. Suppose that M admits a smooth metric h which satisfies*

$$\|\alpha_h\|_{(L^2,h)} = \frac{1}{4\pi}\|s_h\|_{(L^2,h)}.$$

Then, (1) *M carries the geometric structure* $\mathbf{R} \times H^2$ *and furthermore* (2) $L = F \otimes K_M^{\pm 1}$. *Here, F is a complex line bundle with a flat connection and* $K_M^{\pm 1} \to M$ *is a complex line bundle naturally induced from the* (*anti-*)*canonical line bundle* $K_{H^2}^{\pm 1}$ *over* H^2 *by the quotient map*: $\mathbf{R} \times H^2 \to M$.

In the above theorem, $\|\alpha_h\|_{(L^2,h)}$ is the L^2-norm of the harmonic representative of α, and $\|s_h\|_{(L^2,h)}$ is the L^2-norm of the scalar curvature for the given metric h. The statement (1) is also proved in [6]. The statement (2) follows from comparing the first Chern classe of L with the first Chern class of $F \otimes K_M^{\pm 1}$.

We call $\alpha = c_1(L)$ a monopole class, when corresponding 3-dimensional Seiberg-Witten equations (or monopole equations)

$$\begin{cases} c(*F_A) = \Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \,\mathrm{Id}_W \\ D_A\Phi = 0 \end{cases}$$

have a solution for all Riemannian metrics h on M. We denote by $\mathscr{S}$ the set of the solutions to the monopole equations, which is invariant under the gauge action

$$(A, \Phi) \mapsto (A + g^{-1}\,dg, g^{-1}\Phi), \quad g \in \mathscr{G} = \Gamma(M; U(1)).$$

Therefore we can consider the moduli space $\mathscr{M} = \mathscr{S}/\mathscr{G}$. In our case, $\mathscr{M}$ is described as follows.

Theorem 1.2 ([7]). *Let M be an oriented closed* 3*-manifold carrying the geometric structure* $\mathbf{R} \times H^2$ *with the* (*anti-*)*canonical line bundle* $K_M^{\pm 1}$. *Suppose* $b_1(M) > 1$. *It follows then that* (1) *the moduli space of the solutions to the monopole equations associated with the class* $\alpha = c_1(K_M^{\pm 1})$ *and the metric h such that* $\pi^*h = dt^2 \oplus a^2 g_H$ *consists of a single point and is transversal at this point and that* (2) $\alpha = c_1(K_M^{\pm 1})$ *is a monopole class.*

In this theorem, π is the quotient map $\pi : \mathbf{R} \times H^2 \to M$, a is a positive constant and g_H is a hyperbolic metric. The transversality of the moduli space is equivalent to the surjectivity of the map

$$T_{(A,\Phi)}(a, \varphi) = \left(c(i * da) - \varphi \otimes \Phi^* - \Phi \otimes \varphi^* + \frac{1}{2}(\langle \varphi, \Phi \rangle + \langle \Phi, \varphi \rangle)\,\mathrm{Id}_W, D_A\varphi + ic(a)\Phi\right)$$

which is the linearization of the 3-dimensional Seiberg-Witten equations. This surjectivity follows from direct computation ([7]).

It is known that $\mathcal{M}$ is a 0-dimensional compact oriented manifold. So the Seiberg-Witten invariant is defined by counting the points of the moduli space with sign ([2]). Therefore Theorem 1.2 implies that

$$SW(M, K_M^{\pm 1}) = \pm 1.$$

Notice that the metric independence of the invariant follows from well-known cobordism argument. In the case that $b_1(M) = 0$ or 1, we need a so-called wall crossing formula ([12]). Since this argument strays from our purpose, we omit it in this article.

As is well known, Seiberg-Witten-Floer homology and Seiberg-Witten invariant are closely related to each other. For example, by Proposition 3.3.12 in [12], we can compute the Seiberg-Witten invariant $SW(M, L)$ as the Euler characteristic of the $\mathbf{Z}_\ell$-graded Seiberg-Witten-Floer homology $\chi(HF_*(M, L; \mathbf{Z}_\ell))$ for an oriented closed 3-manifold with a fixed complex line bundle L associated with a $Spin(3)^c$ structure.

Our aim of this article is to compute the Seiberg-Witten-Floer homology of an oriented closed 3-manifold which carries the geometric structure $\mathbf{R} \times H^2$.

We are going to introduce the solutions to the 3-dimensional Seiberg-Witten equations as the critical points of the Chern-Simons-Dirac functional

$$C(A, \Phi) = \frac{1}{2}\int_M (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_M \langle \Phi, D_A \Phi \rangle \, dv.$$

Since this functional is not invariant under the gauge action, we add a suitable condition. This condition induces $\tilde{\mathcal{M}}$ which is a $\mathbf{Z}$-covering of the moduli space $\mathcal{M}$ of the solutions to the 3-dimensional Seiberg-Witten equations. By the observation of the structure of $\tilde{\mathcal{M}}$, we define Seiberg-Witten-Floer homology and compute it for our case as follows.

Main theorem. *Let M be an oriented closed 3-manifold carrying the geometric structure $\mathbf{R} \times H^2$ with the (anti-)canonical line bundle $L = K_M^{\pm 1}$. Suppose $b_1(M) > 1$. Then, the Seiberg-Witten-Floer homology of M is computed as follows.*

$$HF_k(M, L) \cong \begin{cases} \mathbf{Z} & (k = dm) \\ \{0\} & (k \neq dm), \end{cases}$$

where $d = \min_g |\langle c_1(L) \cup [g], [M] \rangle| (\neq 0)$, $[g]$ is the cohomology class of the form $\frac{1}{2\pi i} g^{-1}\, dg$ for $g \in \mathcal{G} = \Gamma(M; U(1))$ and $m \in \mathbf{Z}$.

Remark. (1) In Theorem 1.2 and Main Theorem, M has the structure of a Seifert bundle η over a base orbifold B with $e(\eta) = 0$ and $\chi(B) < 0$, where $e(\eta)$ is the orbifold Euler class and $\chi(B)$ is the Euler characteristic ([18]). Notice that B can be not only orientable but also non-orientable although M is oriented ([16]).

(2) Seiberg-Witten-Floer homology for the Seifert fibered homology spheres are computed in [14].

In general, the computation of d is not easy. However, in the case that the structure of M is simple, we can compute d as follows.

PROPOSITION 1.3. *Under the assumption of Main Theorem, let* $M = S^1 \times \Sigma$, *where* Σ *is a closed Riemann surface whose genus* $g_\Sigma \geq 2$. *Then,* $d = 2(g_\Sigma - 1)$.

COROLLARY 1.4.

$$HF_k(S^1 \times \Sigma, K^{\pm 1}_{S^1 \times \Sigma}) \cong \begin{cases} \mathbf{Z} & (k = 2(g_\Sigma - 1)m) \\ \{0\} & (k \neq 2(g_\Sigma - 1)m). \end{cases}$$

Remark. Seiberg-Witten-Floer homology of $S^1 \times \Sigma$ for other $Spin(3)^c$ structures is described with its algebraic aspects in [15].

2. Chern-Simons-Dirac functional

This section is mainly due to [12]. We are going to review the basic properties of the Chern-Simons-Dirac functional.

Let M be an oriented closed 3-manifold. Then there exists a $Spin(3)^c$ structure on M defining the principal $Spin(3)^c$-bundle P associated with the tangent bundle TM. Let W be the spinor bundle associated with P and $L = \det(W)$ be the determinant line bundle of W. For a unitary connection A on L and a section Φ of W, we define the Chern-Simons-Dirac functional as follows.

DEFINITION 2.1. The Chern-Simons-Dirac functional on the space $\mathscr{A} = \mathscr{C} \times \Gamma(W)$, where $\mathscr{C}$ is the space of unitary connections on L and $\Gamma(W)$ is the space of smooth sections of W, is defined as

$$C(A, \Phi) = \frac{1}{2}\int_M (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_M \langle \Phi, D_A \Phi \rangle \, dv.$$

Here, A_0 is a fixed smooth connection, F_A is the curvature form of A and D_A is the Dirac operator twisted with A, namely,

$$D_A : \Gamma(W) \xrightarrow{\nabla_A} \Gamma(T^*M \otimes W) \xrightarrow{c} \Gamma(W),$$

where ∇_A is the spin connection on W and $c : T^*M \to End(W)$ denotes the Clifford multiplication.

We can deduce the 3-dimensional Seiberg-Witten equations from the gradient of the functional C.

PROPOSITION 2.2.

$$\nabla C(A,\Phi) = \left(-*F_A + c^{-1}\left(\Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \,\mathrm{Id}_W\right), D_A\Phi\right),$$

where $*$ *is the Hodge star operator.*

Proof. Set $A = A_0 + ia$, $a \in \Omega^1(M)$. Computing directly, we obtain

$$\begin{aligned}
&\frac{d}{dt}\bigg|_{t=0} C(A + t\dot{A}, \Phi + t\dot{\Phi}) \\
&\quad = \frac{1}{2}\int_M i\dot{a} \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_M ia \wedge i\, d\dot{a} \\
&\qquad + \frac{1}{2}\int_M \langle \Phi, ic(\dot{a})\Phi\rangle\, dv + \int_M \mathrm{Re}\langle \dot{\Phi}, D_A\Phi\rangle\, dv \\
&\quad = \frac{1}{2}\int_M i\dot{a} \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_M i\dot{a} \wedge i\, da \\
&\qquad + \frac{1}{2}\int_M \langle \Phi, ic(\dot{a})\Phi\rangle\, dv + \int_M \mathrm{Re}\langle \dot{\Phi}, D_A\Phi\rangle\, dv \\
&\quad = \int_M i\dot{a} \wedge F_A + \frac{1}{2}\int_M \langle \Phi, ic(\dot{a})\Phi\rangle\, dv + \int_M \mathrm{Re}\langle \dot{\Phi}, D_A\Phi\rangle\, dv \\
&\quad = -\int_M \langle i\dot{a}, *F_A\rangle\, dv + \int_M \left\langle i\dot{a}, c^{-1}\left(\Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2\,\mathrm{Id}_W\right)\right\rangle dv \\
&\qquad + \int_M \mathrm{Re}\langle \dot{\Phi}, D_A\Phi\rangle\, dv. \\
&\quad = \int_M \left\langle \dot{A}, -*F_A + c^{-1}\left(\Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2\,\mathrm{Id}_W\right)\right\rangle dv \\
&\qquad + \int_M \mathrm{Re}\langle \dot{\Phi}, D_A\Phi\rangle\, dv. \qquad \square
\end{aligned}$$

It is clear that the critical points of C are exactly the solutions to the 3-dimensional Seiberg-Witten equations. Moreover we can show that the irreducible solution studied in [7] is a non-degenerate critical point. We call a solution (A,Φ) irreducible, when Φ is not identically zero.

PROPOSITION 2.3. *Let* (A,Φ) *be the irreducible solution to the* 3*-dimensional Seiberg-Witten equations with* $\nabla_A\Phi = 0$, *namely, a critical point of* C, *with the* (*anti-*)*canonical line bundle* $L = K_M^{\pm 1}$. *Then,* (A,Φ) *is a non-degenerate critical point of* C.

Proof. Let $H_{(A,\Phi)}$ be the Hessian operator of C at a critical point (A,Φ). Set $(A_s,\Phi_s)=(A,\Phi)+s(ia,\varphi)$. For

$$C(A_s,\Phi_s)=\frac{1}{2}\int_M (A_s-A_0)\wedge(F_{A_s}+F_{A_0})+\frac{1}{2}\int_M \langle \Phi_s, D_{A_s}\Phi_s\rangle\, dv,$$

we may collect the second order terms of s to compute the Hessian operator. The first term includes the term $\frac{s^2}{2}\int_M ia\wedge ida$. The second term includes the terms

$$\frac{s^2}{2}\left(\int_M \langle \varphi, ic(a)\Phi\rangle\, dv+\int_M \langle \Phi, ic(a)\varphi\rangle\, dv+\int_M \langle \varphi, D_A\varphi\rangle\, dv\right).$$

Therefore C includes the terms

$$\frac{s^2}{2}\left(\int_M ia\wedge\left(i\,da-*c^{-1}\left(\varphi\otimes\Phi^*+\Phi\otimes\varphi^*-\frac{1}{2}(\langle\varphi,\Phi\rangle+\langle\Phi,\varphi\rangle)\,\mathrm{Id}_W\right)\right)\right.$$
$$\left.+\int_M \langle\varphi, D_A\varphi+ic(a)\Phi\rangle\, dv\right).$$

Hence we obtain

$$\langle H_{(A,\Phi)}(a,\varphi),(a,\varphi)\rangle$$
$$=\left\langle ia, i\,da-*c^{-1}\left(\varphi\otimes\Phi^*+\Phi\otimes\varphi^*-\frac{1}{2}(\langle\varphi,\Phi\rangle+\langle\Phi,\varphi\rangle)\,\mathrm{Id}_W\right)\right\rangle$$
$$+\langle\varphi, D_A\varphi+ic(a)\Phi\rangle.$$

In [7], we have already shown that the linearization of the 3-dimensional Seiberg-Witten equations at the solution (A,Φ) with $\nabla_A\Phi=0$, namely,

$$T_{(A,\Phi)}(a,\varphi)$$
$$=\left(c(i*da)-\varphi\otimes\Phi^*-\Phi\otimes\varphi^*+\frac{1}{2}(\langle\varphi,\Phi\rangle+\langle\Phi,\varphi\rangle)\,\mathrm{Id}_W, D_A\varphi+ic(a)\Phi\right)$$

is surjective. It is obvious that $H_{(A,\Phi)}(a,\varphi)$ is equivalent to $T_{(A,\Phi)}(a,\varphi)$. Hence the critical point (A,Φ) is non-degenerate. □

Next we observe how C changes under the gauge action.

Proposition 2.4.

$$C(A+g^{-1}\,dg, g^{-1}\Phi)=C(A,\Phi)+4\pi^2\langle c_1(L)\cup[g],[M]\rangle,$$

where $[g]$ *is the cohomology class of the form* $\frac{1}{2\pi i}g^{-1}\,dg$.

Proof. By definition, we get

$$\begin{aligned} &C(A+g^{-1}\,dg, g^{-1}\Phi) \\ &= \frac{1}{2}\int_M (A+g^{-1}\,dg - A_0)\wedge(F_{A+g^{-1}\,dg}+F_{A_0}) + \frac{1}{2}\int_M \langle g^{-1}\Phi, D_{A+g^{-1}\,dg}(g^{-1}\Phi)\rangle\,dv \\ &= \frac{1}{2}\int_M (A-A_0)\wedge(F_A+F_{A_0}) + \frac{1}{2}\int_M g^{-1}\,dg\wedge(2F_{A_0}+i\,da) + \frac{1}{2}\int_M \langle\Phi, D_A\Phi\rangle\,dv \\ &= C(A,\Phi) + \int_M g^{-1}\,dg\wedge F_{A_0} = C(A,\Phi) + 4\pi^2\int_M \frac{i}{2\pi}F_{A_0}\wedge\frac{1}{2\pi i}g^{-1}\,dg \\ &= C(A,\Phi) + 4\pi^2\langle c_1(L)\cup[g],[M]\rangle. \qquad \square \end{aligned}$$

To make C invariant under the gauge action, we consider the space $\mathscr{B}_L = \mathscr{A}/\mathscr{G}_L$, where

$$\mathscr{G}_L = \{g\in\mathscr{G} \mid \langle c_1(L)\cup[g],[M]\rangle = 0\}$$

is a subgroup of $\mathscr{G}$. The next proposition implies that the space $\mathscr{B}_L$ is a covering space of $\mathscr{B} = \mathscr{A}/\mathscr{G}$ with fiber $\mathbf{Z}$.

PROPOSITION 2.5. *Let $\mathscr{G}_L$ be a subgroup of $\mathscr{G}$ given by*

$$\mathscr{G}_L = \{g\in\mathscr{G} \mid \langle c_1(L)\cup[g],[M]\rangle = 0\}.$$

Then, $\mathscr{G}/\mathscr{G}_L \cong \{0\}$ or $d\mathbf{Z}$, where $d = \min_g|\langle c_1(L)\cup[g],[M]\rangle|$, $g\in\mathscr{G}$, $g\notin\mathscr{G}_L$.

Proof. It is obvious that if $\mathscr{G} = \mathscr{G}_L$, then $\mathscr{G}/\mathscr{G}_L = \{0\}$. So we suppose $\mathscr{G}_L \subsetneq \mathscr{G}$ and consider the following sequence:

$$\mathscr{G} \xrightarrow{\lambda} H^1(M;\mathbf{Z}) \xrightarrow{\varphi} \mathbf{Z}, \quad \lambda(g) = \frac{1}{2\pi i}g^{-1}\,dg, \quad \varphi(\eta) = \langle c_1(L)\cup\eta,[M]\rangle.$$

Composing λ and φ, we obtain a homomorphism $\psi = \varphi\circ\lambda : \mathscr{G}\to\mathbf{Z}$ whose kernel is

$$\operatorname{Ker}\psi = \{g\in\mathscr{G} \mid \langle c_1(L)\cup\lambda(g),[M]\rangle = 0\} = \mathscr{G}_L.$$

Therefore we get $\mathscr{G}/\mathscr{G}_L \cong \operatorname{Im}\psi \subset \mathbf{Z}$. Since $\operatorname{Im}\psi$ is a nontrivial subgroup of $\mathbf{Z}$, we easily see that $\operatorname{Im}\psi = \{dm \mid m\in\mathbf{Z}\}$, where $d = \min_g|\langle c_1(L)\cup\lambda(g),[M]\rangle|$, $g\in\mathscr{G}$, $g\notin\mathscr{G}_L$. $\square$

Remark. Since $L = \det(W)$, $W = W_0\otimes L_1$, $W_0 = M\times\mathbf{C}^2$, we obtain $L = L_1^2$ so that $c_1(L) = c_1(L_1^2) = 2c_1(L_1)$. Therefore $\langle c_1(L)\cup\eta,[M]\rangle = 2\langle c_1(L_1)\cup\eta,[M]\rangle$, $\eta\in H^1(M;\mathbf{Z})$, which implies that d is an even number. We are going to examine this number in Section 4.

By the above proposition, we can consider a $\mathbf{Z}$-covering space $\tilde{\mathscr{M}} = \mathscr{S}/\mathscr{G}_L$ of $\mathscr{M} = \mathscr{S}/\mathscr{G}$. In the infinite dimensional Morse theory, we cannot always define Morse index. So we define relative Morse index

$$\mu(\tilde{a}) - \mu(\tilde{b}) \quad \tilde{a}, \tilde{b} \in \tilde{\mathscr{M}}$$

as the spectral flow of H along a path which connects two critical points $\tilde{a} = [A_{\tilde{a}}, \Phi_{\tilde{a}}]$ and $\tilde{b} = [A_{\tilde{b}}, \Phi_{\tilde{b}}]$. This is well defined as follows.

Proposition 2.6. *The spectral flow of the Hessian operator H of C around a loop in $\mathscr{B}_L$ is zero.*

Proof. For the proof of the statement, it is sufficient to consider a loop in $\mathscr{B}_L$, but we consider a loop in $\mathscr{B}$ for the later use.

Let $[A(t), \Phi(t)]_{t \in [0,1]}$ be a loop in $\mathscr{B}$ such that $(A(1), \Phi(1)) = (A(0) + g^{-1}\,dg, g^{-1}\Phi(0))$, $g \in \mathscr{G}$. Therefore we identify $(A(0), \Phi(0))$ with $(A(1), \Phi(1))$ and glue $M \times \{0\}$ to $M \times \{1\}$ so that we regard $M \times [0,1]$ as $M \times S^1$. Let $\hat{L}$ be a complex line bundle over $M \times S^1$ such that $\hat{L}|_{M \times \{t\}} = L$ and $\hat{A}$ be a unitary connection on $\hat{L}$ such that $\hat{A}|_{M \times \{t\}} = A(t)$. We assume that $\hat{A}$ satisfies so-called temporal gauge condition, namely, it has no dt component.

To compute the spectral flow of $H_{[A(t),\Phi(t)]}$ on the space $\mathscr{B} = \mathscr{A}/\mathscr{G}$, we consider the following $\mathscr{G}$-equivariant extention $\tilde{H}_{(A(t),\Phi(t))}$ on the space $\mathscr{A}$:

$$\tilde{H}_{(A,\Phi)} = \begin{pmatrix} H_{(A,\Phi)} & G_{(A,\Phi)} \\ G^*_{(A,\Phi)} & 0 \end{pmatrix},$$

where G and G^* are the infinitesimal gauge transformation and its adjoint with respect to the L^2-inner product:

$$G_{(A,\Phi)}(u) = (du, -iu\Phi), \quad G^*_{(A,\Phi)}(a, \varphi) = \delta a - i\,\mathrm{Im}\langle \Phi, \varphi \rangle.$$

Therefore we get

$$\mathrm{SF}(H_{[A(t),\Phi(t)]})_{t \in [0,1]} = \mathrm{SF}(\tilde{H}_{(A(t),\Phi(t))})_{t \in [0,1]}.$$

According to Theorem 7.4 in [1], the spectral flow along $(A(t), \Phi(t))_{t \in [0,1]}$ is computed as follows:

$$\mathrm{SF}(\tilde{H}_{(A(t),\Phi(t))})_{t \in [0,1]} = \mathrm{Index}\left(\frac{\partial}{\partial t} + \tilde{H}_{(A(t),\Phi(t))}\right).$$

Taking notice the forms of H and G^*, we obtain

$$\begin{aligned} \mathrm{Index}\left(\frac{\partial}{\partial t} + \tilde{H}_{(A(t),\Phi(t))}\right) &= \mathrm{Index}\left(\left(\frac{\partial}{\partial t} + *d\right) + \left(\frac{\partial}{\partial t} + D_A\right) + \delta\right) \\ &= \mathrm{Index}(d^+ + D_{\hat{A}} + \delta), \end{aligned}$$

where $d^+ : \Omega^1(M \times S^1) \to \Omega^{2+}(M \times S^1)$ and $D_{\hat{A}}$ is the twisted Dirac operator for $\Gamma(M \times S^1; \pi^* W)$. Notice that the natural projection $\pi : M \times S^1 \to M$ induces $\pi^* W \cong W^+ \cong W^-$, where $W^\pm$ are positive and negative spinor bundles over $M \times S^1$. For the 4-dimensional Seiberg-Witten theory, see [9], [13].

Since the Euler number $\chi(M) = 0$ and the first Pontrjagin class $p_1(M \times S^1) = 0$, the Euler number and the signature of $M \times S^1$ are

$$\chi(M \times S^1) = \chi(M) \cdot \chi(S^1) = 0, \quad \sigma(M \times S^1) = \frac{1}{3} \int_{M \times S^1} p_1(M \times S^1) = 0$$

so that $\text{Index}(d^+ + \delta) = \frac{1}{2}(\chi + \sigma) = 0$. Finally, we compute

$$\text{Index}(D_{\hat{A}}) = \int_{M \times S^1} \hat{\mathscr{A}}(M \times S^1) \cdot ch(\pi^* W)\Big|_{Vol} = \frac{1}{2} \int_{M \times S^1} c_1(\hat{L}) \wedge c_1(\hat{L}).$$

The first equality is due to Atiyah-Singer index theorem. Here, $\hat{\mathscr{A}}$ is the $\hat{\mathscr{A}}$-class and ch is the Chern character. Since $F_{\hat{A}} = \dfrac{dA}{dt} \wedge dt + F_{A(t)}$, we obtain $F_{\hat{A}} \wedge F_{\hat{A}} = 2F_{A(t)} \wedge \dfrac{dA}{dt} \wedge dt$. Therefore we get

$$\begin{aligned}
\frac{1}{2} \int_{M \times S^1} c_1(\hat{L}) \wedge c_1(\hat{L}) &= \frac{-1}{8\pi^2} \int_{M \times S^1} F_{\hat{A}} \wedge F_{\hat{A}} = \frac{-1}{4\pi^2} \int_{M \times S^1} F_{A(t)} \wedge \frac{dA}{dt} \wedge dt \\
&= \frac{-1}{4\pi^2} \int_M \left(F_{A(t)} \wedge \int_{S^1} dA(t) \right) = \frac{-1}{2\pi i} \int_M c_1(L) \wedge g^{-1}\, dg \\
&= -\int_M c_1(L) \wedge \frac{1}{2\pi i} g^{-1}\, dg = -\langle c_1(L) \cup [g], [M] \rangle.
\end{aligned}$$

If $g \in \mathscr{G}_L$, then $\langle c_1(L) \cup [g], [M] \rangle = 0$, namely, $SF(H_{[A(t), \Phi(t)]})_{t \in [0,1]} = 0$. This implies that relative Morse index $\mu(\tilde{a}) - \mu(\tilde{b})$ is independent of the choice of paths connectiong $\tilde{a}$ and $\tilde{b}$. Hence the spectral flow is well defined in $\mathscr{B}_L$. $\square$

Remark. In case $g \in \mathscr{G}$ and $g \notin \mathscr{G}_L$, we consider

$$-\langle c_1(L) \cup [g], [M] \rangle \equiv 0 \pmod{\ell}, \quad \text{where } \ell = g.c.d. |\langle c_1(L) \cup [g], [M] \rangle|.$$

Hence we can define relative Morse index by mod ℓ in $\mathscr{B}$.

3. Seiberg-Witten-Floer homology

By Proposition 2.6, for $\tilde{a}, \tilde{b} \in \tilde{\mathscr{M}}$, we can define relative Morse index $\mu(\tilde{a}) - \mu(\tilde{b})$ so that Floer complex is defined as follows.

Definition 3.1. *For a fixed $\tilde{a}_0 \in \tilde{\mathscr{M}}$, we define Floer complex FC_* as follows:*

$$FC_k = \{\tilde{a} \in \tilde{\mathscr{M}} \mid \mu(\tilde{a}) - \mu(\tilde{a}_0) = k\}.$$

Definition 3.2. The boundary operator ∂_k is defined as follows:

$$\partial_k : FC_k \to FC_{k-1}, \quad \partial_k \tilde{a} = \sum_{\mu(\tilde{b}) = \mu(\tilde{a}) - 1} n_{\tilde{a}\tilde{b}} \tilde{b}, \quad \tilde{b} \in \tilde{\mathscr{M}},$$

where $n_{\tilde{a}\tilde{b}}$ is given by counting the number of paths connecting $\tilde{a}$ and $\tilde{b}$ with sign.

It is shown that $\partial_k \circ \partial_{k+1} = 0$ in [3]. So we can define Seiberg-Witten-Floer homology as follows.

Definition 3.3. For (FC_*, ∂_*) and the fixed complex line bundle L associated with a $Spin(3)^c$-structure on M, we define Seiberg-Witten-Floer homology of M as follows:

$$HF_k(M, L) = \text{Ker } \partial_k / \text{Im } \partial_{k+1}.$$

Now we are in a position to prove Main Theorem.

Proof of Main Theorem. Let $\tilde{a} = [A_{\tilde{a}}, \Phi_{\tilde{a}}]$ be any point different from $\tilde{a}_0 = [A_{\tilde{a}_0}, \Phi_{\tilde{a}_0}]$ in $\tilde{\mathscr{M}}$. Since $\mathscr{M}$ consists of a single point by Theorem 1.2, we obtain $(A_{\tilde{a}}, \Phi_{\tilde{a}}) = (A_{\tilde{a}_0} + g^{-1}\, dg, g^{-1}\Phi_{\tilde{a}_0})$, $g \in \mathscr{G}$, $g \notin \mathscr{G}_L$. By the same argument of Proposition 2.5 and Proposition 2.6, we can compute the relative Morse index as follows.

$$\begin{aligned} \mu(\tilde{a}) - \mu(\tilde{a}_0) &= \mathrm{SF}(H_{[A(t), \Phi(t)]})_{t \in [0,1]} = \mathrm{Index}\left(\frac{\partial}{\partial t} + \tilde{H}_{(A(t), \Phi(t))}\right) \\ &= -\langle c_1(L) \cup [g], [M] \rangle = dm, \end{aligned}$$

where $(A(0), \Phi(0)) = (A_{\tilde{a}_0}, \Phi_{\tilde{a}_0})$, $(A(1), \Phi(1)) = (A_{\tilde{a}}, \Phi_{\tilde{a}})$, $d = \min_g |\langle c_1(L) \cup [g], [M] \rangle|$, $m \in \mathbf{Z} \setminus \{0\}$. Hence the Floer complex is given by

$$FC_k = \begin{cases} \mathbf{Z}\langle \tilde{a} \rangle & (k = dm) \\ \mathbf{Z}\langle \tilde{a}_0 \rangle & (k = 0) \\ \{0\} & (k \neq 0, dm). \end{cases}$$

By the remark of Proposition 2.5, d is an even number, hence we obtain the sequence

$$\cdots \longrightarrow 0 \xrightarrow{\partial_{dm+1}} FC_{dm} \xrightarrow{\partial_{dm}} 0 \longrightarrow \cdots$$

so that

$$HF_{dm}(M, L) = \text{Ker } \partial_{dm} / \text{Im } \partial_{dm+1} \cong \mathbf{Z}, \quad HF_k(M, L) \cong \{0\} \quad (k \neq dm).$$

Notice that this result also holds for the case $m = 0$. $\square$

Remark. As stated in the remark of Proposition 2.6, we can define relative Morse index by mod ℓ in $\mathscr{B}$. Consequently, we can define the $\mathbf{Z}_\ell$-graded Seiberg-

Witten-Floer homology $HF_k(M, L; \mathbf{Z}_\ell)$. In our case, it is easily computed because the moduli space $\mathscr{M}$ consists of a single point a_0. Hence we obtain

$$FC_k = \begin{cases} \mathbf{Z}_\ell\langle a_0\rangle & (k = 0) \\ \{0\} & (k \neq 0) \end{cases}$$

so that

$$HF_0(M, L; \mathbf{Z}_\ell) = \mathrm{Ker}\, \partial_0/\mathrm{Im}\, \partial_1 \cong \mathbf{Z}_\ell, \quad HF_k(M, L; \mathbf{Z}_\ell) = \{0\} \quad (k \neq 0).$$

This implies

$$\chi(HF_*(M, L; \mathbf{Z}_\ell)) = \sum_k (-1)^k \dim HF_k = (-1)^0 \dim HF_0 = 1.$$

On the other hand, we have already shown that $SW(M, L) = \pm 1$ in [7]. Taking the suitable orientation of the moduli space, we get $SW(M, L) = 1$. These values give the special case that the formula

$$\chi(HF_*(M, L; \mathbf{Z}_\ell)) = SW(M, L)$$

stated in [12] holds.

4. The computation of d

Suppose that $M = (\mathbf{R} \times H^2)/\Gamma$. To compute

$$d = \min_g |\langle c_1(L) \cup [g], [M]\rangle| = \min_g \left| \int_M c_1(L) \wedge \frac{1}{2\pi i} g^{-1}\, dg \right|, \quad g \in \mathscr{G},\ g \notin \mathscr{G}_L,$$

we consider $\tilde{g} \in \tilde{\mathscr{G}} = \Gamma(\mathbf{R} \times H^2; U(1))$ such that $\tilde{g} = g \circ \pi$, where π is the quotient map $\pi : \mathbf{R} \times H^2 \to M$. Therefore we get $g(\pi(\gamma(p))) = g(\pi(p))$, namely, $\tilde{g}(\gamma(p)) = \tilde{g}(p)$, for any $p = (t, z) \in \mathbf{R} \times H^2$ and $\gamma \in \Gamma$. This implies that $\tilde{g}$ is Γ-invariant. Conversely, the Γ-invariant $\tilde{g}$ induces $g \in \mathscr{G} = \Gamma(M; U(1))$.

On the other hand, from the exact sequence

$$0 \longrightarrow 2\pi\mathbf{Z} \longrightarrow \mathbf{R} \xrightarrow{e^{i(\cdot)}} U(1) \longrightarrow 0,$$

we obtain the cohomology exact sequence

$$\cdots \longrightarrow H^0(\mathbf{R} \times H^2; \mathbf{R}) \xrightarrow{e^{i(\cdot)}} H^0(\mathbf{R} \times H^2; U(1)) \longrightarrow H^1(\mathbf{R} \times H^2; 2\pi\mathbf{Z}) \longrightarrow \cdots.$$

Since $\mathbf{R} \times H^2$ is contractible, $H^1(\mathbf{R} \times H^2; 2\pi\mathbf{Z}) = \{0\}$ so that $e^{i(\cdot)}$ is surjective. Therefore for any $\tilde{g} \in \tilde{\mathscr{G}} = \Gamma(\mathbf{R} \times H^2; U(1))$, there exists $\tilde{u} \in \Gamma(\mathbf{R} \times H^2; \mathbf{R})$ such that $\tilde{g} = e^{i\tilde{u}}$. Since $\tilde{g}$ is Γ-invariant, we get $e^{i\tilde{u}(\gamma(t,z))} = e^{i\tilde{u}(t,z)}$, namely, $\tilde{u}(\gamma(t,z)) = \tilde{u}(t,z) + 2\pi k_{\tilde{u},\gamma}$ for some $k_{\tilde{u},\gamma} \in \mathbf{Z}$.

Now we are ready to prove Proposition 1.3 and Corollary 1.4.

Proof of Proposition 1.3 and Corollary 1.4. For $M = S^1 \times \Sigma$ with the geometric structure $\mathbf{R} \times H^2$, let $S^1 = \mathbf{R}/\Gamma_{\mathbf{R}}$ where $\Gamma_{\mathbf{R}} = \langle \gamma_1^n \,|\, \gamma_1 : t \mapsto t + 1\rangle \cong \mathbf{Z}$ and

$\Sigma = H^2/\Gamma_{H^2}$ where $\Gamma_{H^2} \subset PSL(2, \mathbf{R})$ acts properly discontinuously and without fixed points on H^2 and Σ is compact. From the compactness of Σ, as for H^2-component, it is sufficient to consider $\tilde{u}$ on a fundamental domain. Therefore for $\gamma_1 \in \Gamma_{\mathbf{R}}$ and $\gamma_2 \in \Gamma_{H^2}$, we assume that $\tilde{u}(\gamma(t,z)) = \tilde{u}(\gamma_1^n(t), \gamma_2(z)) = \tilde{u}(t+n, z)$. Hence we obtain that $\tilde{u}(t+n,z) - \tilde{u}(t,z) = 2n\pi k_{\tilde{u}}$, namely, $\tilde{u}(n,z) - \tilde{u}(0,z) = 2n\pi k_{\tilde{u}}$ which is independent of z. Here, for simplicity, we denote by $k_{\tilde{u}}$ the integer $k_{\tilde{u},\gamma}$. As a result, by using $\frac{1}{2\pi i}\tilde{g}^{-1}\,d\tilde{g} = \frac{1}{2\pi}\,d\tilde{u}$ instead of $\frac{1}{2\pi i}g^{-1}\,dg$, we obtain

$$\begin{aligned} d &= \min_{\tilde{u}} \left| \int_{S^1\times\Sigma} \pm c_1(K_\Sigma) \wedge \frac{1}{2\pi}\,d\tilde{u} \right| \\ &= \min_{\tilde{u}} \left| \int_{S^1} \frac{1}{2\pi}\,d\tilde{u} \int_\Sigma c_1(K_\Sigma) \right| = \min_{\tilde{u}} \left| \frac{1}{2\pi} \int_0^1 d\tilde{u} \cdot \chi(\Sigma) \right| \\ &= \min_{\tilde{u}} \left| \frac{1}{2\pi} (\tilde{u}(1,z) - \tilde{u}(0,z))(2 - 2g_\Sigma) \right| = \min_{\tilde{u}}\ 2(g_\Sigma - 1)|k_{\tilde{u}}|. \end{aligned}$$

If $k_{\tilde{u}} = 0$, then $d = 0$ which contradicts that $g \notin \mathscr{G}_L$. Hence $\min_{\tilde{u}}|k_{\tilde{u}}| \neq 0$. Moreover we can take $\min_{\tilde{u}}|k_{\tilde{u}}| = 1$ as follows. Define $\tilde{u}(t,z) = 2\pi t$. It is obvious that

$$\tilde{u}(t+1, z) = 2\pi(t+1) = 2\pi t + 2\pi = \tilde{u}(t,z) + 2\pi \cdot 1,$$

namely, $k_{\tilde{u}} = 1$. Therefore we obtain that $d = \min_{\tilde{u}} 2(g_\Sigma - 1)|k_{\tilde{u}}| = 2(g_\Sigma - 1)$ and the following representation:

$$HF_k(S^1 \times \Sigma, K_{S^1\times\Sigma}^{\pm 1}) \cong \begin{cases} \mathbf{Z} & (k = 2(g_\Sigma - 1)m) \\ \{0\} & (k \neq 2(g_\Sigma - 1)m). \end{cases} \qquad \square$$

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