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REGULAR SEPARATION WITH PARAMETER OF COMPLEX ANALYTIC SETS

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Abstract

The aim of this paper is to prove that a pair of analytic sets $X, Y \subset \mathbf{C}_z^m \times \mathbf{C}_w^n$ is locally regularly separated with a uniform exponent α in the fibres taken over a proper projection $\pi(z,w) = z$ of $X \cap Y$ (under the assumption that $X \cap Y$ has pure dimension): for all $z \in \pi(X \cap Y) \cap U$, $\mathrm{dist}(w, Y^z) \geq \mathrm{const.dist}(w, (X \cap Y)^z)^\alpha$ when $w \in X^z \cap V$, where $U \times V$ is a neighbourhood of a point $a \in X \cap Y$ such that $\pi(a)$ is regular in $\pi(X \cap Y)$. As an application of this we obtain a parameter version of the Łojasiewicz inequality for c-holomorphic mappings. Both results are a complex counterpart of the main result of [ŁW] from the subanalytic case, extended in this paper by a bound on the uniform exponent.

1. Preliminaries

The regular separation inequality was first proved by S. Łojasiewicz (see [Ł1]) for semi-analytic sets (it was an essential element of his solution to L. Schwartz' famous Division Problem). Then it was established for subanalytic sets by H. Hironaka (see [H]).

A parameter version of the regular separation Łojasiewicz inequality is well-known in the real case for subanalytic bounded sets, see [ŁW]. It is thence clear that it holds also for complex analytic sets with proper projection, but till now there are no such theorems proved. We shall fill this gap and we will achieve this aim using only complex analytic geometry tools.

By a result of [Ł1] (see also [Ł2]) any two complex locally analytic sets $X, Y \subset \mathbf{C}^m$ are *regularly separated* at any point $a \in X \cap Y$ i.e. there is a neighbourhood $U \ni a$ and positive constants $\alpha, c > 0$ such that

$$(*) \qquad \mathrm{dist}(z, X) + \mathrm{dist}(z, Y) \geq c \cdot \mathrm{dist}(z, X \cap Y)^\alpha, \quad z \in U,$$

where the distance is computed in any of the usual norms on $\mathbf{C}^m$. Note that if $a \notin \mathrm{int}(X \cap Y)$ (the interior being computed in $\mathbf{C}^m$), then in view of the fact

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that $\operatorname{dist}(z,X)+\operatorname{dist}(z,Y)\le 2\operatorname{dist}(z,X\cap Y)$, there must be $\alpha\ge 1$. Moreover, if U is small enough to have $\operatorname{dist}(z,X\cap Y)<1$, then $(*)$ holds also with any exponent $\ge\alpha$. It is thus interesting to study the minimal exponent, i.e. the Łojasiewicz exponent of X and Y at $a\in X\cap Y$, denoted by $\mathscr{L}(X,Y;a)$, being the greatest lower bound of all exponents $\alpha>0$ for which $(*)$ holds (see e.g. [CgT], [S]).

Condition $(*)$ is clearly a condition on germs and is biholomorphically invariant. Therefore, using local coordinates it can be carried over to manifolds. Note also that even though the exponent α may change from point to point, on compact neighbourhoods it can be chosen independently of $a\in X\cap Y$.

The following version of lemma 1.2 from [CgT] (being itself a variant of [Ł1] p. 84) will be useful for our purposes:

Lemma 1.1. *Let X, Y be closed subsets of an open set $\Omega\subset\mathbf{C}^m$ and let $a\in X\cap Y$, $\alpha\ge 1$. Then X, Y satisfy condition $(*)$ with α iff there is a neighbourhood $U\ni a$ such that*

$$\operatorname{dist}(z,Y)\ge\mathrm{const}\cdot\operatorname{dist}(z,X\cap Y)^{\alpha},\quad z\in X\cap U.$$

Proof. We need only prove the 'if' part. We treat two cases separately:
(i) If $a\in\overline{X\setminus Y}$, we refer the reader to [CgT].
(ii) If $a\notin\overline{X\setminus Y}$, then there is a neighbourhood $V\ni a$ such that $V\cap X\subset Y$. Let $B(a,r)$ denote the open ball with center a and radius r given by the norm used to compute the distance. Let $r>0$ be such that $B(a,2r)\subset V$ and $\operatorname{dist}(z,X\cap Y)<1$ for $z\in B(a,r)$. Then for all $z\in B(a,r)$ we have

$$\operatorname{dist}(z,X\cap Y)=\operatorname{dist}(z,X\cap Y\cap V)=\operatorname{dist}(z,X\cap V)=\operatorname{dist}(z,X).$$

Therefore, since $\alpha\ge 1$, for all $z\in B(a,r)$,

$$\operatorname{dist}(z,X\cap Y)^{\alpha}\le\operatorname{dist}(z,X\cap Y)\le\operatorname{dist}(z,X)+\operatorname{dist}(z,Y),$$

which completes the proof. □

As a matter of fact we shall be dealing with isolated fibres. Note however that one of the main consequences of the regular separation with parameter in the real case is a parameter version of Whitney's property for subanalytic sets which brings an interesting metric information about the geometric structure of such sets. On the other hand, our approach permits to obtain some nice bounds on the uniform exponent which could be very useful in applications.

2. Regular separation with parameter

Since all our considerations are local, we confine ourselves to the case of $\mathbf{C}^N$. Let X, Y be locally analytic subsets of $\mathbf{C}^m_z\times\mathbf{C}^n_w$ such that $0\in X\cap Y$. Let $\pi(z,w)=z$ be the natural projection. For any $z\in\pi(X\cap Y)$ we denote the *fibre of X over z* by $X^z:=\{w\in\mathbf{C}^n\mid(z,w)\in X\}$. Finally, let $\mu_0(\pi|_{X\cap Y})$

denote the (generalized) multiplicity of $\pi|_{X\cap Y}$ (computed as the upper limit $\limsup_{z\to 0} \#\pi^{-1}(z)\cap X\cap Y$—see [Ch] for details). By $\deg_0 X$ we mean the classical degree (Lelong number) of the analytic set X at $0\in X$. The aim of this part is to prove the following theorem:

Theorem 2.1. *Suppose $X\cap Y$ has pure dimension k and π is proper on $X\cap Y$. Then there exists a neighbourhood $U\times V$ of $0\in\mathbf{C}^m\times\mathbf{C}^n$ and a positive constant $s\geq 1$ such that for all $z\in\pi(X\cap Y)\cap U$,*

$$\operatorname{dist}(w, Y^z)\geq c(z)\cdot\operatorname{dist}(w,(X\cap Y)^z)^s,\quad w\in X^z\cap V,$$

for some $c(z)>0$. Moreover,

$$s\leq \deg_0 \pi(X\cap Y)\cdot\mu_0(\pi|_{X\cap Y})\cdot\mathscr{L}(X,Y;0).$$

To obtain this result we shall need among others a sharpened version of proposition 2.2 from [CgT] (one can derive its proof from the one given in [CgT], see also [Ł2] where the methods used in [CgT] were first introduced):

Proposition 2.2. *Let D be an m-dimensional connected complex manifold and $A\subset D\times\mathbf{C}^n_w$ an analytic closed set of pure dimension m with proper projection $\pi(z,w)=z$ onto D. Then there exists a holomorphic mapping $F: D\times\mathbf{C}^n\to\mathbf{C}^r$ with $r=d(n-1)+1$, where d is the multiplicity (sheet number) of the branched covering $\pi|_A$, such that*

(1) $F^{-1}(0)=A$;

(2) *For any point $z\in D$ for which there are exactly d pairwise different points $w^1,\ldots,w^d$ such that $(z,w^j)\in A$, we have*

$$|F(z,w)|\geq\prod_{j=1}^{d}|w-w^j|,\quad \textit{for all } w\in\mathbf{C}^n;$$

(3) $|F(z,w)|\geq\operatorname{dist}(w,A^z)^d$, *for* $(z,w)\in D\times\mathbf{C}^n$.

The following lemma is quite clear:

Lemma 2.3. *Let $F:\Omega\to\mathbf{C}^r$ be locally lipschitz in an open set $\Omega\subset\mathbf{C}^m$. Then for any $a\in F^{-1}(0)$ there is a neighbourhood $U\ni a$ such that*

$$|F(z)|\leq\operatorname{const}\cdot\operatorname{dist}(z,F^{-1}(0)),\quad \textit{for } z\in U.$$

Proof of theorem 2.1. Take a neighbourhood $U\subset\mathbf{C}^m$ such that $\bigcup_1^r Z_j=\pi(X\cap Y)\cap U$ is the decomposition into irreducible components of the germ $\pi(X\cap Y)$ at zero. By Remmert's theorem each Z_j has pure dimension k. Let W_j denote the union of those irreducible components of $(X\cap Y)\cap(U\times\mathbf{C}^n)$ which are projected onto Z_j. Shrinking U if necessary, we may assume that there is a neighbourhood $V\subset\mathbf{C}^n$ of zero such that for any j and each $z\in Z_j$ we have $\operatorname{dist}(w, (W_j)^z) < 1$ for each $w \in V$ (this follows easily from the properness of $\pi|_{W_j}$). We assume also that $U \times V$ is such that X and Y satisfy in it condition $(*)$ with an exponent $\alpha \geq 1$. By a recent result of Spodzieja [S] the latter can be taken to be equal to $\mathscr{L}(X, Y; 0)$.

Over the connected manifold $\operatorname{Reg} Z_j$, the projection $\pi|_{W_j}$ is a branched covering with multiplicity p_j. Let $\sigma_\pi^j \subset \operatorname{Reg} Z_j$ denote its critical set. Note that $\sigma_\pi^j \cup \operatorname{Sng} Z_j$ is closed.

Without loss of generality we may assume that coordinates in $\mathbf{C}^m$ are chosen in such a way, that the natural projection ρ onto the first k coordinates is proper on Z_j for all j and realizes the respective degrees $q_j := \deg_0 Z_j$. We may assume too that $U = U' \times U'' \subset \mathbf{C}_x^k \times \mathbf{C}_y^{m-k}$. Since ρ is a branched covering on each Z_j with multiplicity q_j, $\rho \circ \pi$ is proper on $W_j \cap (U \times \mathbf{C}^n)$ and has multiplicity $p_j q_j$ as a branched covering. Let $\sigma_j \subset U'$ denote its critical set. One easily checks that $\sigma_j \subset \sigma_\rho^j \cup \rho(\sigma_\pi^j)$, where σ_ρ^j is critical for $\rho|_{Z_j}$. Moreover, $\Sigma_j := \sigma_\rho^j \cup \rho(\sigma_\pi^j)$ is closed and nowheredense in U' (though it need not be analytic). Note that the points of the fibres over U' vary continuously.

Take the mapping F_j from proposition 2.2 describing the set W_j. Then for any point $x \in U' \backslash \Sigma_j$ we have exactly q_j pairwise different points $y^1, \ldots, y^{q_j}$ such that $(x, y^l) \in Z_j$, and for each such a point y^l we have exactly p_j pairwise different points $w^{l1}, \ldots, w^{lp_j}$ such that $(x, y^l, w^{l\kappa}) \in W_j$. Thus by proposition 2.2 (2) we have for each l fixed

$$(\#) \qquad |F_j(x, y^l, w)| \geq \prod_{\iota, \kappa} |(y^l - y^\iota, w - w^{\iota\kappa})|, \quad \text{for all } w \in \mathbf{C}^n.$$

We may assume that the considered norm $|\cdot|$ is the ℓ_1 norm i.e. sum of moduli. Then if $\iota = l$, we obtain $|(y^l - y^\iota, w - w^{\iota\kappa})| = |(0, w - w^{l\kappa})| = |w - w^{l\kappa}|$. Otherwise we have $|(y^l - y^\iota, w - w^{\iota\kappa})| \geq |y^l - y^\iota| > 0$ and so there is a constant $c_j(x, y^l) = \prod_{\iota \neq l} |y^l - y^\iota| > 0$ such that

$$(\#\#) \qquad |F_j(x, y^l, w)| \geq c_j(x, y^l) \operatorname{dist}(w, (W_j)^{(x, y^l)})^{p_j}, \quad \text{for all } w \in \mathbf{C}^n.$$

We have to extend this inequality to the whole of Z_j. Take a point $z \in Z_j$ such that $\rho(z) \in \Sigma_j$. We have either $\rho(z) \in \sigma_\rho^j$ or $\rho(z) \notin \sigma_\rho^j$. If the latter occurs, then approximating $\rho(z)$ by any sequence of points not belonging to Σ_j we extend (##) by continuity. If however $\rho(z)$ is critical for $\rho|_{Z_j}$, then approximating it once again but using (#) this time, we obtain 'at worst' $(\prod_\kappa |w - w^{l\kappa}|)^{q_j}$ as a lower bound of it, when $w \in V$. Summing up, we may write for each j (remember how was chosen the set V)

$$|F_j(z, w)| \geq c_j(z) \operatorname{dist}(w, (W_j)^z)^{p_j q_j} \geq c_j(z) \operatorname{dist}(w, (X \cap Y)^z)^{p_j q_j}, \quad w \in V.$$

Shrinking U if necessary, we may assume lemma 2.3 is satisfied for $G := |F_1| \cdot \ldots \cdot |F_r|$ in $U \times V$. Note that $G^{-1}(0) = (X \cap Y) \cap (U \times V)$. So as not to have any problem with computing the distances we may once again shrink both U and V and we get finally for $(z, w) \in (U \cap \bigcup_j Z_j) \times V$,

$$c(z)\ \mathrm{dist}(w, (X \cap Y)^z)^\beta \le G(z,w) \le \mathrm{const} \cdot \mathrm{dist}((z,w), X \cap Y)$$
$$\le \mathrm{const} \cdot (\mathrm{dist}((z,w), X) + \mathrm{dist}((z,w), Y))^{1/\alpha},$$

where $\beta := \sum_j p_j q_j$ and $c(z) = (\min_j c_j(z))^r > 0$ depends only on z. Now since $\mathrm{dist}((z,w), X) \le \mathrm{dist}(w, X^z)$ (and similarly for Y), we obtain the result sought after with $\beta\alpha$ as exponent.

It remains to observe that $\beta \le (\max_j p_j) \sum_j \deg_0 Z_j$ and it is clear that $\mu_0(\pi|_{X\cap Y}) = \max_j p_j$, if $V \cap (X \cap Y)^0 = \{0\}$. □

THEOREM 2.4. *Under the hypothesis of theorem* 2.1, *if zero is a regular point of* $\pi(X \cap Y)$, *then* $c(z)$ *can be chosen independent of* $z \in \pi(X \cap Y) \cap U$, *for some neighbourhood* U *of* a.

Proof. We proceed as in the precedent proof, this time, however, the germ $\pi(X \cap Y)$ is irreducible and there is a single one mapping F given by proposition 2.2. Moreover, F can be taken directly on the manifold $W \cap Z$, where W is a neighbourhood of zero in which $Z := \pi(X \cap Y)$ is a k-dimensional complex connected submanifold. Therefore, $p := \mu_0(\pi|_{X\cap Y})$ is well-defined without using upper limits (see [Ch]). Then we need just one more lemma:

LEMMA 2.5. *If* D *is a* k*-dimensional complex submanifold of an open set* $\Omega \subset \mathbf{C}^m$ *and* $F : D \times \mathbf{C}^n \to \mathbf{C}^r$ *is holomorphic, then for any* $a \in F^{-1}(0)$ *there is a neighbourhood* $U \times V \ni a$ *such that*

$$|F(z,w)| \le \mathrm{const} \cdot \mathrm{dist}((z,w), F^{-1}(0))$$

for $(z,w) \in (U \cap D) \times V$, *where the distance is computed in* $\mathbf{C}^m \times \mathbf{C}^n$.

Proof of the lemma. Fix $a = (a_D, a') \in F^{-1}(0)$. Let $\Phi : \mathbf{E}^k \times \mathbf{E}^{m-k} \to W$, where $\mathbf{E}$ is the unit disc in $\mathbf{C}$, be a biholomorphism onto an open neighbourhood W of a_D, such that $\Phi(\mathbf{E}^k \times \{0\}^{m-k}) = W \cap D$. Take then $\Psi := \Phi \times \mathrm{id}_{\mathbf{C}^n}$. Due to the generalized mean value theorem, there is a neighbourhood $\tilde{U} \times V \subset \mathbf{E}^m \times \mathbf{E}^n$ of $\Psi^{-1}(a)$ and a constant $c > 0$ such that

$$|F(\Psi(u,w))| \le c \cdot \mathrm{dist}((u,w), \Psi^{-1}(F^{-1}(0))), \quad (u,w) \in [\tilde{U} \cap (\mathbf{E}^k \times \{0\}^{m-k})] \times V.$$

Let $L > 0$ be the Lipschitz constant of $(\Psi)^{-1}$ in $\tilde{U} \times V$. Take $U \times V := \Psi(\tilde{U} \times V)$. Then it is easy to see that

$$|F(z,w)| \le cL \cdot \mathrm{dist}((z,w), F^{-1}(0)), \quad (z,w) \in (U \cap D) \times V$$

and the proof is completed. □

To finish the proof of theorem 2.4 we observe that there exists a neighbourhood $U \times V$ of $0 \in \mathbf{C}^n$ (with $U \subset W$) such that on the one hand X and Y satisfy condition $(*)$ in $U \times V$ with an exponent $\alpha \ge 1$, while on the other lemma 2.5 is satisfied for F in $U \times V$. Thus we have for $(z,w) \in (U \cap Z) \times V$,

$$\operatorname{dist}(w, (X \cap Y)^z)^p \le |F(z,w)| \le \mathrm{const} \cdot \operatorname{dist}((z,w), X \cap Y)$$
$$\le \mathrm{const} \cdot (\operatorname{dist}((z,w), X) + \operatorname{dist}((z,w), Y))^{1/\alpha}.$$

Now since $\operatorname{dist}((z,w), X) \le \operatorname{dist}(w, X^z)$ we obtain the required result with $p\alpha$ as exponent. □

Remark 2.6. We should remark that both theorems from this section deal in fact with 'analytically parametrized' families of analytic sets having isolated intersections. It is however more convenient to consider the problem as we presented it, i.e. adopting the point of view involving fibres of analytic sets.

Moreover, unlike in the real case from [ŁW], we are able to obtain also a uniform constant $c(z) = c > 0$ under the additional hypothesis of theorem 2.4 (smoothness of the parameter set).

3. Łojasiewicz inequality with parameter

For the convenience of the reader we recall the definition of a c-holomorphic mapping (this notion goes back to Remmert [R]). Let A be a complex analytic subset of an open set $\Omega \subset \mathbf{C}^m$.

Definition 3.1 ([Ł2], [Wh]). A mapping $f : A \to \mathbf{C}^n$ is called *c-holomorphic* if it is continuous and the restriction of f to the subset $\operatorname{Reg} A$ of regular points is holomorphic. We denote by $\mathcal{O}_c(A, \mathbf{C}^n)$ the vector space of c-holomorphic mappings, and by $\mathcal{O}_c(A)$ the ring of c-holomorphic functions.

This is a way of generalizing the notion of holomorphicity to sets having singularities. A well-known theorem states that a mapping defined in an open set is holomorphic if and only if it is continuous and its graph is an analytic set (it is then a submanifold). We have a similar result for c-holomorphic mappings (cf. [Wh] 4.5Q), which motivates this generalization:

Theorem 3.2. *A mapping $f : A \to \mathbf{C}^n$ is c-holomorphic iff it is continuous and its graph $\Gamma_f := \{(x, f(x)) \mid x \in A\}$ is an analytic subset of $\Omega \times \mathbf{C}^n$.*

By this theorem the zero set of a c-holomorphic function is analytic.

By [D1] each c-holomorphic mapping satisfies the *Łojasiewicz inequality*: for all $a \in f^{-1}(0)$ there exists a neighbourhood $U \ni a$ and positive constants $\alpha, c > 0$ such that

$$|f(z)| \ge c \cdot \operatorname{dist}(z, f^{-1}(0))^{\alpha}, \quad z \in A \cap U.$$

In [D2] we checked that if A has pure dimension k, $n = 1$ and f does not vanish on any irreducible component of A, then its zero set has pure dimension $k-1$, unless it is void. Thus the following corollary to theorem 2.1 is peculiarly useful in such a case, when we project $f^{-1}(0)$ properly onto $\mathbf{C}^{k-1}$.

THEOREM 3.3. *Let A be a pure k-dimensional analytic subset of an open set $U \times V \subset \mathbf{C}_x^r \times \mathbf{C}_y^{m-r}$ and let $\pi(x, y) = x$ be the natural projection onto U. Let $f : A \to \mathbf{C}_w^n$ be c-holomorphic non-constant and such that $f^{-1}(0)$ has pure dimension $p \leq r$ and π is proper on it. Then for each $a \in f^{-1}(0)$ there exists a neighbourhood $G \times H$ of a and an exponent $\alpha > 0$ such that for all $x \in \pi(f^{-1}(0)) \cap G$,*

$$|f(x, y)| \geq c(x) \operatorname{dist}(y, (f^{-1}(0))^x)^\alpha, \quad y \in H \cap A^x,$$

with some $c(x) > 0$. The latter can be chosen independent of x if $\pi(a)$ is regular in $\pi(f^{-1}(0))$.

Proof. We may assume that $a = 0 \in f^{-1}(0)$. It is easy to see that Γ_f has pure dimension k. Put $X := A \times \{0\}^n$ and $Y := \Gamma_f$. Obviously $X \cap Y = f^{-1}(0) \times \{0\}^n$. It is also clear that $(y, w) \in (X \cap Y)^x$ iff $w = 0$, $(x, y) \in A$ and $f(x, y) = 0$ (i.e. $y \in (f^{-1}(0))^x$). Therefore $\operatorname{dist}((y, 0), (X \cap Y)^x) = \operatorname{dist}(y, (f^{-1}(0))^x)$.

Assume that the norm in consideration is the ℓ_1 norm. Then for $(x, y, 0) \in X$ we obtain $\operatorname{dist}((y, 0), Y^x) \leq |f(x, y)|$.

Now applying theorem 2.2 we get a neighbourhood $G \times H \times W$ of $0 \in \mathbf{C}^r \times \mathbf{C}^{m-r} \times \mathbf{C}^n$ and an exponent $\alpha > 0$ such that for $x \in \pi(f^{-1}(0)) \cap G$,

$$\operatorname{dist}((y, 0), Y^x) \geq c(x) \operatorname{dist}((y, 0), (X \cap Y)^x)^\alpha, \quad (y, 0) \in (H \times W) \cap X^x,$$

with some $c(x) > 0$. This yields the required assertion.

Obviously if zero is regular in $\pi(f^{-1}(0))$, then theorem 2.2 gives the second part of the assertion. $\square$

Remark 3.4. Applying theorem 2.2 we easily get the following inequality for the exponent α:

$$\alpha \leq \deg_{\pi(a)} \pi(f^{-1}(0)) \cdot \mu_a(\pi|_{f^{-1}(0)}) \cdot \mathscr{L}(\Gamma_f, A \times \{0\}^n; a)$$

which in case A is open becomes (cf. theorem 2.5 in [D1])

$$\alpha \leq \deg_{\pi(a)} \pi(f^{-1}(0)) \cdot \mu_a(\pi|_{f^{-1}(0)}) \cdot \mathscr{L}(f; a),$$

where $\mathscr{L}(f; a)$ is the Łojasiewicz exponent of f at a (see e.g. [D1]).

Note that we are actually dealing with c-holomorphically parametrized families of c-holomorphic mappings. However, the approach involving a single one c-holomorphic mapping is obviously more convenient.

We shall also give here, in a special case, another proof of the theorem above, making no use this time of theorem 2.2:

THEOREM 3.5. *Let A be a pure k-dimensional analytic subset of the open set $U \times V \subset \mathbf{C}_x^r \times \mathbf{C}_y^{m-r}$. Assume that $f : A \to \mathbf{C}^n$ is a non-constant c-holomorphic mapping such that $f^{-1}(0)$ has pure dimension r and the natural projection $\pi(x, y) = x$ is a d-sheeted branched covering on it. If*

(1) *either* $k = m$,

(2) *or* $k \leq m$ *and* $r = k - n$,

then for each $a \in f^{-1}(0)$ *there exists a neighbourhood* $G \times H$ *of* a *and positive constants* $c, \alpha > 0$ *such that for all* $x \in G$,

$$|f(x, y)| \geq c \cdot \mathrm{dist}(y, (f^{-1}(0))^x)^\alpha, \quad y \in H \cap A^x.$$

Proof. Applying proposition 2.2 we obtain a mapping $F : U \times V \to \mathbf{C}^p$ such that $F^{-1}(0) = f^{-1}(0)$ and

$$|F(x, y)| \geq \mathrm{dist}(y, (f^{-1}(0))^x)^d, \quad (x, y) \in U \times V.$$

We have $A \cap F_j^{-1}(0) \supset \bigcap_{\iota=1}^n f_\iota^{-1}(0)$ for $j = 1, \dots, p$.

Now, assumption (1) means that in fact A is open and f is holomorphic in it, while assumption (2) is exactly the assumption of the Nullstellensatz for c-holomorphic functions from [D2]. Thus in both cases we may apply the local c-holomorphic Nullstellensatz to the functions F_j (restricted to A) with respect to $(f_1, \dots, f_n)$. Namely, when j is fixed, for each $a \in f^{-1}(0)$ there exists a neighbourhood $U_j \times V_j \subset U \times V$ of a, a positive integer k_j and n c-holomorphic functions $\tilde{h}_{j\nu}$ on $A \cap (U_j \times V_j)$ such that

$$F_j^{k_j} = \sum_{\nu=1}^n \tilde{h}_{j\nu} f_\nu, \quad \text{on } A \cap (U_j \times V_j).$$

Let $G \times H := \bigcap_{j=1}^p U_j \times V_j$ and $k := \prod_{j=1}^n k_j$. Shrinking G and H if necessary, we may assume that for $j = 1, \dots, p$,

$$F_j^k = \sum_{\nu=1}^n h_{j\nu} f_\nu, \quad \text{on } A \cap (G \times H),$$

and the functions $|h_{j\nu}|$ are all bounded by $M > 0$.

Assume that the norm in consideration is the maximum norm. Now for $(x, y) \in A \cap (G \times H)$ we have on the one hand $|F(x, y)|^k \geq \mathrm{dist}(y, (f^{-1}(0))^x)^{dk}$, while on the other for all $j = 1, \dots, n$,

$$|F_j(x, y)|^k \leq \sum_{\nu=1}^n |h_{j\nu}(x, y)| \cdot |f_\nu(x, y)| \leq Mn|f(x, y)|.$$

This gives the result sought after. □

Remark 3.6. In view of the results of [D2] and with its notation, we have in the theorem above, under the assumption (2), the following estimate of the uniform exponent:

$$\alpha \leq \mu_a(\pi|_{f^{-1}(0)}) \cdot \deg_a Z_f.$$

Here, $Z_f := \Gamma_f \cdot (\mathbf{C}^m \times \{0\})$ is the proper intersection cycle of zeroes of f (see [Ch] and [D2]).

Similarly to the real case, we obtain most of the time a constant $c(x) > 0$ depending on the parameter defining the fibres. It may be chosen independent of x only when there are no singularities in the analytic set of parameters (this is not true in the real case). It hardly seems possible to obtain a uniform constant $c > 0$ in the general setting.

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Added in proof. By the revised version of [D2] recently published in Ann. Polon. Math., theorem 3.5 holds true also in the case when A is a locally irreducible analytic set with no additional assumptions on r.

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