K. KONNO
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ON CERTAIN FIBRED RATIONAL SURFACES

Kazuhiro Konno

Abstract

Fibred rational surfaces with a certain extremal property are classified.

Introduction

This is a continuation of [**3**] in which a systematic study of fibred rational surfaces is done. Let X be a non-singular projective rational surface and $f : X \to \mathbf{P}^1$ a relatively minimal fibration of curves of genus $g \geq 2$. We denote by F a general fibre of f. Then $K_X + F$ is nef and $h^0(X, K_X + F) = g$ by [**3**, Lemma 1.1]. The rational map associated to the complete linear system $|K_X + F|$ was studied in [**3**, Proposition 1.1] when it is generically finite onto its image: It is a morphism if $(K_X + F)^2 \leq 2g - 4$ that is birational onto the image if $(K_X + F)^2 \leq 2g - 5$. See also §1 for further properties.

In this article, we consider the following two cases and give the complete description of the corresponding fibred rational surfaces.

(A) $(K_X + F)^2 = 2g - 5$, $g \geq 4$, and F is either trigonal or plane quintic.

(B) $(K_X + F)^2 = 2g - 4$ and $|K_X + F|$ induces a morphism of degree 2 onto the image.

Fibred rational surfaces as in (A) are classified into two types and are described in Theorems 2.8 and 2.9, respectively. Those as in (B) are determined in Theorems 3.4, 3.9 and 3.10.

The reason why we are interested in these special cases is in the jumping phenomena of $(K_X + F)^2$ observed in [**3**, §1] as follows (see also §1): If F is either trigonal or plane quintic, then either $(K_X + F)^2 = g - 2$ or $(K_X + F)^2 \geq 2g - 5$. If F is hyperelliptic, then either $(K_X + F)^2 = 0$ or $(K_X + F)^2 \geq 2g - 4$. In both cases, the first possibility is known to occur and easily described (see, e.g., [**3**, Remarks 1.1 and 1.3]). Therefore, it is natural to ask what happens in the second region. The cases (A) and (B) respectively correspond to the smallest possible value of $(K_X + F)^2$ to be investigated.

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The organization of the paper is as follows. In §1, we recall some results in [**3**] in order to summarize the basic facts on $|K_X + F|$ which clarify the meaning of (A), (B). In §2, we shall determine all the fibred rational surfaces with the property (A). Since F is either trigonal or plane quintic, the quadric hull of F is a surface of minimal degree by the Enriques-Petri theorem. As F moves in the pencil, such surfaces trace a threefold of minimal degree through the image of X by the birational morphism defined by $|K_X + F|$. This enables us to describe the structure of $f : X \to \mathbf{P}^1$ in Theorems 2.8 and 2.9. Among other things, we show that the g_3^1 on F is induced from a pencil of elliptic curves on X when $g \geq 7$. In §3, by using the double covering method [**2**], we shall show that f as in (B) is necessarily a hyperelliptic fibration when $g \geq 5$, and list in Theorem 3.4 the possible branch locus as well as the pencil inducing f, on the rational surface downstairs. On the other hand, when $g = 3$ or 4, non-hyperelliptic fibrations also appear (Theorems 3.9 and 3.10). If $g = 3$ (resp. $g = 4$), then we can find a pencil of curves in $|-2K_Y|$ (resp. $|-3K_Y|$) giving us a non-hyperelliptic fibration f, where Y is the weak del Pezzo surface of degree 2 (resp. 1) obtained as the reduction of X (see §1 for the definition of the reduction).

Throughout the paper, we shall work over $\mathbf{C}$ and use the following notation. We denote by Σ_d the Hirzebruch surface of degree d. Let Δ_0 and Γ be a minimal section and a fibre of Σ_d, respectively. For a subvariety Z of $\mathbf{P}^n$, we denote by $\mathrm{Quad}(Z)$ the intersection of all hyperquadrics through Z, and call it the *quadric hull* of Z. If there are no hyperquadrics through Z, then we put $\mathrm{Quad}(Z) = \mathbf{P}^n$. A non-singular projective surface S is called a weak del Pezzo surface if $-K_S$ is nef and big. For two divisors D_1, D_2 on a non-singular variety, $D_1 \sim D_2$ means that D_1 and D_2 are linearly equivalent.

1. Preliminaries

In this section, we summarize the results in [**3**, §1] to fix the notation and give the background for our problem.

Let X be a non-singular projective rational surface with a relatively minimal fibration $f : X \to \mathbf{P}^1$ whose general fibre F is a non-singular projective curve of genus $g \geq 2$. Then $K_X + F$ is nef and the restriction map $H^0(X, K_X + F) \to H^0(F, \omega_F)$ is an isomorphism by [**3**, Lemma 1.1]. In particular, $h^0(X, K_X + F) = g$.

If $|K_X + F|$ is composed of a pencil, then f is a hyperelliptic fibration and $(K_X + F)^2 \geq 2g - 2$ holds for $g \geq 3$ by [**3**, Lemma 1.3]. If $|K_X + F|$ defines a generically finite rational map $\Phi : X \to \mathbf{P}^{g-1}$ onto the image W, then the following hold by [**3**, Proposition 1.1].

(1) If $(K_X + F)^2 \leq 2g - 4$, then $|K_X + F|$ is free from base points, and $\Phi : X \to W$ is a birational morphism except in the case where $(K_X + F)^2 = 2g - 4$ and $\deg \Phi = 2$.

(2) If $(K_X + F)^2 \leq 2g - 5$, then the graded ring $R(X, K_X + F) = \bigoplus_{n \geq 0} H^0(X, n(K_X + F))$ is generated in degree one. Furthermore, W has at most rational double points as the singularity.

(3) If $(K_X+F)^2 \le 2g-6$, then the homogeneous ideal of $W \subset \mathbf{P}^{g-1}$ is generated by quadrics. If $(K_X+F)^2 = 2g-5$ and $g \ge 4$, then it is generated by quadrics and cubics.

Since Φ induces the canonical map of F, one sees immediately from the above facts that either $(K_X+F)^2 = 0$ or $(K_X+F)^2 \ge 2g-4$ holds when F is hyperelliptic. Note that we have $(K_X+F)^2 \ge \deg W \ge g-2$ when $\deg \Phi = 1$, since W is an irreducible non-degenerate surface in $\mathbf{P}^{g-1}$. By using (3), one can show, as we did in [**3**, Theorem 1.1], that F regarded as a canonical curve is cut out by quadrics provided that $g-1 \le (K_X+F)^2 \le 2g-6$. Hence, by the Enriques-Petri theorem, we see that either $(K_X+F)^2 = g-2$ or $(K_X+F)^2 \ge 2g-5$ holds when F is trigonal or plane quintic.

Let $\mu : X \to Y$ be the blowing-down of all the (-1)-curves E satisfying $(K_X+F)E = 0$ and put $G = \mu_* F$. Then $\mu^*(K_Y+G) = K_X+F$ and Φ factors through Y. The original fibration f is obtained from a pencil $\Lambda_f \subset |G|$ by eliminating the base points. We call the pair (Y, G) the *reduction* of (X, F). Since the properties (1)–(3) also hold for K_Y+G, it is often convenient to consider (Y, G) instead of (X, F).

2. Fibrations with $(K_X+F)^2 = 2g-5$

In this section, we study the case $(K_X+F)^2 = 2g-5$, $g \ge 4$, assuming that F is either trigonal or plane quintic. We let (Y, G) be the reduction of (X, F). Then f corresponds to a pencil $\Lambda_f \subset |G|$ and the graded ring $\bigoplus_{m \ge 0} H^0(Y, m(K_Y+G))$ is generated in degree one. Let W be the image in $\mathbf{P}^{g-1}$ of Y by the birational morphism defined by $|K_Y+G|$. Then it is projectively normal and has at most rational double points as the singularity. For these facts, see [**3**] or §1.

A general member $G \in \Lambda_f$ can be regarded as the canonical curve in $\mathbf{P}^{g-1}$, because the restriction map $H^0(Y, K_Y+G) \to H^0(G, K_G)$ is an isomorphism. With this identification, we have

$$\begin{array}{ccc} G & \subset & W \\ \cap & & \cap \\ \mathrm{Quad}(G) & \subset & \mathrm{Quad}(W) \end{array}$$

in $\mathbf{P}^{g-1}$.

Lemma 2.1. *The quadric hull of W is a threefold of minimal degree $g-3$ in $\mathbf{P}^{g-1}$.*

Proof. Since G is either trigonal or plane quintic, $\mathrm{Quad}(G)$ is a surface of minimal degree $g-2$ in $\mathbf{P}^{g-1}$ by the Enriques-Petri theorem. It is a surface different from W for the reason of degrees. Furthermore, $\mathrm{Quad}(G)$ moves in $\mathrm{Quad}(W)$ as G moves in Λ_f. Therefore, $\mathrm{Quad}(W)$ has a component of dimension bigger than or equal to 3 that contains W.

The multiplication map $\mathrm{Sym}^2 H^0(Y, K_Y + G) \to H^0(Y, 2(K_Y + G))$ is surjective and $h^0(Y, 2(K_Y + G)) = 4g - 6$. Hence there are $(g-3)(g-4)/2$ independent hyperquadrics through W defining $\mathrm{Quad}(W)$. It follows that $\mathrm{Quad}(W)$ is a threefold of minimal degree. □

From the classification of varieties of minimal degree (see e.g. [**1**]), we know that $\mathrm{Quad}(W)$ is either (i) $\mathbf{P}^3$ $(g = 4)$, or (ii) a hyperquadric $(g = 5)$, or (iii) a cone over the Veronese surface $(g = 7)$, or (iv) a rational normal scroll $(g \geq 5)$.

Lemma 2.2. *If $g = 4$, then Y is a weak del Pezzo surface of degree* 3 *and $G \in |-2K_Y|$.*

Proof. If $g = 4$, then $\mathrm{Quad}(W) = \mathbf{P}^3$ and W is a cubic surface with at most rational double points. Hence $-K_Y$ is induced from the hyperplane bundle. It follows $K_Y + G = -K_Y$ and we have $G \in |-2K_Y|$. Conversely, we get a non-hyperelliptic curve of genus 4 as a hyperquadric section of a cubic surface. □

Assume that $g \geq 5$.

Lemma 2.3. $\mathrm{Quad}(W)$ *is a rational normal scroll of dimension three.*

Proof. We first assume that $g = 5$ and $\mathrm{Quad}(W)$ is a non-singular hyperquadric. Then the Picard group of $\mathrm{Quad}(W)$ is generated by the hyperplane class. It follows that any hypersurface on it is of even degree. Since $\deg W = 5$ is odd, this is impossible.

We next assume that $g = 7$ and $\mathrm{Quad}(W)$ is a cone over the Veronese surface. Then by considering the linear system of hyperplanes through the vertex of $\mathrm{Quad}(W)$, we have $K_Y + G \sim \Delta + 2\ell$, where ℓ denotes the transform to Y of a line on $\mathbf{P}^2$ and Δ is the divisorial part of the inverse image of the vertex. Since a general hyperplane does not pass through the vertex, we have $(K_Y + G)\Delta = 0$. Then $9 = (K_Y + G)^2 = 2\ell(K_Y + G)$, which is impossible. □

Since $\mathrm{Quad}(W)$ is a rational normal scroll of dimension three, it has a ruling by planes which is unique when $g \geq 6$. By pulling it back, we obtain a pencil $|D|$ of curves on Y. Let $\rho : \hat{Y} \to Y$ be a minimal succession of blowing-ups that eliminates the base points of $|D|$. Then we have a fibration $\psi : \hat{Y} \to \mathbf{P}^1$ induced by the variable part $|\hat{D}|$ of $\rho^*|D|$. Let $\mathscr{E}$ be the subbundle of $\psi_* \mathcal{O}_{\hat{Y}}(\rho^*(K_Y + G))$ generically generated by its global sections. Then it is of rank three and we can write it as

$$\mathscr{E} = \mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(c)$$

with non-negative integers a, b, c satisfying $0 \leq a \leq b \leq c$ and $a + b + c = g - 3$. The composite $\psi^*\mathscr{E} \hookrightarrow \psi^*\psi_*\mathcal{O}_{\hat{Y}}(\rho^*(K_Y + G)) \to \mathcal{O}_{\hat{Y}}(\rho^*(K_Y + G))$ induces the morphism $\hat{Y} \to \mathbf{P}(\mathscr{E})$ which is birational onto the image $\hat{W}$. Let H and Γ

denote a tautological divisor and a fibre of $\mathbf{P}(\mathscr{E})$, respectively. The Picard group of $\mathbf{P}(\mathscr{E})$ is generated by H and Γ, and we have $H^3 = (g-3)H^2\Gamma$ in the Chow ring. Clearly $\mathrm{Quad}(W)$ (resp. W) is the image of $\mathbf{P}(\mathscr{E})$ (resp. $\hat{W}$) by the morphism defined by $|H|$.

LEMMA 2.4. *$\hat{W}$ is linearly equivalent to either*
(I) $3H-(g-4)\Gamma$, *or*
(II) $2H+\Gamma$
on $\mathbf{P}(\mathscr{E})$. If (I) *is the case, then either $D^2=0$ or $D^2=1$, $g \le 7$. If* (II) *is the case, then $D^2=0$.*

Proof. We put $\hat{W} \sim \alpha H + \beta\Gamma$. Since W is of degree $2g-5$, we have $(\alpha H + \beta\Gamma)H^2 = 2g-5$, that is, $\beta = 2g-5-\alpha(g-3)$. Let C be a general member of $|K_Y+G|$. Then it is of genus $g-3$. The image $\hat{C}$ of C in $\hat{W}$ is a section of $\hat{W}$ with a general member of $|H|$. Since the canonical bundle of $\hat{C}$ is induced from $(\alpha-2)H + (3g-10-\alpha(g-3))\Gamma$, we have

$$((\alpha-2)H + (3g-10-\alpha(g-3))\Gamma)H(\alpha H + (2g-5-\alpha(g-3))\Gamma) = 2g-8$$

which is equivalent to $(\alpha-3)(\alpha-2)(g-3)=0$. Hence we have either (I) or (II).

We have $(K_Y+G)D = \rho^*(K_Y+G)\hat{D} = \alpha$. It follows from Hodge's index theorem that $\alpha^2 \ge (K_Y+G)^2 D^2$. Since $\alpha = 2$ or 3 and $(K_Y+G)^2 = 2g-5$, we have either $D^2=0$, or $\alpha=3$, $D^2=1$ and $g \le 7$. We have $K_Y+G \equiv 3D$ when $g=7$ and $D^2=1$, where the symbol $\equiv$ means the numerical equivalence. □

We say that the fibration f is of type (I) or type (II) according to whether the linear equivalence class of $\hat{W}$ is as in (I) or (II).

We let $\hat{Q}$ denote the proper transform of $\mathrm{Quad}(G)$ to $\mathbf{P}(\mathscr{E})$.

LEMMA 2.5. *$\hat{Q}$ is linearly equivalent to either $H+\Gamma$ or $2H-(g-4)\Gamma$, $(g=5,6)$. In the former case, the projection map of $\mathbf{P}(\mathscr{E})$ presents $\hat{Q}$ as a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$. Furthermore, $\hat{Q} \sim H+\Gamma$ when f is of type* (I) *and $\hat{Q} \sim 2H-(g-4)\Gamma$ when f is of type* (II).

Proof. Since $\mathrm{Quad}(G)$ is of degree $g-2$ in $\mathbf{P}^{g-1}$, we have $\hat{Q}H^2 = g-2$. Then it is easy to see that the possible linear equivalence class of $\hat{Q}$ is as stated. It is well-known that $\mathrm{Quad}(G)$ is non-singular when $g \ge 5$. If $\hat{Q} \sim H+\Gamma$, then $\hat{Q}$ is also non-singular and $K_{\hat{Q}}$ is the restriction of $-2H+(g-4)\Gamma$ by the adjunction formula. It follows that $K_{\hat{Q}}^2 = 8$ which implies that $\hat{Q}$ is a Hirzebruch surface. Furthermore, the projection map of $\mathbf{P}(\mathscr{E})$ gives $\hat{Q}$ a pencil of lines.

Now, G regarded as a canonical curve is a curve of degree $2g-2$ contained in the intersection $\mathrm{Quad}(G) \cap W$. Hence $\hat{Q}\hat{W}H \ge 2g-2$. It follows that we have $\hat{Q} \sim H+\Gamma$ when $\hat{W} \sim 3H-(g-4)\Gamma$, and $\hat{Q} \sim 2H-(g-4)\Gamma$ when $\hat{W} \sim 2H+\Gamma$. In either case, we have $G = \hat{Q} \cap \hat{W}$. □

We choose sections Z_0, Z_1 and Z_2 of $[H - a\Gamma]$, $[H - b\Gamma]$ and $[H - c\Gamma]$, respectively, such that (Z_0, Z_1, Z_2) forms a system of homogeneous coordinates on fibres of $\mathbf{P}(\mathscr{E})$. We let (t_0, t_1) be a system of homogeneous coordinates on the base curve $\mathbf{P}^1$. In addition, we sometimes use the following notation especially when $a = 0$. Let Δ be the section defined by $Z_1 = Z_2 = 0$ and $\sigma : \tilde{\mathbf{P}} \to \mathbf{P}(\mathscr{E})$ the blowing-up along Δ. Then $\tilde{\mathbf{P}}$ is isomorphic to the total space of the $\mathbf{P}^1$-bundle $\varpi : \mathbf{P}(\mathscr{O}(a\Gamma) \oplus \mathscr{O}(\Delta_0 + c\Gamma)) \to \Sigma_{c-b}$, and $\sigma^* H$ gives us the tautological line bundle. The exceptional divisor H_∞ for σ is linearly equivalent to $\sigma^* H - \varpi^*(\Delta_0 + c\Gamma)$. When $\mathrm{Quad}(W)$ is a cone over a surface scroll, $\tilde{\mathbf{P}}$ is obtained by blowing up the vertex and H_∞ is nothing more than its inverse image.

Lemma 2.6. *If f is of type* (I), *then $b \le a + 1$ and $2b + 1 \ge a + c$. If f is of type* (II), *then* $(g; a, b, c) = (6; 1, 1, 1), (6; 0, 1, 2), (5; 0, 1, 1)$.

Proof. We first consider the case that $\hat{W} \sim 3H - (g - 4)\Gamma$ and $\hat{Q} \sim H + \Gamma$. Then the equation of $\hat{W}$ is of the form $\sum_{i,j} \psi_{ij}(t) Z_0^{3-i-j} Z_1^i Z_2^j$, where $\psi_{ij}(t)$ is a homogeneous form of degree $(3 - i - j)a + ib + jc - (g - 4)$, and the sum is taken over all non-negative integers i, j satisfying $0 \le i + j \le 3$. Recall that we have $a + b + c = g - 3$. If $2b + 1 < a + c$, then the equation can be divided by Z_2, which is impossible because of the irreducibility of $\hat{W}$. If $b > a + 1$, then all the terms containing Z_0 disappear, which implies that $\hat{W}$ has a triple curve along the section Δ. We show that it is inadequate. Since $\hat{W}$ has a triple curve along Δ, its proper transform $\tilde{W}$ on $\tilde{\mathbf{P}}$ is linearly equivalent to $\varpi^*(3\Delta_0 + (2c - b - a + 1)\Gamma)$. This implies that it is the restriction of the $\mathbf{P}^1$-bundle $\tilde{\mathbf{P}}$ to a curve on the base Σ_{c-b} linearly equivalent to $3\Delta_0 + (2c - b - a + 1)\Gamma$. Note that the dualizing sheaf of the curve is induced by $\Delta_0 + (c - a - 1)\Gamma$. Hence $\tilde{W}$ is either a $\mathbf{P}^1$-bundle over an irrational curve or it is non-normal along several fibres. Both cases are impossible, because W is a normal rational surface. This shows $b \le a + 1$. In particular, we cannot have $a = b = 0$ when $g \ge 5$.

We show that $|D|$ has a base point when $a = 0$ for the later use. Assume that $a = 0$. Then $b = 1$, $c = g - 4$ and $g \le 7$. In the equation of $\hat{W}$, ψ_{01}, the coefficient of $Z_0^2 Z_2$, has to be a non-zero constant. It follows that $\hat{W}$ is non-singular in a neighborhood of Δ. Then the proper transform of $\hat{W}$ by σ is linearly equivalent to $2\sigma^* H + \varpi^* \Delta_0$. Since $(\sigma^* H - \varpi^*(\Delta_0 + (g - 4)\Gamma))^2 \cdot (2\sigma^* H + \varpi^* \Delta_0) = -1$, we see that Δ is a (-1)-curve on $\hat{W}$. Then Δ induces on $\hat{Y}$ a (-1)-curve E with $E\hat{D} = 1$. Therefore, $D^2 = 1$.

We next consider the case that $\hat{W} \sim 2H + \Gamma$ and $\hat{Q} \sim 2H - (g - 4)\Gamma$. Then the equation of $\hat{Q}$ is of the form $\sum_{i,j} \varphi_{ij}(t) Z_0^{2-i-j} Z_1^i Z_2^j$, where $\varphi_{ij}(t)$ is a homogeneous form in t_0, t_1 of degree $(2 - i - j)a + ib + jc - (g - 4)$ and the sum is taken over all non-negative integers i, j satisfying $0 \le i + j \le 2$. We claim that $\min\{2b,\ a + c\} \ge g - 4$. This can be seen as follows. If $2b < g - 4$, then the equation can be divided by Z_2 and $\hat{Q}$ is reducible, which is absurd. If $a + c < g - 4$, then $\hat{Q}$ is singular along the curve Δ defined by $Z_1 = Z_2 = 0$. This is impossible when $a > 0$, because then $\hat{Q} = \mathrm{Quad}(G)$ which is non-singular. When $a = 0$, we blow up $\mathbf{P}(\mathscr{E})$ along Δ. The proper transform $\tilde{Q}$ of $\hat{Q}$ is linearly equivalent to $\varpi^*(2\Delta_0 + (2c - g + 4)\Gamma)$ on $\tilde{\mathbf{P}}$. This implies that $\tilde{Q}$ is a $\mathbf{P}^1$-bundle over the curve linearly equivalent to $2\Delta_0 + (2c - g + 4)\Gamma$. Since $(\Delta_0 + c\Gamma) \cdot (2\Delta_0 + (2c - g + 4)\Gamma) = g - 2$, we see that $\tilde{Q} \simeq \Sigma_{g-2}$ and $H_\infty \cap \tilde{Q}$ is the minimal section of $\tilde{Q}$. Since $\mathrm{Quad}(W)$ is obtained from $\tilde{\mathbf{P}}$ by contracting H_∞, we see that $\mathrm{Quad}(G)$ is obtained from Σ_{g-2} by contracting the minimal section. This is absurd, since $\mathrm{Quad}(G)$ is non-singular. Therefore, $\min\{2b, a + c\} \geq g - 4$. Since $a + b + c = g - 3$, this condition is satisfied only when $g \leq 6$ and we have $(g; a, b, c) = (6; 1, 1, 1), (6; 0, 1, 2), (5; 0, 1, 1)$. We study $\hat{Q}$ more closely for the later use. Note that $\hat{Q}$ is non-singular, since it can be checked directly by examining the equation that it is non-singular in a neighborhood of Δ. By the adjunction formula, $K_{\hat{Q}}$ is induced from $-H - \Gamma$. Then $K_{\hat{Q}}^2 = (-H - \Gamma)^2 \cdot (2H - (g - 4)\Gamma) = g + 2$. It follows that $\hat{Q}$ is a Hirzebruch surface when $g = 6$, and it is a Hirzebruch surface blown up at a point when $g = 5$. The projection map of $\mathbf{P}(\mathscr{E})$ gives $\hat{Q}$ the structure of a conic bundle. □

We put $L = K_Y + G$.

2.1. Type (I). Assume that $\hat{W} \sim 3H - (g - 4)\Gamma$.

If $a > 0$, then $\hat{W} = W$ and we have $K_Y \sim -D$ and $G \sim L + D$. Therefore, Y is a minimal rational elliptic surface. Since $GD = 3$, f is a trigonal fibration.

We assume that $\mathrm{Quad}(W)$ is singular, that is, $a = 0$. Then $b = 1$, $c = g - 4$, $g \leq 7$ and $D^2 = 1$ as we have already seen. Recall that $\hat{W}$ is non-singular in a neighborhood of Δ defined by $Z_1 = Z_2 = 0$ in $\mathbf{P}(\mathscr{E})$. Hence $\hat{W}$ has at most rational double points, because so is W. Then $|\hat{D}|$ is a pencil of elliptic curves and it follows that $\mathscr{E} = \psi_* \mathscr{O}_{\hat{Y}}(\rho^* L)$. Since L is normally generated, we see that $\hat{W}$ is isomorphic to $\mathrm{Proj}(\bigoplus_{n \geq 0} \psi_*(\mathscr{O}_{\hat{Y}}(n\rho^* L)))$. Since $\hat{Y}$ is a rational elliptic surface whose canonical bundle is induced by $-\hat{D}$, we see that Y is a weak del Pezzo surface of degree 1 with $-K_Y = D$. We have $G \in |L + D|$. Note that $G \in |-4K_Y|$ when $g = 7$, because $K_Y + G \sim 3D$ in this case.

We have $K_{\hat{Y}} \hat{D} = 0$. It follows that $K_Y D = -1$ and $GD = 4$, because $3 = \rho^*(K_Y + G)\hat{D} = (K_Y + G)D$. Let $\hat{G}$ be the proper transform of G by ρ and E the exceptional (-1)-curve. Since G is non-singular, we have either $\hat{G} \sim \rho^* G - E$ or $\hat{G} \sim \rho^* G$. Then we have either $\hat{G}\hat{D} = 3$ or $\hat{G}\hat{D} = 4$. Recall that E is the inverse image of Δ.

In order to see that the g_3^1 on G is induced from $|D|$, it suffices to show the following:

Lemma 2.7. *If* $\mathrm{Quad}(W)$ *is a cone over a Hirzebruch surface, then* G *is a trigonal curve passing through the base point of* $|D|$. *In particular, the reduction map* $\mu : X \to Y$ *factors through* $\hat{Y}$.

Proof. Assume that G is a trigonal curve. Then $\hat{G}\hat{D} = 3$, because the projection map of $\mathbf{P}(\mathscr{E})$ induces the ruling of $\hat{Q}$ as we have already seen. It follows that G passes through the base point of $|D|$.

We assume that G is a plane quintic ($g = 6$) and show that this is impossible. Since $\hat{Q}$ is isomorphic to a Hirzebruch surface, it has to be the Veronese surface blown up at a point. Hence $\hat{Q} \simeq \Sigma_1$ and the ruling of $\hat{Q}$ induces on $\hat{G}$ either a g_4^1 or a g_5^1 according to whether the canonical curve passes through the center of the blowing-up or not. From $\hat{G}\hat{D} \le 4$, we conclude that $\hat{G}\hat{D} = 4$. The minimal section of $\hat{Q}$ arises from the vertex of $\mathrm{Quad}(W)$. Hence it must be Δ. Since $\hat{G}\hat{D} = 4$ occurs only when the image of $\hat{G}$ meets the minimal section of $\hat{Q}$, we must have $\hat{G}E = 1$ and $\hat{G} \sim \rho^*G - E$. But this implies that $GD = 5$, which is inadequate. Therefore, G is not a plane quintic curve. $\square$

We remark that the restriction map $H^0(\mathbf{P}(\mathscr{E}), H + \Gamma) \to H^0(\hat{W}, H + \Gamma)$ is surjective. We have shown the following:

THEOREM 2.8. *The fibration of type* (I) *is obtained as follows. Let $\hat{W}$ be a surface on $\mathbf{P}(\mathscr{E})$ linearly equivalent to $3H - (g-4)\Gamma$ with at most rational double points. Let Λ be a pencil in $|H + \Gamma|_{\hat{W}}$ whose general member is non-singular. Then X is obtained by resolving singular points of $\hat{W}$ as well as $2g + 1$ base points of Λ.*

2.2. Type (II). We assume that $\hat{W} \sim 2H + \Gamma$. Then ρ is the identity map and D is a non-singular rational curve with $LD = 2$. It follows that $GD = 4$ and $\mathscr{E} = \psi_*\mathscr{O}_Y(L)$. Since L is normally generated, we see that $\hat{W}$ is isomorphic to $\mathrm{Proj}(\bigoplus_{n\ge 0} \psi_*(\mathscr{O}_Y(nL)))$. In particular $\hat{W}$ has at most rational double points. Hence K_Y is obtained as the pull-back of $-H + (g-4)\Gamma$. Then $K_Y^2 = -2g + 11$ and $\Lambda_f \subset |2L - (g-4)D|$. Though the restriction map $H^0(\mathbf{P}(\mathscr{E}), 2H - (g-4)\Gamma) \to H^0(\hat{W}, 2H - (g-4)\Gamma)$ is not surjective, any general member of Λ_f is obtained by cutting $\hat{W}$ by a member of $|2H - (g-4)\Gamma|$. It follows that Λ_f is induced by a pencil in $|2H - (g-4)\Gamma|_{\hat{W}}$.

If $(g; a, b, c) = (6; 1, 1, 1)$, then we can identify $\mathrm{Quad}(W)$ with $\mathbf{P}^1 \times \mathbf{P}^2$ and $\mathrm{Quad}(G)$ is the product of $\mathbf{P}^1$ with a conic curve because it is linearly equivalent to $2H - 2\Gamma$. Hence $\mathrm{Quad}(G) \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and G is a trigonal curve of bi-degree $(3, 4)$, since $W \sim 3\Gamma + 2(H - \Gamma)$. We have $h^0(Y, -K_Y) = 0$.

If $(g; a, b, c) = (6; 0, 1, 2)$, then $\hat{Q}$ is isomorphic to Σ_1 which has Δ as the minimal section. This can be seen as follows. Consider the equation of $\hat{Q}$ as in the proof of Lemma 2.6 where we have shown that $\hat{Q}$ is a Hirzebruch surface. Then φ_{01} and φ_{20}, the coefficients of Z_0Z_2 and Z_1^2, are both non-zero constants, and we see that $\hat{Q}$ contains Δ. We blow $\mathbf{P}(\mathscr{E})$ up along Δ to get $\mathbf{P}(\mathscr{O} \oplus \mathscr{O}(\Delta_0 + 2\Gamma)) \to \Sigma_1$. The proper transform of $\hat{Q}$ is linearly equivalent to $\sigma^*H + \varpi^*\Delta_0$. Then $(\sigma^*H - \varpi^*(\Delta_0 + 2\Gamma))^2(\sigma^*H + \varpi^*\Delta_0) = -1$. This shows that Δ is a (-1)-curve on $\hat{Q}$ and, thus, $\hat{Q} \simeq \Sigma_1$. Since $\mathrm{Quad}(G)$ is obtained from $\hat{Q}$ by contracting Δ, we see that it is isomorphic to $\mathbf{P}^2$. Therefore, G is isomorphic to a plane quintic curve. We remark that it passes through the vertex of $\mathrm{Quad}(W)$, because $GD = 4$. Note also that $|-K_Y|$ is non-empty. Assume that Λ_f is spanned by G and G' whose images are defined in $\hat{W}$ by $\sum_{ij} \varphi_{ij} Z_0^{2-i-j} Z_1^i Z_2^j = 0$ and $\sum_{ij} \varphi'_{ij} Z_0^{2-i-j} Z_1^i Z_2^j = 0$, respectively. Then Λ_f is induced by the pencil $\sum_{ij}(\lambda_0\varphi_{ij} + \lambda_1\varphi'_{ij})Z_0^{2-i-j}Z_1^iZ_2^j = 0$, where $(\lambda_0 : \lambda_1) \in \mathbf{P}^1$. Therefore, it has a remarkable member at $(\lambda_0 : \lambda_1) = (\varphi'_{01} : -\varphi_{01})$ which is of hyperelliptic type with one node, and a reducible member corresponding to $(\lambda_0 : \lambda_1) = (\varphi'_{20} : -\varphi_{20})$.

If $(g; a, b, c) = (5; 0, 1, 1)$, then $\hat{Q}$ is Σ_0 blown up one point, because the proper transform is linearly equivalent to $\sigma^*H + \varpi^*\Delta_0$ on $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\Delta_0 + \Gamma)) \to \Sigma_0$. Since the canonical image of G is linearly equivalent to $3\Delta_0 + 5\Gamma$ in $\mathrm{Quad}(G) \simeq \Sigma_1$, we see that the composite of the inverse of $\hat{Q} \to \mathrm{Quad}(G)$ and $\hat{Q} \to \Sigma_0$ is nothing but the elementary transformation with center a point not lying on the minimal section of $\mathrm{Quad}(G)$. Since $GD = 4$, the transformation must be performed at a point on G. Note that $-K_Y$ moves in a pencil.

We remark that when $g = 5$ we cannot distinguish fibrations of types (I) and (II) in the sense that Y has both pencils: Recall that $\mathrm{Quad}(W)$ is a quadric of rank four in $\mathbf{P}^4$ and it has two rulings by planes. Then Y also has two induced pencils one of which represents Y as a surface of type (I) and another represents it as a surface of type (II).

We have shown the following:

Theorem 2.9. *The fibration of type* (II) *is obtained as follows. Let $\hat{W}$ be a surface on $\mathbf{P}(\mathscr{E})$ linearly equivalent to $2H + \Gamma$ with at most rational double points, where $\mathscr{E} \simeq \mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(c)$ and $(g; a, b, c) = (5; 0, 1, 1), (6; 1, 1, 1), (6; 0, 1, 2)$. Let Λ be a pencil in $|2H - (g-4)\Gamma|_{\hat{W}}$ whose general member is non-singular. Then X is obtained by resolving singular points of $\hat{W}$ as well as* 12 *base points of Λ.*

Let f be a fibration of type (II) *with $g = 6$.*

(1) *If $(a, b, c) = (1, 1, 1)$, then any non-singular non-hyperelliptic fibre of f is a curve of bi-degree $(3, 4)$ on $\mathbf{P}^1 \times \mathbf{P}^1$.*

(2) *If $(a, b, c) = (0, 1, 2)$, then any non-singular non-hyperelliptic fibre of f is a plane quintic curve.*

It should be noticed that the threefold scroll of type $(0, 1, 2)$ is a specialization of that of type $(1, 1, 1)$, while trigonal curves cannot specialize to a quintic curve. Therefore, the above two kinds of fibrations cannot deform to one another.

Example 2.10. Here we give a down-to-earth construction of fibrations of type (II). Let d be an integer with $0 \le d \le 3$. We choose 9 distinct points $p_1, \dots, p_9$ on Σ_d and let $\tau : Y \to \Sigma_d$ be the blowing-up with center $p_1, \dots, p_9$. We put $e_i = \tau^{-1}(p_i)$, $1 \le i \le 9$, and $L_0 = \tau^*(2\Delta_0 + (d+3)\Gamma) - \sum_{i=1}^9 e_i$. Let $(t_0 : t_1)$ be a system of homogeneous coordinates on $\mathbf{P}^1$.

We assume that the p_i's are in sufficiently general position so that there are no curves in $|2\Delta_0 + (d+2)\Gamma|$ passing through all those points. Since $h^0(\Sigma_d, 2\Delta_0 + (d+3)\Gamma) = 12$, we may assume that $h^0(Y, L_0) = 3$ and that $|L_0|$ is free from base points. Since $L_0^2 = 3$, we have a morphism $\phi : Y \to \mathbf{P}^2$ of degree 3. Let ζ_0, ζ_1, ζ_2 be a basis for $H^0(Y, L_0)$ and consider a non-singular curve G defined by a quadratic form in the ζ_i's. Then G is of genus 6 and the restriction of ϕ to G is a morphism of degree 3 onto a conic curve. It follows that G is a trigonal curve. We choose two such curves G_0 and G_1, and let Λ be the pencil spanned by them. We blow Y up at $G_0 \cap G_1$ to get a fibration $f : X \to \mathbf{P}^1$ induced by Λ. We have a morphism $Y \to \mathbf{P}(\mathscr{E})$ over $\mathbf{P}^1$ by putting $Z_i = \zeta_i$ $(0 \le i \le 2)$, where $\mathscr{E} = \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 3}$. Hence this is a fibration of type (II) with $(a, b, c) = (1, 1, 1)$.

We slightly modify the construction by putting $p_1, \dots, p_9$ in a special position. Let $C_1 \in |2\Delta_0 + (d+3)\Gamma|$ and $C_2 \in |2\Delta_0 + (d+2)\Gamma|$ be general members. We assume that they meet at distinct 10 points and put $C_1 \cap C_2 = \{p_0, p_1, \dots, p_9\}$. There exists a member $C_0 \in |2\Delta_0 + (d+4)\Gamma|$ which passes through $p_1, \dots, p_9$ but not p_0. Let $\tau : Y \to \Sigma_d$ and L_0 be as above. Then $|L_0|$ has a transversal base point at p_0, because it comes from the net spanned by $C_2 + \Gamma_1$, $C_2 + \Gamma_2$ and C_1, where Γ_1, Γ_2 are distinct fibres. Let ζ_j be the section defining the proper transform of C_j, $0 \le j \le 2$. Then we have a morphism $Y \to \mathbf{P}(\mathscr{E})$ over $\mathbf{P}^1$ by putting $Z_i = \zeta_i$ $(0 \le i \le 2)$, where $\mathscr{E} = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2)$. Let $G \in |2L_0|$ be a non-singular member which passes through p_0 simply. More precisely, we let G be defined by the equation of the form $c_0\zeta_0\zeta_2 + c_1\zeta_1^2 + c_2\zeta_1\zeta_2 + c_3\zeta_2^2 = 0$, where c_0 and c_1 are non-zero constants and c_2, c_3 are the homogeneous forms of degree 1 and 2 in t_0, t_1, respectively. Eliminating the base point of $|L_0|$, we see that G is mapped birationally onto a plane quintic curve. We choose two such curves G_0, G_1 and consider the pencil spanned by them. Then the induced fibration is of type (II) with $(a, b, c) = (0, 1, 2)$.

3. Fibrations with $(K_X + F)^2 = 2g - 4$

We assume that $(K_X + F)^2 = 2g - 4$ and that the rational map Φ induced by $|K_X + F|$ is a generically finite map onto the image $W \subset \mathbf{P}^{g-1}$. By [**3**, Proposition 1.1], $|K_X + F|$ is free from base points. We further assume that Φ is of degree 2 as a morphism onto W. Since $(K_X + F)^2 = 2g - 4$, we see that $W \subset \mathbf{P}^{g-1}$ is a surface of minimal degree $g - 2$.

In the course of the study, we freely use the results in [**2**] for double coverings.

3.1. Branch loci and hyperelliptic case. Let (Y, G) be the reduction of (X, F). Then f corresponds to a pencil $\Lambda_f \subset |G|$ and Φ factors through Y. We denote by $\varphi : Y \to W$ the induced morphism of degree 2. Recall that $|K_Y + G|$ induces the canonical map of G. Let C be the image of G. It follows that C is a rational normal curve of degree $g - 1$ or it is a canonical curve of degree $2g - 2$. In either case, $\varphi_* G$ is of degree $2g - 2$ as an algebraic cycle.

From the list of surfaces of minimal degree (see, e.g., [**1**]), we see that W is either (i) $\mathbf{P}^2$ $(g = 3)$, (ii) the Veronese surface in $\mathbf{P}^5$ $(g = 6)$, or (iii) a rational normal scroll $(g \ge 4)$. If (iii) is the case, then there is an integer d with $d + g$

even and $0 \le d \le g-2$ such that W is the image of Σ_d by the morphism defined by $\left|\Delta_0 + \dfrac{d+g-2}{2}\Gamma\right|$. In particular, we have $d = g-2$ if and only if W is a cone over the rational normal curve.

3.1.1. Assume that $g = 3$ and $W = \mathbf{P}^2$. We have $K_Y + G = \varphi^*\ell$, where ℓ denotes a line on $\mathbf{P}^2$. We denote by R and B the ramification divisor and the branch locus of φ, respectively. Then $R = K_Y - \varphi^* K_{\mathbf{P}^2} \sim K_Y + 3(K_Y + G)$ and $B = \varphi_* R \sim \varphi_* K_Y + 6\ell$. We have $(\varphi_* K_Y)\ell = K_Y \varphi^*\ell = K_Y(K_Y + G) = -2$. Hence $\varphi_* K_Y = -2\ell$ and $B \in |4\ell|$. Let Y_0 be the finite double covering of $\mathbf{P}^2$ with branch locus B constructed in the total space of $[2\ell]$. Then the dualizing sheaf of Y_0 is induced from $K_{\mathbf{P}^2} + B/2 = -\ell$. By a well-known formula for double coverings [**2**], we get $\omega_{Y_0}^2 = 2$ and $\chi(\mathcal{O}_{Y_0}) = 1$. Since $\chi(\mathcal{O}_Y) = 1$, we see that B has at most simple triple points by considering the canonical resolution. Then Y_0 has at most rational double points as its singularities and Y is the minimal resolution of Y_0. In particular, we have $K_Y = -\varphi^*\ell$. It follows that Y is a weak del Pezzo surface of degree 2. Furthermore, since $K_Y + G = \varphi^*\ell$, we have $G \in |-2K_Y|$ and $\varphi_* G = 4\ell$. If G is hyperelliptic, then it should be obtained as a double covering of a conic curve, because $\varphi_* G = 2(2\ell)$. Since the branch locus is a quartic curve, we indeed have a hyperelliptic curve of genus 3 in this way by the Hurwitz formula.

3.1.2. We assume that $g = 6$ and W is the Veronese surface. We shall show that this cannot happen. Indeed, let ℓ be a line on $\mathbf{P}^2$. Since W is the image of $\mathbf{P}^2$ under the morphism defined by $|2\ell|$, we have $K_Y + G = 2\varphi^*\ell$. It follows that $K_Y\varphi^*\ell = (1/2)K_Y(K_Y + G) = -1$. This implies that $K_Y\varphi^*\ell + (\varphi^*\ell)^2 = 1$, which is impossible because it must be even. Therefore, W cannot be the Veronese surface.

3.1.3. We assume that $g \ge 4$ and that W is a rational normal scroll.

LEMMA 3.1. *Suppose that W is a cone over a rational normal curve. Then $\varphi : Y \to W$ can be lifted to a morphism $h : Y \to \Sigma_{g-2}$ of degree 2, except possibly when $g = 4$.*

Proof. If W is a cone over a rational normal curve, then we get Σ_{g-2} by blowing up the vertex w of W. We let Δ denote the divisorial part of the inverse image on Y of w. Then $(K_Y + G)\Delta = 0$. By pulling back to Y the linear system of hyperplanes through w, we have a linear pencil $|D|$ such that $K_Y + G \sim \Delta + (g-2)D$. Since $2g - 4 = (K_Y + G)^2 = (g-2)(K_Y + G)D$, we have $(K_Y + G)D = 2$. Then $2 = (K_Y + G)D = \Delta D + (g-2)D^2$. Since $g \ge 4$ and $\Delta D \ge 0$, we get $D^2 = 0$ except when $(g, D^2, \Delta D) = (4, 1, 0)$. In the exceptional case, we have $\Delta = 0$ by Hodge's index theorem. If $D^2 = 0$, then $\Delta \ne 0$ and we can lift $\varphi : Y \to W$ to $h : Y \to \Sigma_{g-2}$. □

Assume for a moment that $Y \to W$ can be lifted to $Y \to \Sigma_{g-2}$ even when $g = 4$. For simplicity, we use the symbol φ to denote its lift h. Let R and B denote the ramification divisor and the branch locus of φ, respectively. Then $R = K_Y - \varphi^* K_{\Sigma_d} = K_Y + \varphi^*(2\Delta_0 + (d+2)\Gamma)$ and $B = 6\Delta_0 + (g+3d+2)\Gamma - \varphi_* G$, because $K_Y + G = \varphi^*(\Delta_0 + (d+g-2)/2\Gamma)$. Since B is divisible by 2 in $\mathrm{Pic}(\Sigma_d)$, we can put $\varphi_* G = 2\alpha\Delta_0 + 2\beta\Gamma$ with non-negative integers α and β. Recall that we have $(\varphi_* G)(\Delta_0 + (d+g-2)/2\Gamma) = 2g-2$. It follows $2\beta = -\alpha(g-2-d) + 2g-2$ and we have $B \sim (6-2\alpha)\Delta_0 + (2d+2+(\alpha-1)(g-2-d))\Gamma)$. We have $\Delta_0(B-\Delta_0) \geq 0$ and $\Delta_0\varphi_* G \geq 0$, because B does not have multiple components and the support of $\varphi_* G$ is irreducible. Then we have the following numerical possibilities:

(i) $\alpha = 0$: $d = 0,\ g = 4,\ B \sim 6\Delta_0,\ \varphi_* G \sim 6\Gamma$
(ii) $\alpha = 1$: $d \leq 2,\ B \sim 4\Delta_0 + (2d+2)\Gamma,\ \varphi_* G \sim 2\Delta_0 + (g+d)\Gamma$
(iii) $\alpha = 2$: $d \leq 1,\ B \sim 2\Delta_0 + (g+d)\Gamma,\ \varphi_* G \sim 4\Delta_0 + (2d+2)\Gamma$
(iv) $\alpha = 3$: $d = 0,\ g = 4,\ B \sim 6\Gamma,\ \varphi_* G \sim 6\Delta_0$

The first and the last alternatives are impossible, because the support of $\varphi_* G$ is irreducible of degree $g-1$ or $2g-2$. Note further that, in the second alternative, Δ_0 is a component of B when $d = 2$.

LEMMA 3.2. *In the above situation, $f : X \to \mathbf{P}^1$ is a hyperelliptic fibration and there are the following two cases*:

(1) $B \sim 2\Delta_0 + (g+d)\Gamma,\ \varphi_* G \sim 2(2\Delta_0 + (d+1)\Gamma)$ ($g+d$ *is even*, $d = 0, 1$)
(2) $B \sim 4\Delta_0 + (2d+2)\Gamma,\ \varphi_* G \sim 2\left(\Delta_0 + \dfrac{g+d}{2}\Gamma\right)$ ($g+d$ *is even*, $d = 0, 1, 2$)

Proof. Assume that G is non-hyperelliptic. Then $\varphi_* G$ is a canonical curve of genus g. If (ii) is the case, then $|\Gamma|$ gives $\varphi_* G$ a g_2^1 and $\varphi_* G$ is a hyperelliptic curve, which is inadequate. If (iii) is the case, then $d = 0$ is inadequate for the same reasoning by considering the ruling $|\Delta_0|$. Therefore, we have (ii) with $d = 1$, that is, $W = \Sigma_1$ and $\varphi_* G \sim 4\Delta_0 + 4\Gamma$. But then $\varphi_* G$ is isomorphic to a plane quartic curve, which is also inadequate because $g \geq 4$. Hence G is a hyperelliptic curve. $\square$

In particular, we see that W cannot be a cone over a rational normal curve when $g \geq 5$, since $d \leq 2 < g-2$. We let Y_0 be the double covering of Σ_d with branch locus B and Y^* its canonical resolution. If (1) above is the case, then we have $K_{\Sigma_d} + B/2 \sim -\Delta_0 + ((g+d)/2 - 2)\Gamma$. It follows that $\chi(\mathcal{O}_{Y_0}) = 1$. Hence B has at most simple triple points and $Y^* = Y$. Then $K_Y^2 = \omega_{Y_0}^2 = -2g+8$. G is obtained as a double covering of a rational curve linearly equivalent to $2\Delta_0 + (d+1)\Gamma$. Since $B(2\Delta_0 + (d+1)\Gamma) = 2g+2$, the curve thus obtained is in fact a hyperelliptic curve of genus g. If (2) above is the case, then we have $K_{\Sigma_d} + B/2 \sim -\Gamma$ and $\chi(\mathcal{O}_{Y_0}) = 1$. Hence B has at most simple triple points and $Y^* = Y$. We have $K_Y^2 = \omega_{Y_0}^2 = 0$ which implies that Y is a rational elliptic surface. Furthermore, G is obtained as a double covering of a rational curve linearly equivalent to $\Delta_0 + (g+d)/2\Gamma$. Since $B(\Delta_0 + (g+d)/2\Gamma) = 2g+2$, the curve thus obtained is in fact a hyperelliptic curve of genus g.

We should be more careful when $d = 2$ in (2). Then Δ_0 is a component of B and we write $B_0 = B - \Delta_0$. We remark that B_0 and Δ_0 are disjoint. Then the (-2)-curve Δ_0 produces a (-1)-curve E on Y. In fact, we have $\varphi^*\Delta_0 = 2E$. Since $\Delta_0(\Delta_0 + (g/2+1)\Gamma) = g/2 - 1$, we get $GE = g/2 - 1$. When $g = 4$, we get $(K_Y + G)E = 0$, which is inadequate because (Y, G) is the reduction of (X, F). This shows that $Y \to W$ cannot be lifted to a morphism $Y \to \Sigma_2$ when $g = 4$ and W is a quadric cone.

Now, we study the case that $g = 4$ and W is a quadric cone. We use the same notation as in the proof of Lemma 3.1. Recall that we have $D^2 = 1$. We let $\rho : \hat{Y} \to Y$ be the blowing up at the base point of $|D|$. We put $\hat{D} = \rho^*D - E$, where E denotes the exceptional (-1)-curve. Then $\rho^*(K_Y + G) = 2E + 2\hat{D}$ and $2E$ is the inverse image of the vertex of W. We can lift the induced morphism $\hat{Y} \to W$ to a morphism $h : \hat{Y} \to \Sigma_2$ such that $2E = h^*\Delta_0$, $\hat{D} = h^*\Gamma$. Let $\hat{R}$ and B denote the ramification divisor and the branch locus of h, respectively. Then $\hat{R} = K_{\hat{Y}} - h^*K_{\Sigma_2}$ and $B \sim h_*(\rho^*K_Y + E) + 2(2\Delta_0 + 4\Gamma)$. Since $\rho^*(K_Y + G) = h^*(\Delta_0 + 2\Gamma)$, we have $B \sim 6\Delta_0 + 12\Gamma - h_*\rho^*G + h_*E$. We have $Eh^*(\Delta_0 + 2\Gamma) = 0$ and $E\hat{D} = Eh^*\Gamma = 1$. It follows that $h_*E = \Delta_0$ and $B \sim 7\Delta_0 + 12\Gamma - h_*\rho^*G$. We put $h_*\rho^*G \sim \alpha\Delta_0 + \beta\Gamma$. Since $h_*\rho^*G(\Delta_0 + 2\Gamma) = 2g - 2 = 6$, we have $\beta = 6$. Then $B \sim (7-\alpha)\Delta_0 + 6\Gamma$. Since it must be divided by 2 in the Picard group, we see that α is odd. Let $\hat{G}$ be the proper transform of G by ρ. Since G is non-singular, we have either $\hat{G} \sim \rho^*G$ or $\hat{G} \sim \rho^*G - E$. Then we get $\alpha = 3$, since the support of $h_*\hat{G}$ is irreducible and $\Delta_0(B - \Delta_0) \geq 0$. Therefore, we have either $h_*\hat{G} \sim 3\Delta_0 + 6\Gamma$ or $h_*\hat{G} \sim 2\Delta_0 + 6\Gamma$ according as $\hat{G} \sim \rho^*G$ or not.

Lemma 3.3. *If W is a quadric cone, then $B \sim 4\Delta_0 + 6\Gamma$ and Y is a weak del Pezzo surface of degree one and $G \in |-3K_Y|$. Furthermore, there are the following two cases*:

(1) *G is hyperelliptic and $h_*\hat{G} \sim 2\Delta_0 + 6\Gamma$.*

(2) *G is non-hyperelliptic and $h_*\hat{G} \sim 3\Delta_0 + 6\Gamma$.*

Proof. We can show that B has at most triple points and that the canonical resolution Y^* is isomorphic to $\hat{Y}$ as before. We have the decomposition $B = \Delta_0 + B_0$ with Δ_0 and B_0 being disjoint. Then B is non-singular in a neighbourhood of Δ_0, and Δ_0 produces a (-1)-curve E on $\hat{Y}$. By contracting E, we get Y. We have $K_{\hat{Y}} = -h^*\Gamma = -\hat{D}$. It follows that $\rho^*K_Y + E + \hat{D} \sim 0$, that is, $\rho^*(K_Y + D) \sim 0$. Hence $K_Y \sim -D$ and Y is a weak del Pezzo surface of degree 1. Since $K_Y + G \sim 2D$, we have $G \in |-3K_Y|$. The rest may be clear. □

If G is hyperelliptic, then $\hat{G}$ is a double covering of a member of $|\Delta_0 + 3\Gamma|$. We have $B(\Delta_0 + 3\Gamma) = 10 = 2 \times 4 + 2$. Hence we surely get a hyperelliptic curve of genus 4 in this way. Furthermore, we have $\Delta_0(\Delta_0 + 3\Gamma) = 1$. Hence

$\hat{G}$ meets E normally at a point and it is blown down to a non-singular hyperelliptic curve of genus 4 on Y. We remark that the reduction map $\mu : X \to Y$ factors through $\hat{Y}$.

In sum, we have shown the following:

THEOREM 3.4. *Assume that* $f : X \to \mathbf{P}^1$ *is a hyperelliptic fibration with* $(K_X + F)^2 = 2g - 4$, $g \geq 3$. *Then it is obtained from one of the following datum.*

(1) *A double covering of* $\mathbf{P}^2$ *with branch locus a quartic curve and the pull-back of a pencil of conic curves* $(g = 3)$.

(2) *A double covering of* Σ_d *with branch locus linearly equivalent to* $2\Delta_0 + (g + d)\Gamma$, *and the pull-back of a pencil of curves in* $|2\Delta_0 + (d + 1)\Gamma|$, *where* $d = 0, 1$ *and* $g + d$ *is even* $(g \geq 4)$.

(3) *A double covering of* Σ_d *with branch locus linearly equivalent to* $4\Delta_0 + (2d + 2)\Gamma$, *and the pull-back of a pencil of curves in* $\left|\Delta_0 + \frac{g+d}{2}\Gamma\right|$, *where* $d = 0, 1, 2$ *and* $g + d$ *is even* $(g \geq 4)$.

In all cases, the branch locus has at most simple triple points as its singularity.

3.2. Non-hyperelliptic case. As we have already seen, if $f : X \to \mathbf{P}^1$ is a non-hyperelliptic fibration, then $g = 3$ or 4. We shall study these cases separately.

3.2.1. $g = 3$. The branch locus B is a quartic curve with at most simple triple points, and $\varphi_* G$ is also a quartic curve which is non-singular for a generic choice of $G \in \Lambda_f$. Put $C = \varphi(G)$ and assume that it is non-singular. We also assume that $C \neq B$.

CLAIM 3.5. *C meets B at non-singular points of B.*

Proof. Recall that Y_0 has at most rational double points. If e is a (-2)-curve on Y lying over a rational double points of Y_0, then we have $(K_Y + G)e = 0$. Since $K_Y e = 0$, we get $Ge = 0$. This implies that G does not meet any (-2)-curves coming from rational double points of Y_0. If C passes through a singular point of B, then, in the course of the canonical resolution, the proper transform of C has an intersection with an exceptional curve over that point. Since such an exceptional curve produces a (-2)-curve on Y, from what we have just seen, we conclude that C does not pass through any singular points of B. □

CLAIM 3.6. *C contacts B at every intersection point.*

Proof. The previous claim shows that $\varphi : Y \to \mathbf{P}^2$ is finite over a neighbourhood of C. Since $C \neq B$, $\varphi^* C$ contains G as a component of multiplicity one. If C meets B normally at a point, then $\varphi^* C$ is irreducible and it would follow $G = \varphi^* C$. But this is impossible, because we have $G^2 = 8$ and $(\varphi^* C)^2 = 2C^2 = 32$. □

Hence C contacts B at least to the second order at any points of $C \cap B$. We also remark that the order of contact must be even to have a non-singular curve G over C. Then we have $\varphi^* C = G + G'$ with another curve G' isomorphic to G. Since $C \sim 4\ell$ and $G \sim -2K_Y = 2\varphi^*\ell$, we get $G' \sim G$. Let ι be the involution on Y associated to the double covering $Y \to \mathbf{P}^2$. Then $\iota^*\Lambda_f$ is also a pencil of non-hyperelliptic curves and $G' \in \iota^*\Lambda_f$.

Claim 3.7. $\Lambda_f \neq \iota^*\Lambda_f$.

Proof. Recall that φ is induced by the relative canonical map of f, which is a birational morphism onto the image in our case, followed by the projection. Assume that $G' \in \Lambda_f$. Then the covering transformation group $\langle \iota \rangle$ of $\varphi : Y \to \mathbf{P}^2$ acts on $\Lambda_f \simeq \mathbf{P}^1$ as an automorphism group of order 2. There are two members G_1 and G_2 of Λ_f fixed by this action. Then $G_1 + G_2$ is a part of the ramification divisor. Since $R = 2\varphi^*\ell$, this is impossible. Hence $\iota^*\Lambda_f \neq \Lambda_f$. □

Claim 3.8. $R \in \Lambda_f$.

Proof. Recall that $R \sim 2\varphi^*\ell$. Let $\xi \in H^0(Y, 2[\varphi^*\ell])$ be the equation of R. We may assume that $\iota^*\xi = -\xi$. We have the eigen space decomposition with respect to the action of $\langle \iota \rangle$:

$$H^0(Y, 2\varphi^*\ell) \simeq H^0(\mathbf{P}^2, 2\ell) \oplus H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2})$$

in which ξ generates the last summand, the (-1)-eigen space. The module of Λ_f is a 2-dimensional linear subspace of $H^0(Y, 2\varphi^*\ell)$. It follows that it would be spanned by invariant sections if R were not a member of Λ_f. This is impossible, because $\varphi^*C = G + G'$ and $G' = \iota^*G$. Therefore, $R \in \Lambda_f$. □

We have shown that the module of Λ_f is generated by ξ and a section η with $\iota^*\eta = \eta$. Namely, Λ_f is spanned by R and the pull-back to Y of a (possibly singular) conic curve. Note that B and the conic curve cannot have any common components. Hence

Theorem 3.9. *If $f : X \to \mathbf{P}^1$ is a non-hyperelliptic fibration of genus* 3, *then it is obtained from a pencil* (*without fixed components*) *of quartic curves on $\mathbf{P}^2$ spanned by a reduced but not necessarily irreducible quartic curve B and the double of a conic curve* Δ.

The double conic produces on X a fibre of hyperelliptic type. This can be seen as follows. We already know that Y_0 is a hypersurface of degree 4 in the weighted projective space $\mathbf{P}(1,1,1,2)$ defined by

$$v^2 = a_4(u_0, u_1, u_2),$$

where (u_0, u_1, u_2, v) is a system of coordinates with $\deg u_i = 1$, $\deg v = 2$. Since $G \in |-2K_Y|$, it is given by cutting Y_0 with a hypersurface defined by

$cv + b_2(u_0, u_1, u_2) = 0$, where c is a constant and b_2 denotes a quadratic form in u_0, u_1, u_2. It follows that non-hyperelliptic Λ_f is given by the family of curves in $\mathbf{P}(1,1,1,2)$:

$$\begin{cases} v^2 = a_4(u_0, u_1, u_2) \\ \lambda_0 v = \lambda_1 b_2(u_0, u_1, u_2), \end{cases} \quad ((\lambda_0 : \lambda_1) \in \mathbf{P}^1)$$

Therefore, the member corresponding to $(0:1)$ is of hyperelliptic type.

Similarly, if we replace the second equation by $\lambda_0 b_2(u_0, u_1, u_2) = \lambda_1 b_2'(u_0, u_1, u_2)$, then we get a hyperelliptic Λ_f. Such a hyperelliptic pencil is a specialization of a non-hyperelliptic one.

3.2.2. $g = 4$. As usual, we let $G \in \Lambda_f$ be a general member and put $h(\hat{G}) = C$. Since B is non-singular in a neighbourhood of Δ_0, every (-2)-curve contracted by h is disjoint from E. Hence we can argue as in the previous case to see that C does not pass through any singular points of B and it contacts B at least to the second order at every point of $C \cap B$. Then we have $h^*C = \hat{G} + \hat{G}'$ with another non-singular curve $\hat{G}'$ isomorphic to $\hat{G}$. Since $C \sim 3(\Delta_0 + 2\Gamma)$ and $2E = h^*\Delta_0$, we have $h^*C \sim 6E + 6\hat{D}$. Recall that $\hat{G} = \rho^*G \sim 3\rho^*D \sim 3E + 3\hat{D}$. It follows that $\hat{G}' \sim \hat{G}$. If Λ_f is invariant under the action of $\mathbf{Z}_2$, then similarly as in the case of $g = 3$, we can find two members G_1, G_2 of Λ_f such that $\rho^*(G_1 + G_2)$ is a part of the ramification divisor. Since $\hat{R} \sim 4E + 3\hat{D}$, this is impossible. Note that $\hat{R}$ is of the form $E + R_0$, where $2R_0 = h^*B_0$. We have the following decomposition of $H^0(\hat{Y}, h^*(2\Delta_0 + 3\Gamma))$ into the (± 1)-eigen spaces under the action of the covering transformation group of h:

$$H^0(\hat{Y}, h^*(2\Delta_0 + 3\Gamma)) \simeq H^0(\Sigma_2, 2\Delta_0 + 3\Gamma) \oplus H^0(\Sigma_2, \mathcal{O}_{\Sigma_2})$$

We remark that the (-1)-eigen space is generated by the equation of $\hat{R}$ and that we have $|2\Delta_0 + 3\Gamma| = \Delta_0 + |\Delta_0 + 3\Gamma|$. Then it can be shown that $\rho^*\Lambda_f$ is spanned by R_0 and E plus a curve obtained by the pull-back of a member of $|\Delta_0 + 3\Gamma|$ similarly as in the previous case. This yields the following:

Theorem 3.10. *If $f : X \to \mathbf{P}^1$ is a non-hyperelliptic fibration of genus 4, then it is obtained from a pencil of curves on Σ_2 spanned by a reduced irreducible curve $B_0 \in |3\Delta_0 + 6\Gamma|$ and $\Delta_0 + 2\Delta$, where $\Delta \in |\Delta_0 + 3\Gamma|$.*

It is possible to write down the equation defining Λ_f. Let $\{u_0, u_1\}$ be a basis for $H^0(Y, -K_Y) \simeq \mathbf{C}^2$. Two curves $u_0 = 0$ and $u_1 = 0$ meet at the base point P of $|D|$. We have for $m \geq 2$

$$h^0(Y, -mK_Y) = \frac{1}{2}m(m+1) + 1$$

In particular, $H^0(Y, -2K_Y) \simeq \mathbf{C}^4$ has an element v linearly independent from the three products u_iu_j. Since $-2K_Y \sim K_Y + G$ and $\mathrm{Bs}|K_Y + G| = \emptyset$, v does not vanish at P. Then (u_0, u_1, v) defines the double covering $Y \to W = \mathbf{P}(1,1,2)$.

We have $h^0(Y, -3K_Y) = 7$. We can find 6 elements $u_i u_j u_k$, $u_i v$ in $H^0(Y, -3K_Y)$. Clearly they are linearly independent. Hence there is a new element $w \in H^0(Y, -3K_Y)$ which does not vanish at P. It is not so hard to see that the 4 elements u_0, u_1, v, w generate the anti-canonical ring $R(Y, -K_Y)$. Since $w^2 \in H^0(Y, -6K_Y)$, there is a relation of the form

$$w^2 = a_0 v^3 + a_4(u)v + a_6(u), \quad 0 \neq a_0 \in \mathbf{C},$$

that is, the anti-canonical model is a hypersurface of degree 6 in $\mathbf{P}(1,1,2,3)$. Since $G \in |-3K_Y|$ is given by $cw + b_1(u)v + b_3(u) = 0$, this shows that non-hyperelliptic Λ_f is the family of curves defined by

$$\begin{cases} w^2 = a_0 v^3 + a_4(u_0, u_1)v + a_6(u_0, u_1), \\ \lambda_0 w = \lambda_1 (b_1(u_0, u_1)v + b_3(u_0, u_1)), \end{cases} \quad ((\lambda_0 : \lambda_1) \in \mathbf{P}^1)$$

in $\mathbf{P}(1,1,2,3)$. Again, the member corresponding to $(0:1)$ is of hyperelliptic type. When Λ_f is hyperelliptic, the second equation should be replaced by a cubic not involving w.

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Kazuhiro Konno
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043
Japan
E-mail: konno@math.sci.osaka-u.ac.jp