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A NEWTON-LIKE METHOD IN BANACH SPACES UNDER MILD DIFFERENTIABILITY CONDITIONS

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Abstract

The aim of this paper is to discuss the convergence of a third order Newton-like method for solving nonlinear equations $F(x) = 0$ in Banach spaces by using recurrence relations. The convergence of the method is established under the assumption that the second Fréchet derivative of F being ω-continuous given by $\|F''(x) - F''(y)\| \le \omega(\|x - y\|)$, $x, y \in \Omega$, where ω be a nondecreasing function on $\mathbf{R}_+$ and Ω any open set. This ω-continuity condition is milder than the usual Lipschitz/Hölder continuity condition. To get a priori error bounds, a family of recurrence relations based on two parameters depending on the operator F is also derived. Two numerical examples are worked out to show that the method is successful even in cases where Lipschitz/Hölder continuity condition fails but ω-continuity condition is satisfied. In comparison to the work of Wu and Zhao [15], our method is more general and leads to better results.

1. Introduction

Let $F : \Omega \subseteq \mathbf{X} \to \mathbf{Y}$ be a nonlinear operator on an open convex subset Ω of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. The most well known second order iterative methods used to solve $F(x) = 0$ are Newton's method and it's variants. The Kantorovich theorem [10, 14], provides sufficient conditions to ensure convergence of these methods. A lot of research [4, 5, 8, 15] has been carried out to provide improvements in these methods, their applications and convergence. Third order one point iterative methods [2, 4, 5] are used in many applications. They can also be used in stiff systems [11], where a quick convergence is required. A very restrictive condition of one point iteration of order N is that they depend explicitly on the first $N - 1$ derivatives of F. All these higher order derivatives are very difficult to compute. Multi point third order iterative methods [6, 7, 12, 15, 16] use information at a number of points have also gained importance recently.

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Wu and Zhao [15] have studied the convergence of a third order Newton-like iterative method for solving $F(x) = 0$. This method involves only the value of F and it's first derivative F' is given by

$$\left.\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - \left(\frac{F'(x_n) + F'(y_n)}{2}\right)^{-1} F(x_n) \end{aligned}\right\} \tag{1}$$

The convergence of this method is established by using majorizing functions. They studied it's semilocal convergence under the assumption that F'' is Lipschitz continuous. We shall consider iterative method (1) and establish it's convergence based on recurrence relations under weaker continuity conditions. Hernández and Salanova [9], Hernández [7] studied the convergence of Chebyshev's method and second derivative free version Chebyshev's method by using recurrence relations under Hölder continuity condition on F''. Ye and Li [16], studied the convergence of Euler-Halley method under similar conditions. However, the Lipschitz/Hölder continuity condition on the second derivative of F may be violated in many problems.

Example. Consider the following nonlinear integral equation of mixed type [3]:

$$F(x)(s) = x(s) + \sum_{i=1}^{m} \int_a^b k_i(s,t) l_i(x(t))\, dt - u(s), \quad s \in [a,b]$$

where $-\infty < a < b < \infty$, u, l_i, and k_i, for $i = 1, 2, \ldots, m$ are known functions and x is a continuous function.

If $l_i''(x(t))$ is (L_i, p_i)-Hölder continuous in Ω, $L_i \geq 0$, $p_i \in (0,1]$ for $i = 1, 2, \ldots, m$, then we have

$$\|F''(x) - F''(y)\| \leq \sum_{i=1}^{m} L_i \|x - y\|^{p_i}, \quad x, y \in \Omega.$$

Here, F'' is not Hölder continuous, when sup-norm is used.

Recently, Ezquerro and Hernández [2] and Hernández and Romero [8] considered the more generalized condition

$$\|F''(x) - F''(y)\| \leq \omega(\|x - y\|), \quad x, y \in \Omega. \tag{2}$$

where $\omega(x)$ is a nondecreasing continuous real function for $x > 0$, such that $\omega(0) \geq 0$, on F'' to study the semilocal convergence of Halley's method and a family of third order iterative method respectively.

In this paper, the convergence of a third order Newton-like method (1) for solving nonlinear equations $F(x) = 0$ is discussed. The convergence of the method is established by using recurrence relations under the assumption that the second Fréchet derivative of F satisfy the ω-continuity condition given by (2). This ω-continuity condition is milder than the usual Lipschitz/Hölder continuity condition. A family of recurrence relations based on two constants depending on the operator F is derived to establish a priori error bounds. Two numerical examples are worked out to show that the method is successful even in cases where Lipschitz/Hölder continuity condition fails. In comparison to the work of Wu and Zhao [15], our method is more general and leads to better results.

The paper is organized as follows. In section 2, three real sequences are constructed and their properties are studied. The recurrence relations for our third order Newton-like method are derived in section 3. The convergence analysis based on these recurrence relations of the method is given in section 4. In section 5, some numerical examples are worked out and the results obtained are compared with the results of [15] for a particular case. Finally, conclusions form the section 6.

2. Construction of real sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ and their properties

In this section, we shall discuss the construction of three real sequences and study their properties in order to study the convergence of the iterative method (1) for solving the nonlinear operator equation

$$F(x) = 0. \tag{3}$$

Let F be twice Fréchet differentiable operator in Ω and $\mathscr{BL}(\mathbf{Y}, \mathbf{X})$ be the set of bounded linear operators from $\mathbf{Y}$ into $\mathbf{X}$. It is assumed that $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{BL}(\mathbf{Y}, \mathbf{X})$ exists at some point $x_0 \in \Omega$ and the following conditions hold on F.

(4)
- C1. $\|F'(x_0)^{-1}\| \le \beta$,
- C2. $\|F'(x_0)^{-1}F(x_0)\| \le \eta$,
- C3. $\|F''(x)\| \le M, \quad \forall x \in \Omega$,
- C4. $\|F''(x) - F''(y)\| \le \omega(\|x - y\|), \quad \forall x, y \in \Omega$, where $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous and non-decreasing function such that $\omega(0) \ge 0$,
- C5. There exists a continuous and non-decreasing function $h : [0, 1] \to \mathbf{R}_+$ such that, $\omega(tx) \le h(t)\omega(x)$, with $t \in [0, 1]$ and $x \in \mathbf{R}_+$.

Note that the above condition C5 of (4) does not involve any restriction, since as a consequence of ω is non-decreasing function, there always exists a function h such that $h(t) = 1$. We can consider $h(t) = \sup_{x>0} \omega(tx)/\omega(x)$ to sharpen the error bounds for a particular case. The above condition C4 of (4) is milder than the Lipschitz/Hölder continuity condition as this condition reduced to Lipschitz and Hölder condition, if we consider $\omega(x) = Nx$ and $\omega(x) = Nx^p$, $p \in (0, 1]$, respectively.

Let $a_0 = M\beta\eta$, $b_0 = \beta\eta\omega(\eta)$ be two parameters. For $n \in \mathbf{Z}_+$, let us define three real sequences

$$c_n = f(a_n)g(a_n, b_n), \quad a_{n+1} = a_n f(a_n)c_n, \quad b_{n+1} = b_n f(a_n)c_n h(c_n) \tag{5}$$

where

$$f(x) = (2-x)/(2-3x) \tag{6}$$

$$g(x,y) = \left[\frac{x(4-x)(3-x)}{(2-x)^2} + Ky\right], \quad K \in \mathbf{R}_+ \tag{7}$$

Let r_0 be the smallest positive zero of the polynomial $r(x) = -x^3 + 16x^2 - 24x + 4$, then $r_0 = 0.1905960896\ldots$. We shall now establish a number of properties of the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$. This will require the following lemma.

LEMMA 1. *Let f and g be functions defined by equations* (6) *and* (7) *respectively, then for* $x \in (0, r_0]$

(i) *f is increasing and* $f(x) > 1$,

(ii) *g is increasing in both arguments for* $y > 0$,

(iii) $f(\delta x) < f(x)$ *and* $g(\delta x, \delta y) < \delta g(x,y)$, *for* $\delta \in (0,1)$.

Proof. The proof is simple and hence omitted.

LEMMA 2. *Let f and g be functions defined by equations* (6) *and* (7) *respectively and* $h(t) \le 1$, $\forall t \in [0,1]$. *Let us define a function*

$$\Phi(x) = \frac{(-x^3 + 16x^2 - 24x + 4)}{K(2-x)^2} \tag{8}$$

If $a_0 \in (0, r_0]$ *and* $0 \le b_0 \le \Phi(a_0)$, *then*

(i) $c_n f(a_n) \le 1$,

(ii) $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *are decreasing and* $a_n < 1$, $c_n < 1$ $\forall n$.

Proof. Now from definitions of f and g, we have

$$c_n f(a_n) = f(a_n)^2 g(a_n, b_n) \le 1$$

$$\text{or iff,} \quad \left(\frac{2-a_n}{2-3a_n}\right)^2 \left(\frac{a_n(4-a_n)(3-a_n)}{(2-a_n)^2} + Kb_n\right) \le 1$$

$$\text{or iff,} \quad b_n \le \frac{(-a_n^3 + 16a_n^2 - 24a_n + 4)}{K(2-a_n)^2} = \Phi(a_n)$$

Induction will be used to prove the lemma. Now for $0 < a_0 \le r_0$, $0 \le b_0 \le \Phi(a_0)$, from above one can easily conclude that $c_0 f(a_0) \le 1$. Thus from equation (5), we obtain

$$a_1 = a_0 f(a_0) c_0 \le a_0 < 1,$$

and as $f(x) > 1$ in $(0, r_0]$ and $h(t) \le 1$, $\forall t \in [0,1]$, we have

$$b_1 = b_0 f(a_0) c_0 h(c_0) \le b_0 f(a_0) c_0 \le b_0$$

Hence

$$c_1 = f(a_1) g(a_1, b_1) \le f(a_0) g(a_0, b_0) = c_0 < c_0 f(a_0) \le 1.$$

Let the statements hold for $n = k$. Then since f and g are increasing functions, we get

$$a_{k+1} = a_k f(a_k) c_k \le a_k < 1.$$

Also as $f(x) > 1$ in $(0, r_0]$ and $h(t) \le 1$, $\forall t \in [0,1]$, we have

$$b_{k+1} = b_k f(a_k) c_k h(c_k) \le b_k f(a_k) c_k \le b_k.$$

and

$$c_{k+1} = f(a_{k+1}) g(a_{k+1}, b_{k+1}) \le f(a_k) g(a_k, b_k) = c_k < 1.$$

Also

$$c_{k+1} f(a_{k+1}) = f(a_{k+1})^2 g(a_{k+1}, b_{k+1}) \le f(a_k)^2 g(a_k, b_k) = c_k f(a_k) \le 1$$

Hence, by induction it holds for all n. This proves the Lemma 2. □

LEMMA 3. *Let us suppose* $a_0 \in (0, r_0)$ *and* $0 < b_0 < \Phi(a_0)$. *Define* $\gamma = a_1/a_0$, *then for* $n \ge 1$ *we have,*

(i) $a_n \le \gamma^{2^{n-1}} a_{n-1} \le \gamma^{2^n - 1} a_0$, *where inequality strictly hold for* $n \ge 2$,

(ii) $b_n < \gamma^{2^{n-1}} b_{n-1} < \gamma^{2^n - 1} b_0$,

(iii) $c_n < \gamma^{2^n} / f(a_0)$

Proof. We will prove (i) and (ii) by induction. Since $a_1 = \gamma a_0$ and $a_1 < a_0$ from lemma 2(i), we get $\gamma < 1$. By lemma 1(i) and lemma 2(i),

$$b_1 = b_0 f(a_0) c_0 h(c_0) = b_0 f(a_0)^2 g(a_0, b_0) h(c_0) \le f(a_0)^2 g(a_0, b_0) b_0 = \gamma b_0$$

Suppose (i) and (ii) hold for $n = k$, then

$$\begin{aligned} a_{k+1} &= a_k f(a_k) c_k = a_k f(a_k)^2 g(a_k, b_k) \\ &< \gamma^{2^{k-1}} a_{k-1} f(\gamma^{2^{k-1}} a_{k-1})^2 g(\gamma^{2^{k-1}} a_{k-1}, \gamma^{2^{k-1}} b_{k-1}) \\ &< \gamma^{2^{k-1}} a_{k-1} f(a_{k-1})^2 \gamma^{2^{k-1}} g(a_{k-1}, b_{k-1}) = \gamma^{2^k} a_k \end{aligned}$$

Also as $f(x) > 1$ in $(0, r_0)$,

$$b_{k+1} = b_k f(a_k) c_k h(c_k) \le b_k f(a_k) c_k = b_k \frac{a_{k+1}}{a_k} < \gamma^{2^k} b_k$$

So

$$a_{k+1} < \gamma^{2^k} a_k < \gamma^{2^k} \gamma^{2^{k-1}} \cdots \gamma^{2^0} a_0 = \gamma^{2^{k+1}-1} a_0.$$

and

$$b_{k+1} < \gamma^{2^k} b_k < \gamma^{2^k} \gamma^{2^{k-1}} \cdots \gamma^{2^0} b_0 = \gamma^{2^{k+1}-1} b_0.$$

Hence by induction the statements (i) and (ii) hold true.

Again

$$\begin{aligned} c_n &= f(a_n) g(a_n, b_n) < f(\gamma^{2^n-1} a_0) g(\gamma^{2^n-1} a_0, \gamma^{2^n-1} b_0) < \gamma^{2^n-1} f(a_0) g(a_0, b_0) \\ &= \gamma^{2^n} / f(a_0) \end{aligned}$$

as $\gamma = a_1/a_0 = f(a_0)^2 g(a_0, b_0)$. Hence (iii) holds. □

3. Recurrence relations

In this section, we shall derive the recurrence relations for the iterative method given by (1) under the assumptions given in previous section.

Now

$$M \|\Gamma_0\| \, \|y_0 - x_0\| \leq M \beta \eta = a_0$$

By our assumption, y_0 exists as $\Gamma_0 = F'(x_0)^{-1}$ exists. Thus

$$\left\| I - \Gamma_0 \frac{F'(y_0) + F'(x_0)}{2} \right\| = \left\| \Gamma_0 \frac{F'(x_0) - F'(y_0)}{2} \right\| \leq \frac{1}{2} M \|\Gamma_0\| \, \|x_0 - y_0\| \leq \frac{a_0}{2} < 1$$

Hence by Banach's theorem [10], $\left(\frac{F'(y_0) + F'(x_0)}{2} \right)^{-1} F'(x_0)$ exists and

$$\left\| \left(\frac{F'(y_0) + F'(x_0)}{2} \right)^{-1} F'(x_0) \right\| \leq \frac{1}{1 - \frac{1}{2} M \|\Gamma_0\| \, \|x_0 - y_0\|} \leq \frac{2}{2 - a_0}$$

Thus

$$\|x_1 - x_0\| \leq \frac{2}{2 - a_0} \|y_0 - x_0\|$$

and

$$\|x_1 - y_0\| \leq \|x_1 - x_0\| + \|x_0 - y_0\| \leq \frac{4 - a_0}{2 - a_0} \|y_0 - x_0\|$$

Also

$$\|\Gamma_0\| \, \|y_0 - x_0\| \omega(\|y_0 - x_0\|) \leq \beta \eta \omega(\eta) = b_0$$

We shall now establish the following inequalities for $n \geq 1$:

$$
(9)\quad \left.\begin{array}{ll}
\text{(I)} & \|\Gamma_n\| = \|F'(x_n)^{-1}\| \le f(a_{n-1})\|\Gamma_{n-1}\|, \\
\text{(II)} & \|y_n - x_n\| = \|\Gamma_n F(x_n)\| \le c_{n-1}\|y_{n-1} - x_{n-1}\|, \\
\text{(III)} & M\|\Gamma_n\|\,\|y_n - x_n\| \le a_n, \\
\text{(IV)} & \|x_{n+1} - x_n\| \le 2/(2 - a_n)\|y_n - x_n\|, \\
\text{(V)} & \|x_{n+1} - y_n\| \le (4 - a_n)/(2 - a_n)\|y_n - x_n\|, \\
\text{(VI)} & \|\Gamma_n\|\,\|y_n - x_n\|\omega(\|y_n - x_n\|) \le b_n,
\end{array}\right\}
$$

The following lemma will be used for this purpose.

LEMMA 4. *Let the sequences* $\{x_n\}$ *and* $\{y_n\}$ *are generated by* (1). *Then* $\forall n \in \mathbf{Z}_+$, *we have*

$$
(10)\quad \begin{aligned}
F(x_{n+1}) = & \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\,dt(x_{n+1} - y_n)^2 \\
& + \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))\,dt(y_n - x_n)(x_{n+1} - y_n) \\
& - \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))\,dt(y_n - x_n)^2 \\
& + \int_0^1 F''(x_n + t(y_n - x_n))(1-t)\,dt(y_n - x_n)^2
\end{aligned}
$$

Using Taylor's method one can easily prove the above lemma.

The conditions (I)–(VI) can be proved by induction. Assume that $x_1 \in \Omega$, then

$$
\|I - \Gamma_0 F'(x_1)\| \le M\|\Gamma_0\|\,\|x_0 - x_1\| \le M\beta\frac{2}{2 - a_0}\|y_0 - x_0\| \le \frac{2a_0}{2 - a_0} < 1
$$

Hence, by Banach's theorem, $\Gamma_1 = F'(x_1)^{-1}$ exists and

$$
(11)\quad \|\Gamma_1\| \le \frac{\|\Gamma_0\|}{1 - M\|\Gamma_0\|\,\|x_0 - x_1\|} \le \frac{\|\Gamma_0\|}{1 - 2a_0/(2 - a_0)} = f(a_0)\|\Gamma_0\|
$$

Also

$$
\begin{aligned}
& \left\| \int_0^1 F''(x_0 + t(y_0 - x_0))(1-t)\,dt(y_0 - x_0)^2 \right. \\
& \qquad \left. - \frac{1}{2}\int_0^1 F''(x_0 + t(y_0 - x_0))\,dt(y_0 - x_0)^2 \right\| \\
& \quad \le \left\| \int_0^1 F''(x_0 + t(y_0 - x_0))(1-t)\,dt(y_0 - x_0)^2 - \frac{1}{2}F''(x_0)(y_0 - x_0)^2 \right\| \\
& \qquad + \left\| \frac{1}{2}\int_0^1 F''(x_0 + t(y_0 - x_0))\,dt(y_0 - x_0)^2 - \frac{1}{2}F''(x_0)(y_0 - x_0)^2 \right\|
\end{aligned}
$$

$$\leq \left\| \int_0^1 [F''(x_0 + t(y_0 - x_0)) - F''(x_0)](1-t)\, dt \right\| \|y_0 - x_0\|^2$$

$$+ \left\| \frac{1}{2} \int_0^1 [F''(x_0 + t(y_0 - x_0)) - F''(x_0)]\, dt \right\| \|y_0 - x_0\|^2$$

$$\leq \int_0^1 \omega(t\|y_0 - x_0\|)(1-t)\, dt \|y_0 - x_0\|^2 + \frac{1}{2} \int_0^1 \omega(t\|y_0 - x_0\|)\, dt \|y_0 - x_0\|^2$$

$$\leq \int_0^1 h(t)(1-t)\, dt \omega(\|y_0 - x_0\|) \|y_0 - x_0\|^2$$

$$+ \frac{1}{2} \int_0^1 h(t)\, dt \omega(\|y_0 - x_0\|) \|y_0 - x_0\|^2$$

$$= K\omega(\|y_0 - x_0\|) \|y_0 - x_0\|^2$$

where $K = \int_0^1 h(t)(1-t)\, dt + \frac{1}{2}\int_0^1 h(t)\, dt$

Hence equation (10) leads us,

$$\|F(x_1)\| \leq \frac{M}{2}\|x_1 - y_0\|^2 + \frac{M}{2}\|y_0 - x_0\|\,\|x_1 - y_0\| + K\omega(\|y_0 - x_0\|)\|y_0 - x_0\|^2$$
$$\leq M\frac{(4-a_0)(3-a_0)}{(2-a_0)^2}\|y_0 - x_0\|^2 + K\omega(\|y_0 - x_0\|)\|y_0 - x_0\|^2$$

From this, we get

$$\begin{aligned}
\|\Gamma_1 F(x_1)\| &\leq \|\Gamma_1\|\,\|F(x_1)\| \\
&\leq f(a_0)\|\Gamma_0\| \left[M\frac{(4-a_0)(3-a_0)}{(2-a_0)^2}\|y_0 - x_0\|^2 \right. \\
&\qquad \left. + K\omega(\|y_0 - x_0\|)\|y_0 - x_0\|^2 \right] \\
&\leq f(a_0)\left[\frac{a_0(4-a_0)(3-a_0)}{(2-a_0)^2} + Kb_0\right]\|y_0 - x_0\| \\
&= f(a_0)g(a_0, b_0)\|y_0 - x_0\| = c_0\|y_0 - x_0\|
\end{aligned} \tag{12}$$

Now

$$M\|\Gamma_1\|\,\|y_1 - x_1\| \leq M\|\Gamma_0\| f(a_0)c_0\|y_0 - x_0\| \leq a_0 f(a_0)c_0 = a_1 \tag{13}$$

As $\Gamma_1 = F'(x_1)^{-1}$ exists, so y_1 exists. Hence,

$$\left\| I - \Gamma_1 \frac{F'(y_1) + F'(x_1)}{2} \right\| \leq \frac{1}{2}M\|\Gamma_1\|\,\|x_1 - y_1\| \leq \frac{a_1}{2} < 1$$

So by Banach's theorem, $\left(\frac{F'(y_1)+F'(x_1)}{2}\right)^{-1}F'(x_1)$ exists and

$$\left\|\left(\frac{F'(y_1)+F'(x_1)}{2}\right)^{-1}F'(x_1)\right\| \le \frac{1}{1-\frac{1}{2}M\|\Gamma_1\|\,\|x_1-y_1\|} \le \frac{2}{2-a_1}$$

Thus,

$$\|x_2-x_1\| \le \frac{2}{2-a_1}\|y_1-x_1\| \tag{14}$$

and

$$\|x_2-y_1\| \le \|x_2-x_1\|+\|x_1-y_1\| \le \frac{4-a_1}{2-a_1}\|y_1-x_1\| \tag{15}$$

Again

$$\begin{aligned}\|\Gamma_1\|\,\|y_1-x_1\|\omega(\|y_1-x_1\|) &\le \|\Gamma_0\|f(a_0)c_0\|y_0-x_0\|\omega(c_0\|y_0-x_0\|)\\ &\le f(a_0)c_0h(c_0)\|\Gamma_0\|\,\|y_0-x_0\|\omega(\|y_0-x_0\|)\\ &\le b_0f(a_0)c_0h(c_0)=b_1\end{aligned} \tag{16}$$

For $n=1$ the conditions (I)–(VI) follows from equations (11)–(16) respectively. Now considering these conditions hold for $n=k$ and $x_k \in \Omega$, proceeding similarly one can easily prove that these conditions also hold for $n=k+1$. Hence, by induction they hold for all n.

4. Convergence analysis

The following theorem will establish the convergence of the sequence $\{x_n\}$ and give a priori error bounds for it. Let us denote $\gamma = a_1/a_0$, $\Delta = 1/f(a_0)$, $R = \frac{2}{(2-a_0)(1-\gamma\Delta)}$, $\mathscr{B}(x_0,R\eta)=\{x\in \mathbf{X} : \|x-x_0\|<R\eta\}$ and $\bar{\mathscr{B}}(x_0,R\eta)=\{x\in \mathbf{X} : \|x-x_0\|\le R\eta\}$.

Theorem 1. *Let F satisfy the conditions given in* (4). *Suppose* $0<a_0\le r_0$ *and* $0\le b_0\le \Phi(a_0)$ *hold, where r_0 be the smallest positive zero of the polynomial* $r(x)=-x^3+16x^2-24x+4$ *and $\Phi(x)$ is the function defined by equation* (8). *If* $\bar{\mathscr{B}}(x_0,R\eta)\subseteq\Omega$, *then starting from x_0, the sequence $\{x_n\}$ defined by method* (1) *converges to a solution x^* of the equation* (3) *with R-order at least* 2 *and x_n, y_n and x^* belonging $\bar{\mathscr{B}}(x_0,R\eta)$ where x^* is the unique solution in* $\mathscr{B}(x_0,2/(M\beta)-R\eta)\cap\Omega$.

Furthermore, the error bounds on x^* is given by

$$\|x^*-x_n\| \le \frac{2\gamma^{2^n-1}}{(2-\gamma^{2^n-1}a_0)}\frac{\Delta^n}{(1-\gamma^{2^n}\Delta)}\eta \tag{17}$$

Proof. It is sufficient to show that $\{x_n\}$ is a Cauchy sequence in order to prove $\{x_n\}$ is convergent.

We have $b_0 = \Phi(a_0) = 0$ and $c_0 f(a_0) = 1$ for $a_0 = r_0$. Hence from (5), $a_n = a_{n-1} = \cdots = a_0$, $c_n = c_{n-1} = \cdots = c_0$ and $b_n = b_{n-1} = \cdots = b_0 = 0$.

Now from (9), we have

$$\|y_n - x_n\| \le c_{n-1}\|y_{n-1} - x_{n-1}\| = c_0\|y_{n-1} - x_{n-1}\| \le \cdots \le c_0^n\|y_0 - x_0\| = \Delta^n\eta$$

and

$$\|x_{n+1} - x_n\| \le \frac{2}{2-a_n}\|y_n - x_n\| \le \frac{2}{2-a_0}\Delta^n\eta$$

Thus

$$\begin{aligned}\|x_{m+n} - x_m\| &\le \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{m+1} - x_m\| \\ &\le \frac{2}{2-a_0}[\Delta^{m+n-1} + \cdots + \Delta^m]\eta = \frac{2\Delta^m}{2-a_0}\left(\frac{1-\Delta^n}{1-\Delta}\right)\eta\end{aligned} \tag{18}$$

Hence if we take $m = 0$, $x_n \in \bar{\mathscr{B}}(x_0, R\eta)$. Similarly one can prove that $y_n \in \bar{\mathscr{B}}(x_0, R\eta)$. Also as $\Delta = 1/f(a_0) < 1$, from (18) one can conclude that $\{x_n\}$ is a Cauchy sequence.

Let $0 < a_0 < r_0$ and $b_0 < \Phi(a_0)$. Now from (9) and lemma 3(iii), for $n \ge 1$, we have

$$\begin{aligned}\|y_n - x_n\| &\le c_{n-1}\|y_{n-1} - x_{n-1}\| \le \cdots \le \|y_0 - x_0\|\prod_{j=0}^{n-1} c_j < \prod_{j=0}^{n-1}(\gamma^{2^j}\Delta)\eta \\ &= \gamma^{2^n-1}\Delta^n\eta,\end{aligned}$$

where $\gamma = a_1/a_0 < 1$ and $\Delta = 1/f(a_0) < 1$. Hence

$$\begin{aligned}\|x_{m+n} - x_m\| &\le \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{m+1} - x_m\| \\ &\le \frac{2}{2-a_{m+n-1}}\|y_{m+n-1} - x_{m+n-1}\| + \cdots + \frac{2}{2-a_m}\|y_m - x_m\| \\ &< \frac{2}{2-a_{m+n-1}}\gamma^{2^{m+n-1}-1}\Delta^{m+n-1}\eta + \cdots + \frac{2}{2-a_m}\gamma^{2^m-1}\Delta^m\eta \\ &< \frac{2\Delta^m}{2-a_m}[\gamma^{2^{m+n-1}-1}\Delta^{n-1} + \cdots + \gamma^{2^m-1}]\eta \\ &< \frac{2\gamma^{2^m-1}\Delta^m}{2-\gamma^{2^m-1}a_0}[\gamma^{2^m[2^{n-1}-1]}\Delta^{n-1} + \cdots + \gamma^{2^m[2-1]}\Delta + 1]\eta\end{aligned}$$

By Bernoulli's inequality, for every real number $x > -1$ and every integer $k \ge 0$, we have $(1+x)^k - 1 \ge kx$. Thus

$$\|x_{m+n} - x_m\| < \frac{2\gamma^{2^m-1}\Delta^m}{(2-\gamma^{2^m-1}a_0)}\frac{1-\gamma^{2^m n}\Delta^n}{(1-\gamma^{2^m}\Delta)}\eta \tag{19}$$

For $m = 0$, we obtain

$$\|x_n - x_0\| < \frac{2}{2 - a_0} \frac{1 - \gamma^n \Delta^n}{1 - \gamma\Delta} \eta < R\eta \tag{20}$$

Hence $x_n \in \mathscr{B}(x_0, R\eta)$. Also $y_n \in \mathscr{B}(x_0, R\eta)$, is evident from the following result.

$$\begin{aligned}
\|y_{n+1} - x_0\| &\le \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \cdots + \|x_1 - x_0\| \\
&\le \|y_{n+1} - x_{n+1}\| + \frac{2}{2 - a_n}\|y_n - x_n\| + \cdots + \frac{2}{2 - a_0}\|y_0 - x_0\| \\
&< \frac{2}{2 - a_{n+1}}\|y_{n+1} - x_{n+1}\| + \cdots + \frac{2}{2 - a_0}\|y_0 - x_0\| \\
&< \cdots < \frac{2}{2 - a_0} \frac{1 - \gamma^{n+1}\Delta^{n+1}}{1 - \gamma\Delta}\eta < R\eta
\end{aligned}$$

Taking limit $n \to \infty$ as in (18) and (20), we get $x^* \in \bar{\mathscr{B}}(x_0, R\eta)$. Now we have to show that x^* is a solution of $F(x) = 0$. We have $\|F(x_n)\| \le \|F'(x_n)\| \, \|\Gamma_n F(x_n)\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded as

$$\|F'(x_n)\| \le \|F'(x_0)\| + M\|x_n - x_0\| < \|F'(x_0)\| + MR\eta.$$

Now taking limit $n \to \infty$ we get $F(x^*) = 0$ as F is continuous.

To show the uniqueness of the zero x^*, let us consider y^* be another root of (3) in $\mathscr{B}(x_0, 2/(M\beta) - R\eta) \cap \Omega$. Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*)) \, dt(y^* - x^*)$$

We have $y^* = x^*$, if the operator $P = \int_0^1 F'(x^* + t(y^* - x^*)) \, dt$ is invertible. From

$$\begin{aligned}
\|I - \Gamma_0 P\| = \|\Gamma_0(F'(x_0) - P)\| &= \left\| \Gamma_0 \int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)] \, dt \right\| \\
&\le M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\| \, dt \\
&\le M\beta \int_0^1 (1 - t)\|x^* - x_0\| + t\|y^* - x_0\| \, dt \\
&< \frac{M\beta}{2}\left(R\eta + \frac{2}{M\beta} - R\eta\right) = 1,
\end{aligned}$$

and by Banach's theorem [10], P is invertible. □

5. Numerical examples

In this section, two numerical examples are worked out for demonstrating the efficacy of the existence and uniqueness theorem given in the previous section.

Example 1. Let $\mathbf{X} = C[a,b]$ be the space of continuous functions on $[a,b]$ and consider the problem of finding the solutions of nonlinear integral equations $F(x) = 0$ of mixed type [3], given by

$$F(x)(s) = x(s) - f(s) - \lambda \int_a^b G(s,t)[x(t)^{2+p} + x(t)^3]\, dt, \quad p \in (0,1],\ \lambda \in \mathbf{R} \tag{21}$$

where f, x are continuous functions such that $f(s) > 0$, $s \in [a,b]$, and the Kernel G is continuous and nonnegative in $[a,b] \times [a,b]$.

Solution. For the solution of the problem, we have taken the norm as sup-norm and $G(s,t)$ as the Green's function

$$G(s,t) = \begin{cases} (b-s)(t-a)/(b-a), & t \le s, \\ (s-a)(b-t)/(b-a), & s \le t, \end{cases}$$

Now in order to apply the existence and uniqueness theorem of the previous section to this problem, we compute the scalars M, β, η and the function $\omega(x)$. The first and second derivatives of F can easily be obtained and given by

$$F'(x)u(s) = u(s) - \lambda \int_a^b G(s,t)[(2+p)x(t)^{1+p} + 3x(t)^2]u(t)\, dt, \quad u \in \Omega$$

$$F''(x)(uv)(s) = -\lambda \int_a^b G(s,t)[(1+p)(2+p)x(t)^p + 6x(t)](uv)(t)\, dt, \quad u, v \in \Omega$$

For $p \in (0,1)$, we must note here that the second derivative F'' does not satisfy the Lipschitz/Hölder continuity condition, as

$$\begin{aligned} &\|F''(x) - F''(y)\| \\ &\quad = \left\| \lambda \int_a^b G(s,t)[(1+p)(2+p)(x(t)^p - y(t)^p) + 6(x(t) - y(t))]\, dt \right\| \\ &\quad \le |\lambda| \max_{s\in[a,b]} \left| \int_a^b G(s,t)\, dt \right| [(1+p)(2+p)\|x(t)^p - y(t)^p\| + 6\|x(t) - y(t)\|] \\ &\quad \le |\lambda|\, \|l\| [(1+p)(2+p)\|x-y\|^p + 6\|x-y\|], \quad \forall x, y \in \Omega, \end{aligned}$$

where

$$\|l\| = \max_{s\in[a,b]} \left| \int_a^b G(s,t)\, dt \right|$$

Thus convergence of methods depending upon lipschitz/Hölder continuity condition on F'' can not be applied. However, it satisfies the ω-continuity condition given by

$$\|F''(x) - F''(y)\| \le \omega(\|x-y\|), \quad \forall x, y \in \Omega,$$

where, $\omega(x) = |\lambda|\,\|l\|[(1+p)(2+p)x^p + 6x]$. This leads to $\omega(tx) \le t^p\omega(x)$, for $p \in (0,1)$ and $t \in [0,1]$. Hence, $h(t) = t^p$, and $K = \int_0^1 h(t)(1-t)\,dt + \frac{1}{2}\int_0^1 h(t)\,dt = \dfrac{p+4}{2(p+1)(p+2)}$.

It is easy to compute

$$\|F(x_0)\| \le \|x_0 - f\| + |\lambda|\,\|l\|[\|x_0\|^{2+p} + \|x_0\|^3]$$

and

$$\|F''(x)\| \le |\lambda|\,\|l\|[(1+p)(2+p)\|x\|^p + 6\|x\|]$$

This gives $M = |\lambda|\,\|l\|[(1+p)(2+p)\|x\|^p + 6\|x\|]$. Also

$$\|I - F'(x_0)\| \le |\lambda|\,\|l\|[(2+p)\|x_0\|^{1+p} + 3\|x_0\|^2]$$

Now if $|\lambda|\,\|l\|[(2+p)\|x_0\|^{1+p} + 3\|x_0\|^2] < 1$, then by Banach's theorem [10], we obtain

$$\|\Gamma_0\| = \|F'(x_0)^{-1}\| \le \frac{1}{1 - |\lambda|\,\|l\|[(2+p)\|x_0\|^{1+p} + 3\|x_0\|^2]} = \beta$$

and

$$\|\Gamma_0 F(x_0)\| \le \frac{\|x_0 - f\| + |\lambda|\,\|l\|[\|x_0\|^{2+p} + \|x_0\|^3]}{1 - |\lambda|\,\|l\|[(2+p)\|x_0\|^{1+p} + 3\|x_0\|^2]} = \eta$$

For $a = 0$ and $b = 1$, we get

$$\|l\| = \max_{s\in[0,1]}\left|\int_0^1 G(s,t)\,dt\right| = 1/8$$

For $\lambda = 1/3$, $p = 1/2$, $f(s) = 1$, and initial point $x_0 = x_0(s) = 1$ in $[0,1]$, we get $\|\Gamma_0\| \le \beta = 1.2973$, $\|\Gamma_0 F(x_0)\| \le \eta = 0.108108$, $\omega(\eta) = 0.0784017$ and $b_0 = \beta\eta\omega(\eta) = 0.0109957$. Now we look for a domain in the form of $\Omega = \mathscr{B}(x_0, S)$ such that

$$\Omega = \mathscr{B}(x_0, S) \subseteq C[0,1] = \mathbf{X}$$

Thus, we get $M = M(S) = 0.15625S^p + 0.25S$ and $a_0 = a_0(S) = M(S)\beta\eta = 0.0219138S^p + 0.03506208S$. To calculate S, from the condition of theorem 1 it is necessary that $\bar{\mathscr{B}}(x_0, R\eta) \subseteq \Omega$. For this it is sufficient to check $S - (R(S)\eta + 1) > 0$ and $\Phi(a_0(S)) - b_0 > 0$. Hence it is necessary that $S \in (1.1425, 4.13138)$ as is evident from fig. 1.

Also $a_0(S) < r_0 = 0.1905960896$, if and only if $S < 4.16104$. Hence if we choose $S = 3$, then we have $\Omega = \mathscr{B}(1,3)$, $M = 1.3125$, $a_0 = 0.184076$ and $b_0 = 0.0109957 < 0.0596816 = \Phi(0.184076)$. Thus the conditions of the theorem 1 is satisfied. Hence a solution of equation (21) exists in the ball $\bar{\mathscr{B}}(1, 0.49741) \subseteq \Omega$ and unique in the ball $\mathscr{B}(1, 0.6771907) \cap \Omega$.

We must also note that if we take $p = 1$, in the equation (21), we get

$$F(x)(s) = x(s) - f(s) - 2\lambda\int_a^b G(s,t)x(t)^3\,dt, \quad \lambda \in \mathbf{R} \tag{22}$$

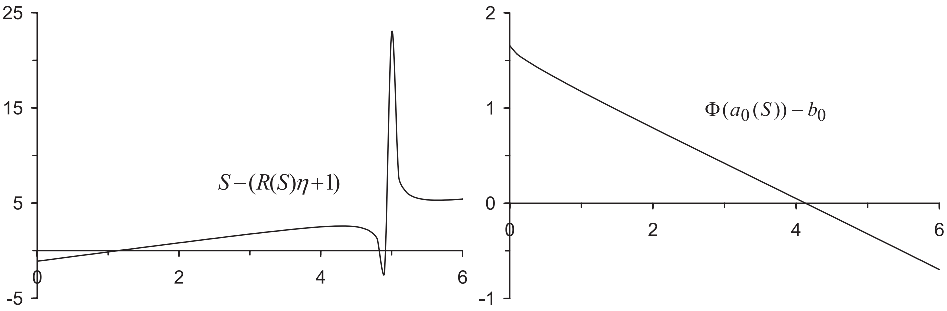

FIGURE 1. Conditions on the parameter S

Similarly proceeding as above one can easily find

$$\|F''(x) - F''(y)\| \leq \omega(\|x - y\|),$$

with $\omega(x) = Nx$ and $N = 12|\lambda|\,\|l\|$. That is,

$$\|F''(x) - F''(y)\| \leq N\|x - y\|,$$

Hence it satisfies the Lipschitz continuity condition. This implies that ω-continuity condition is a generalization of Lipschitz continuity condition. Now, for $\lambda = 1/3$, $f(s) = 1$, and initial point $x_0 = x_0(s) = 1$ in $[0, 1]$, proceeding similarly we obtain $\beta = 1.33333$, $\eta = 0.111111$, $M = 0.5S$, $N = 0.5$. This gives $a_0 = 0.07407408S$, $b_0 = 0.00823045$. Similarly to calculate S, from the condition of theorem 1 it is necessary that $\bar{\mathscr{B}}(x_0, R\eta) \subseteq \Omega$. For this it is sufficient to check $S - (R(S)\eta + 1) > 0$ and $\Phi(a_0(S)) - b_0 > 0$. Hence it is necessary that $S \in (1.15493, 2.5544)$ as evident from the figure 2.

Also $a_0(S) < r_0 = 0.1905960896$, if and only if $S < 2.573047$. Thus for $S = 1.5$, we get $\Omega = \mathscr{B}(1, 1.5)$, $M = 0.75$, $a_0 = 0.111111$ and $b_0 = 0.00823045 < 1.02883045 = \Phi(0.111111)$. Hence the conditions of the theorem 1 are satisfied. Thus a solution of equation (22) exists in $\bar{\mathscr{B}}(1, 0.196209) \subseteq \Omega$ and unique in $\mathscr{B}(1, 1.803791) \cap \Omega$.

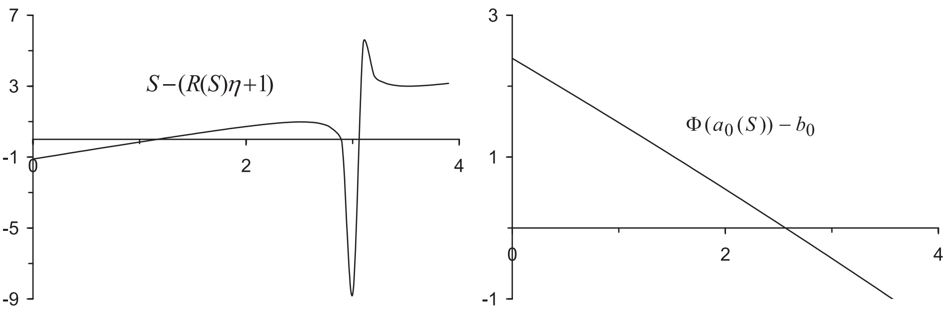

FIGURE 2. Conditions on the parameter S

On comparison with results of Wu and Zhao [15], we find $K = M\left[1 + \frac{5N}{3M^2\beta}\right] = 0.83333334$ and $h = K\beta\eta = 0.123456358025$. This gives $t^* = \frac{1 - \sqrt{1-2h}}{h}\eta = 0.11897488$ and $t^{**} = \frac{1 + \sqrt{1-2h}}{h}\eta = 1.6810296183$. Hence the solution of equation (22) exists in $\bar{B}(1, 0.11897488) \subseteq \Omega$, and unique in $B(1, 1.6810296183) \cap \Omega$, both of these are inferior to our results. •

Example 2. Let $\mathbf{X} = C[0,1]$ be the space of continuous functions on $[0,1]$ and let us consider the integral equations $F(x) = 0$ on $\mathbf{X}$, where

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+p}\, dt, \tag{23}$$

with $s \in [0,1]$, $x, f \in \mathbf{X}$, $p \in (0,1)$ and λ is a real number. These types of integral equations are known as Fredholm type (cf. Davis [1]).

Solution. To find the solutions of these integral equations, we compute the first and the second derivatives of F as

$$F'(x)u(s) = u(s) - \lambda(2+p) \int_0^1 \frac{s}{s+t} x(t)^{1+p} u(t)\, dt, \quad u \in \Omega$$

$$F''(x)(uv)(s) = -\lambda(1+p)(2+p) \int_0^1 \frac{s}{s+t} x(t)^p (uv)(t)\, dt, \quad u, v \in \Omega$$

We must note that for sup-norm,

$$\|F''(x) - F''(y)\| \le N\|x - y\|^p, \quad \forall x, y \in \Omega,$$

where $N = |\lambda|(1+p)(2+p)\log 2$. This implies

$$\|F''(x) - F''(y)\| \le \omega(\|x - y\|), \quad \forall x, y \in \Omega,$$

where $\omega(x) = Nx^p$. As p is a real number in $(0,1]$, F'' is (N,p)-Hölder continuous. Thus the used ω-continuity condition is the generalization of (N,p)-Hölder continuity condition. Now it is easy to compute

$$\|F(x_0)\| \le \|x_0 - f\| + |\lambda| \log 2\|x_0\|^{2+p}$$

and

$$\|F''(x)\| \le |\lambda|(1+p)(2+p)\log 2\|x\|^p$$

Hence, $M = |\lambda|(1+p)(2+p)\log 2\|x\|^p = N\|x\|^p$. Also it is easy to compute

$$\|I - F'(x_0)\| \le |\lambda|(2+p)\log 2\|x_0\|^{1+p}$$

If $|\lambda|(2+p)\log 2\|x_0\|^{1+p} < 1$, then by Banach's theorem [10], we obtain

$$\|\Gamma_0\| = \|F'(x_0)^{-1}\| \le \frac{1}{1 - |\lambda|(2+p)\log 2\|x_0\|^{1+p}} = \beta$$

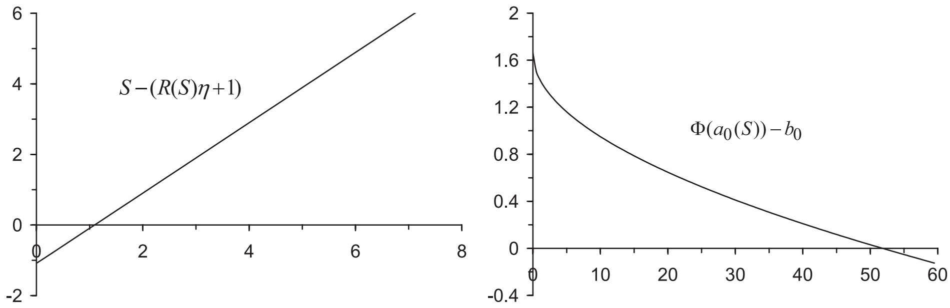

FIGURE 3. Conditions on the parameter S

and

$$\|\Gamma_0 F(x_0)\| \le \frac{\|x_0 - f\| + |\lambda| \log 2\|x_0\|^{2+p}}{1 - |\lambda|(2+p) \log 2\|x_0\|^{1+p}} = \eta$$

Now, for $\lambda = 1/10$, $p = 1/2$, $f(s) = 1$, and initial point $x_0 = x_0(s) = 1$ in the interval $[0,1]$, we get $\|\Gamma_0\| \le \beta = 1.20961$, $\|\Gamma_0 F(x_0)\| \le \eta = 0.0838437$, $N = 0.25993$. Now we look for a domain in the form of $\Omega = \mathscr{B}(x_0, S)$ such that

$$\Omega = \mathscr{B}(x_0, S) \subseteq C[0,1] = \mathbf{X}$$

Thus $M = M(S) = 0.25993S^p$, $a_0 = a_0(S) = M(S)\beta\eta = 0.0263616S^p$ and $b_0 = \beta\eta\omega(\eta) = 0.00763322$. Now to calculate S, from the condition of theorem 1, it is necessary that $\bar{\mathscr{B}}(x_0, R\eta) \subseteq \Omega$. For this it is sufficient to check $S - (R(S)\eta + 1) > 0$ and $\Phi(a_0(S)) - b_0 > 0$. Hence it is necessary that $S \in (1.0935, 51.8179)$ as is evident from the figure 3.

Also $a_0(S) < r_0 = 0.1905960896$, if and only if $S < 52.2738444$. Hence if we choose $S = 51$, then we have $\Omega = \mathscr{B}(1, 51)$, $M = 1.85627$, $a_0 = 0.18826$ and $b_0 = 0.00763322 < 0.02140762 = \Phi(0.18826)$. Thus, the conditions of the theorem 1 are satisfied. Hence, a solution of equation (23) exists in $\bar{\mathscr{B}}(1, 0.4241) \subseteq \Omega$ and unique in the ball $\mathscr{B}(1, 0.466624671) \cap \Omega$. •

6. Conclusions

In this paper, the convergence of a third order Newton-like method (1) for solving nonlinear equation (3) is discussed by using recurrence relations under the assumption that the second Fréchet derivative of F is ω-continuous. This ω-continuous condition is milder than the usual Lipschitz/Hölder continuity condition used to establish the convergence of third order Newton-like methods. A family of recurrence relations based on two constants depending on the operator F is derived to establish a priori error bounds. This approach is simple and efficient in comparison with the work of Wu and Zhao [15]. Numerical examples are worked out to demonstrate the efficacy of our method and our results are improvement over the results of Wu and Zhao [15].

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