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THE AXIOM OF SPHERES IN SEMI-RIEMANNIAN GEOMETRY WITH LIGHTLIKE SUBMANIFOLDS

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Abstract

Let $(\bar{M}, \bar{g})$ be a semi-Riemannain manifold and $(M, g, S(TM), S(TM^{\perp}))$ be its lightlike submanifold. We show that if $\bar{M}$ satisfies the axiom of r-planes & spheres then it is a real space form.

1. Introduction

The notion of axiom of planes for Riemannian manifolds was originally introduced by Elie Cartan [1] as: *A Riemannian manifold M of dimension $m \geq 3$ satisfies the axiom of r-planes if for each point p in M and for every r-dimensional linear subspace T of $T_p(M)$, there exists a r-dimensional totaly geodesic submanifold N of M containing p such that $T_p(N) = T$.* He proved the following:

Theorem (A). *A Riemannian manifold of dimension $m \geq 3$ satisfies the axiom of r-planes for some r, $2 \leq r < m$ if and only if it is a real space form.*

In 1971, D. S. Leung and K. Nomizu [5] generalized this notion by introducing the axiom of r-spheres as: *A Riemannian manifold M of dimension $m \geq 3$ satisfies the axiom of r-spheres if for each point p in M and for every linear subspace T of $T_p(M)$, there exists a r-dimensional totaly umbilical submanifold N of M with parallel mean curvature vector field such that $p \in N$ and $T_p(N) = T$.* They proved the following result to characterize a real space form:

Theorem (B). *A Riemannian manifold of dimension $m \geq 3$ is a real space form if and only if it satisfies the axiom of r-spheres for some r, $2 \leq r < m$.*

L. Graves and K. Nomizu [4] generalized these notions of axioms of planes and spheres for indefinite Riemannian manifolds.

The growing importance of lightlike submanifolds in Mathematical Physics, especially in relativity, motivated the authors to study lightlike submanifolds extensively. Here, in particular we studied the axioms of planes and spheres for

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semi-Riemannian manifolds with lightlike submaifolds. We propose these axioms for semi-Riemannian manifolds as follows:

Axiom of r-planes (r-spheres). *A semi-Riemannian manifold* $\overline{M}$ *of dimension* $m+n \geq 3$ *satisfies the axiom of r-planes (r-spheres) if for each point* $p \in \overline{M}$ *and for every r-dimensional linear subspace* T *of* $T_p(\overline{M})$ *there exists a r-dimensional totally geodesic lightlike submanifold (totally umbilical lightlike submanifold with parallel transversal curvature vector field)* M *such that* $p \in M$ *and* $T_p(M) = T$, $2 \leq r < m+n$ *and we shall show that*

Theorem (C). *Let* $(\overline{M}, \bar{g})$ *be a semi-Riemannain manifold of dimension* $m+n \geq 3$ *and* $(M, g, S(TM), S(TM^{\perp}))$ *be a lightlike submanifold of* $\overline{M}$. *If* $\overline{M}$ *satisfies the axiom of r-spheres,* $2 \leq r < m+n$, *then* $\overline{M}$ *is a real space form.*

Theorem (D). *Let* $(\overline{M}, \bar{g})$ *be a semi-Riemannain manifold of dimension* $m+n \geq 3$ *and* $(M, g, S(TM), S(TM^{\perp}))$ *be a lightlike submanifold of* $\overline{M}$. *If* $\overline{M}$ *satisfies the axiom of r-planes,* $2 \leq r < m+n$, *then* $\overline{M}$ *is a real space form.*

2. Lightlike submanifolds

The used notations and fundamental equations for lightlike submanifolds are refereed from the book [2] by Duggal and Bejancu. Let $(\overline{M}, \bar{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold where $m, n > 1$ and $\bar{g}$ be a semi-Riemannian metric on $\overline{M}$ with constant index $q \in \{1, 2, \dots, m+n-1\}$. Let M be a submanifold of $\overline{M}$ of codimensional n and said to be a lightlike submanifold if it admits a degenerate metric g induced from $\bar{g}$ whose radical distribution $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$ is of rank s, where the orthogonal complement $TM^{\perp}$ of tangent space TM is defined by

$$TM^{\perp} = \bigcup \{u \in T_x\overline{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\}. \tag{1}$$

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM and non-degenerate with respect to $\bar{g}$. Let $S(TM^{\perp})$ be a screen transversal vector bundle which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$.

Since $S(TM^{\perp})$ is a vector subbundle of the orthogonal complement $S(TM)^{\perp}$ of the $S(TM)$ in $T\overline{M}$, and $S(TM)^{\perp}$ and $S(TM^{\perp})$ are non-degenerate, therefore, we have

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is an orthogonal complement of the $S(TM^{\perp})$ in $S(TM)^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $\mathrm{Rad}(TM)$, there exists a local frame $\{N_i\}$ in the orthogonal complement of $S(TM^{\perp})$ in the $S(TM)^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a complementary vector bundle to $\mathrm{Rad}(TM)$ in $S(TM^{\perp})^{\perp}$ on which $\bar{g}$ vanishes and is called a lightlike transversal vector bundle of M which is locally spanned by $\{N_i\}$ and denoted by $\mathrm{ltr}(TM)$.

Consider the vector bundle which is complementary, but not orthogonal to TM in $T\overline{M}|_M$, that is,

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp}), \tag{2}$$

which is called as a transversal vector bundle of M. Then we obtain

$$T\overline{M}|_M = TM \oplus \mathrm{tr}(TM) = S(TM) \perp S(TM^{\perp}) \perp (\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)). \tag{3}$$

Hence we have a local quasi-orthogonal field $\{\xi_i, N_i, X_a, W_\alpha\}$ of frames of $T\overline{M}$ along M, where $\{X_a\}$ and $\{W_\alpha\}$ are orthonormal basis of $\Gamma(S(TM))|_U$ and $\Gamma(S(TM^{\perp}))|_U$ respectively, where U is a coordinate neighborhood of M.

Now a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is

(i) s-lightlike if $s < \min\{m, n\}$,
(ii) Coisotropic if $s = n < m$,
(iii) Isotropic if $s = m < n$,
(iv) Totally lightlike if $s = m = n$.

Suppose $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$. As TM and $\mathrm{tr}(TM)$ are complementary vector subbundles of $T\overline{M}|_M$, the Gauss and Weingarten formulas are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{4}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM),\ V \in \Gamma(\mathrm{tr}(TM)), \tag{5}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belongs to $\Gamma(TM)$, $\Gamma(\mathrm{tr}(TM))$, respectively. It follows that ∇ and ∇^t are linear connections on M and on the vector bundle $\mathrm{tr}(TM)$, respectively. According to the decomposition (2), considering the projection morphisms L and S of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{6}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{7}$$

$$\overline{\nabla}_X W = -A_W X + D^l(X, W) + \nabla^s_X W, \tag{8}$$

where $X, Y \in \Gamma(TM)$, $N \in \Gamma(\mathrm{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using (3)–(8), we have

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{9}$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \tag{10}$$

$$\bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X) \tag{11}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(\mathrm{ltr}(TM))$, $W \in \Gamma(S(TM^{\perp}))$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$. By using linear connections we have the following covariant derivatives as

$$(\nabla_X h^l)(Y, Z) = \nabla^l_X(h^l(Y, Z)) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \tag{12}$$

$$(\nabla_X h^s)(Y, Z) = \nabla^s_X(h^s(Y, Z)) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z). \tag{13}$$

Denoting $\overline{R}$ and R as the curvature tensor of $\overline{\nabla}$ and ∇, respectively, and by using (6)–(8) and (12)–(13), we obtain

$$\begin{aligned}(14)\qquad \bar{R}(X,Y)Z = {} & R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) + (\nabla_X h^s)(Y,Z) \\ & - (\nabla_Y h^s)(X,Z) + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) \\ & + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)).\end{aligned}$$

Let $(M, g, S(TM))$ be an m-dimensional coisotropic submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then we have, [2]

$$\begin{aligned}(15)\qquad \bar{R}(X,Y)Z = {} & R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X \\ & + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z).\end{aligned}$$

For an m-dimensional isotropic submanifold $(M, g, S(TM^\perp))$ of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$, we have

$$\begin{aligned}(16)\qquad \bar{R}(X,Y)Z = {} & R(X,Y)Z + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ & + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) \\ & + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z))\end{aligned}$$

and for a totally lightlike submanifold (M, g) of $(\overline{M}, \bar{g})$, we have

$$(17)\qquad \bar{R}(X,Y)Z = R(X,Y)Z.$$

Definition [3]. A lightlike submsnifold $(M, g, S(TM), S(TM^\perp))$ of a semi-Riemannian manifold $(\overline{M}, \bar{g})$ is totally umbilical if there is a smooth transversal vector field $\mathsf{H} \in \Gamma(\mathrm{tr}(TM))$ on M, called the transversal curvature vector field of M, such that for all $X, Y \in \Gamma(TM)$

$$(18)\qquad h(X,Y) = \mathsf{H} g(X,Y).$$

Using (6)–(8) and (18), it is easy to see that M is totally umbilical if and only if on each coordinate neighborhood u, there exist smooth vector fields $\mathsf{H}^l \in \Gamma(\mathrm{ltr}(TM))$ and $\mathsf{H}^s \in \Gamma(S(TM^\perp))$ such that

$$(19)\qquad h^l(X,Y) = \mathsf{H}^l g(X,Y), \quad D^l(X,W) = 0,$$

$$(20)\qquad h^s(X,Y) = \mathsf{H}^s g(X,Y).$$

The transversal curvature vector field of M is parallel (in $\mathrm{tr}(TM)$) if $\nabla^t_X \mathsf{H} = 0$ or equivalently

$$(21)\qquad \nabla^l_X \mathsf{H}^l = 0 \quad \text{and} \quad \nabla^s_X \mathsf{H}^s = 0.$$

Also, in case of totally umbilical submanifolds, we have [3]

$$(22)\qquad h^l(X,\xi) = 0, \quad h^s(X,\xi) = 0, \quad A^*_\xi \xi = 0, \quad A_W \xi = 0.$$

Then from (11) by replacing X by ξ we have

$$(23)\qquad D^s(\xi, N) = 0.$$

Definition [2]. A lightlike submanifold M of $\overline{M}$ is said to be totally geodesic if any geodesic of M with respect to an induced connections ∇ is a geodesic of $\overline{M}$ with respect to the Levi-Civita connection $\overline{\nabla}$.

Theorem (E) [2]. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of $(\overline{M}, \bar{g})$. Then the following assertions are equivalent*:

(i) *M is totally geodesic.*
(ii) *h^l and h^s vanish identically on M.*
(iii) *A^*_ξ vanishes identically on M, for any $\xi \in \Gamma(\mathrm{Rad}(TM))$, A_W is $\Gamma(\mathrm{Rad}(TM))$-valued for any $W \in \Gamma(S(TM))$ and $D^l(X, SV) = 0$ for any $X \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}(TM))$.*

3. Proof of theorems

To prove the theorems, the following lemmas are required.

Lemma (1). *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical submanifold of a semi-Riemannian manifold $\overline{M}$. Then $\nabla^l_X \mathsf{H}^l = 0$ if and only if $\nabla_X h^l = 0$.*

Proof. Let $X, Y, Z \in \Gamma(TM)$. Then by (12) and (19) we have

$$(24)\quad \begin{aligned}(\nabla_X h^l)(Y, Z) &= \nabla^l_X(g(Y, Z)\mathsf{H}^l) - g(\nabla_X Y, Z)\mathsf{H}^l - g(Y, \nabla_X Z)\mathsf{H}^l \\ &= Xg(Y, Z)\mathsf{H}^l + g(Y, Z)\nabla^l_X \mathsf{H}^l - g(\nabla_X Y, Z)\mathsf{H}^l - g(Y, \nabla_X Z)\mathsf{H}^l \\ &= (\nabla_X g)(Y, Z)\mathsf{H}^l + g(Y, Z)\nabla^l_X \mathsf{H}^l\end{aligned}$$

or

$$(25)\quad (\nabla_X h^l)(Y, Z) = g(Y, Z)\nabla^l_X \mathsf{H}^l.$$

Lemma (2). *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical submanifold of a semi-Riemannian manifold $\overline{M}$. Then $\nabla^s_X \mathsf{H}^s = 0$ if and only if $\nabla_X h^s = 0$.*

Proof. Let $X, Y, Z \in \Gamma(TM)$. Then by (13) and (20) we have

$$(26)\quad \begin{aligned}&(\nabla_X h^s)(Y, Z) \\ &= \nabla^s_X(g(Y, Z)\mathsf{H}^s) - g(\nabla_X Y, Z)\mathsf{H}^s - g(Y, \nabla_X Z)\mathsf{H}^s \\ &= Xg(Y, Z)\mathsf{H}^s + g(Y, Z)\nabla^s_X \mathsf{H}^s - g(\nabla_X Y, Z)\mathsf{H}^s - g(Y, \nabla_X Z)\mathsf{H}^s \\ &= (\nabla_X g)(Y, Z)\mathsf{H}^s + g(Y, Z)\nabla^s_X \mathsf{H}^s\end{aligned}$$

or

$$(27)\quad (\nabla_X h^s)(Y, Z) = g(Y, Z)\nabla^s_X \mathsf{H}^s.$$

LEMMA (3) [4]. *Let* $(\overline{M}, \overline{g})$ *be a semi-Riemannian manifold. If* $g(\overline{R}(X, Y)Z, X) = 0$ *for every orthonormal triple* $X, Y, Z \in \Gamma(TM)$, *then* $\overline{M}$ *has constant sectional curvature.*

Proof of theorem (C). At an arbitrary point $p \in M$ let X, ξ, V be orthonormal at p. Let T be a r-dimensional subspace of $T_p(\overline{M})|_M$ containing X and ξ, transversal to V. By the axiom there exists an r-dimensional totally umbilical lightlike submanifold M with parallel transversal curvature H such that $T_p(M) = T$. Since the transversal curvature vector field is parallel, i.e., $\nabla_X^t \mathsf{H} = 0$, by (21) and Lemma 1 and Lemma 2, we have $(\nabla_X h^l) = 0$ and $(\nabla_X h^s) = 0$ or in particular

(28)
$$(\nabla_X h^l)(X, X) = 0, \quad (\nabla_X h^l)(\xi, X) = 0, \quad (\nabla_X h^s)(X, X) = 0, \quad (\nabla_X h^s)(\xi, X) = 0.$$

Now form (14), for $X, \xi \in \Gamma(TM)$ the transversal form of $\overline{R}(X, \xi)X$ is as below

$$(29) \quad (\overline{R}(X, \xi)X)^N = (\nabla_X h^l)(\xi, X) - (\nabla_\xi h^l)(X, X) + (\nabla_X h^s)(\xi, X) - (\nabla_\xi h^s)(X, X) + D^l(X, h^s(\xi, X)) - D^l(\xi, h^s(X, X)) + D^s(X, h^l(\xi, X)) - D^s(\xi, h^l(X, X)).$$

Since M is totally umbilical therefore by using (19), (23) and (28) in (29), we obtain $(\overline{R}(X, \xi)X)^N = 0$. Hence $g(\overline{R}(X, \xi)X, V) = 0$ then the Theorem follows from the Lemma 3.

Proof of theorem (D). Since there exists a r-dimensional totally geodesic lightlike submanifold on M therefore from Theorem (E) and (29) we have $(\overline{R}(X, \xi)X)^N = 0$. Hence $g(\overline{R}(X, \xi)X, V) = 0$ then the Theorem follows from the Lemma 3.

Remark. From (15), (16) and (17) it is clear that a semi-Riemannian manifold with coisotropic, isotropic or totally lightlike submanifolds, is a real space form if it satisfies the axiom of r-planes and spheres.

REFERENCES

[1] E. CARTAN, Lecons sur la géométrie des espaces de Riemann, Paris, Gauthiers-Villars, 1946.

[2] K. L. DUGGAL AND A. BEJANCU, Lightlike submanifolds of semi-Riemannian manifolds and applications, Mathematics and its applications **364**, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[3] K. L. DUGGAL AND D. H. JIN, Totally umbilical lightlike submsnifolds, Kodai Mathematical Journal **26** (2003), 49–68.

[4] L. GRAVES AND K. NOMIZU, On sectional curvature of indefinite metric I, Math. Ann. **232** (1978), 267–272.

[5] D. S. LEUNG AND K. NOMIZU, The axiom of spheres in Riemannian geometry, J. of Differential Geometry **5** (1971), 487–489.

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